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RIGID ARTINIAN RINGS

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1. Introduction

In [**4**], Maxson studied the properties of a ring R whose only ring endomorphisms $\phi: R \to R$ are the trivial ones, namely the identity map, id_R, and the map 0_R given by $\phi(R) = 0$. We shall say that any such ring is *rigid*, slightly extending the definition used in [**4**] by dropping the restriction that $R^2 \neq 0$. Maxson's most detailed results concerned the structure of rigid artinian rings, and our main aim is to complete this part of his investigation by establishing the following

Theorem. *Let $R(\neq 0)$ be a left-artinian ring. Then R is rigid if and only if*

(i) $R \cong Z_{p^k}$, *the ring of integers modulo a prime power* p^k,

(ii) $R \cong N_2$, *the null ring on a cyclic group of order* 2, *or*

(iii) *R is a rigid field of characteristic zero.*

We shall also show that if $R(\neq 0)$ is any (not necessarily artinian) nilpotent ring, then R is rigid if and only if $R \cong N_2$.

From [**4**] Theorem 3.1, any rigid artinian ring is commutative and is either local or nilpotent. We shall examine these two cases separately.

2. Rigid artinian local rings

Let R be a rigid artinian local ring with radical $\boldsymbol{m}$.

Maxson showed that if char $R = 0$, then R is a field [**4**, Theorem 3.2] and that if char R is a prime p, then $R \cong Z_p$ [**4**, Corollary 3.2].

The remaining possibility is that char $R = p^k$ and char $(R/\boldsymbol{m}) = p$ for some prime p and some integer $k > 1$. In this case there exists an unramified complete discrete valuation ring V of characteristic zero whose residue field is isomorphic to $R/\boldsymbol{m}$. This ring V is uniquely determined up to isomorphism by $R/\boldsymbol{m}$ and is called the *v-ring* with residue field $R/\boldsymbol{m}$. (See [**1**] Lemma 13, p. 79 and Corollary 2, p. 83.) Moreover, by [**1**] Theorem 11, p. 79, R contains a subring C such that $C \cong V/p^kV$, $C \cap \boldsymbol{m} = pC$ and $R = C + \boldsymbol{m}$. We shall call any such subring C a *coefficient ring* of R.

The following theorem, augmented by Maxson's results, determines the structure of a rigid artinian local ring.

Theorem 1. *Let R be a rigid artinian local ring with radical* $\boldsymbol{m}$ *such that* char $R = p^k$ *and* char $(R/\boldsymbol{m}) = p$ *for some prime p and some integer $k > 1$. Then $R \cong Z_{p^k}$.*

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Proof. Since $k>1$, $\boldsymbol{m}\neq 0$ and we can choose a non-zero element $a\in\boldsymbol{m}^{t-1}$, where t is the index of nilpotence of $\boldsymbol{m}$.

Let C be a coefficient ring of R and $\theta\colon R\to R/(pR+\boldsymbol{m}^2)$ be the natural map. We have

$$pC\subseteq C\cap(pR+\boldsymbol{m}^2)\subseteq C\cap\boldsymbol{m}=pC$$

and $C/pC\cong R/\boldsymbol{m}$. It follows that $\theta(C)$ is a field.

Suppose that $\theta(R)\neq\theta(C)$. Then there exist elements $x_1,\ldots,x_n\in R$ such that $\theta(1)$, $\theta(x_1),\ldots,\theta(x_n)$ is a basis for the $\theta(C)$-vector space $\theta(R)$. Now define a map $\phi\colon R\to R$ as follows. Given $r\in R$, there are unique elements $\theta(c_i)\in\theta(C)$ such that

$$\theta(r)=\theta(c_0)\theta(1)+\theta(c_1)\theta(x_1)+\ldots+\theta(c_n)\theta(x_n).$$

Lift $\theta(c_1)$ to $c_i\in C$. Then c_i is unique modulo pC, so that c_1a is uniquely determined by r. Let

$$\phi(r)=r+c_1a$$

and check that ϕ is a C-module homomorphism which acts as the identity on C. Since $R=C+\boldsymbol{m}$, ϕ will be a ring endomorphism if $\phi(mm')=\phi(m)\phi(m')$ for all $m,m'\in\boldsymbol{m}$. But $\theta(\boldsymbol{m}^2)=0$ gives $\phi(mm')=mm'$, whilst $a\boldsymbol{m}=0$ yields $\phi(m)\phi(m')=mm'$. Hence ϕ is a ring endomorphism. Now $\phi(x_1)=x_1+a$ and $\phi(1)=1$, so that ϕ is neither id_R nor 0_R, contradicting the fact that R is rigid.

Thus $\theta(R)=\theta(C)$ and $R=C+pR+\boldsymbol{m}^2$. This gives

$$\boldsymbol{m}=(C\cap\boldsymbol{m})+pR+\boldsymbol{m}^2=pC+pR+\boldsymbol{m}^2=pR+\boldsymbol{m}^2.$$

Nakayama's lemma yields $\boldsymbol{m}=pR$, whence $R=C+pR$. Using the lemma again, we get $R=C$.

If the residue field $R/\boldsymbol{m}$ where imperfect, it would possess a non-trivial derivation map, δ. Applying [**2**, Theorem 1], we could lift δ to a derivation of the v-ring V with residue field $R/\boldsymbol{m}$, and hence to a derivation D of $R=C\cong V/p^kV$. As in [**3** §V], the map $\alpha\colon R\to R$ given by

$$\alpha(r)=r+pD(r)+\frac{p^2}{2!}D^2(r)+\ldots+\frac{p^{k-1}}{(k-1)!}D^{k-1}(r)$$

would be a non-trivial automorphism of R, contradicting the rigidity of R.

Hence $R/\boldsymbol{m}$ is a perfect field and, by [**4**, Theorem 3.3], $R\cong Z_{p^k}$.

3. Rigid nilpotent rings

No artinian hypothesis is needed in the next theorem.

Theorem 2. *Let $R(\neq 0)$ be a rigid nilpotent ring. Then $R\cong N_2$.*

Proof. Suppose that $R^2\neq 0$. Then R has index of nilpotence $t>2$ and there exists $a\in R^{t-2}$ such that $aR\neq 0$. The map $\phi\colon R\to R$ given by $\phi(r)=r+ar$ is a ring endomorphism which is not the identity, but which acts as the identity on the non-zero ideal R^2. This contradicts the rigidity of R.

Hence $R^2=0$. By considering the map $r\to 2r$ on R, it is easy to see that $R\cong N_2$.

Combining Maxson's results with our own, we have the theorem stated in the introduction.

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