This is a ``preproof'' accepted article for *The Bulletin of Symbolic Logic*.
This version may be subject to change during the production process.
DOI: 10.1017/bsl.2025.10125

On the Arithmetical Complexity of Models

Warren Goldfarb
Harvard University

In 1939 Hilbert and Bernays proved, by arithmetizing Gödel's original 1930 proof of the completeness of quantification theory, that every satisfiable sentence of first-order predicate logic has a model over $\mathbb{N}$ in which the predicates are all arithmetically definable. Here and below, by "predicate logic" we mean a language that includes predicate letters of arbitrary adicity but no function signs or individual constants. Once the relevant tools were in place, their proof was shown to yield predicates that are Δ^0_2. Kreisel [1953] and Mostowski [1953] independently showed that the predicates could not always be taken to be recursive; Putnam [1957] strengthened this result by showing that they could not always be taken to be recursively enumerable (r.e.) or co-r.e.

In [1965] Putnam refined the positive result by showing that the predicates could always be taken to be Σ^*_1, that is, Boolean combinations of r.e. predicates. Putnam's proof, however, applied only to logic without identity.

Hensel and Putnam [1969] presented a proof that the result could be extended to first-order predicate logic with identity. This paper was almost completely unnoticed: it was not reviewed in the JSL, it is not listed, although Putnam [1965] is, in bibliographies such as those in Odifreddi [1989] and Hájek and Pudlák [1993].[1] It was therefore unremarked that the paper is nearly unreadable, containing inadequate notation, misleading citations, and cryptic arguments. In particular, the crucial final argument (corresponding to §4 below) is expressed extremely breezily, which masks several deductive gaps. Luckily, the proof can be reframed so as to be rigorous and reasonably transparent. Here is a fully explicit statement of the theorem to be proved:

> Let T be a recursive set of sentences of first-order predicate logic with identity containing a finite number of predicates. If T has an infinite model, then it has a model over $\mathbb{N}$ in which the predicates are Σ^*_1.

In §2 we prove the theorem in the restricted case of a single sentence with no

[1] The question is said to be open in Goldfarb [2018] and Visser [2017].

https://doi.org/10.1017/bsl.2025.10125 Published online by Cambridge University Press

occurrence of '='. This is just a modernized rendition of the argument from Putnam [1965]. The proof of the full theorem occupies §§3-4.

§1. Limit-determination and modulus. An $(n+1)$-place number-theoretic function f *limit-determines* an n-place relation R iff for all $i_1, \ldots, i_n$,

$$R(i_1, \ldots, i_n) \text{ iff } \lim_{y\to\infty} f(i_1, \ldots, i_n, y) = 1$$
$$\neg R(_1, \ldots, i_n) \text{ iff } \lim_{y\to\infty} f(i_1, \ldots, i_n, y) = 0$$

The Limit Lemma, a standard result of recursion theory originally due to Shoenfield, states that a relation is Δ^0_2 iff there is a recursive function that limit-determines it (see Odifreddi [1989] p. 373).

A function f is *k-trial* iff, for all $i_1, \ldots, i_n$, $f(i_1, \ldots, i_n, y)$ changes value at most k times as y increases. Putnam introduced this notion in [1965], and showed that a relation is Σ^*_1 iff for some k there is a recursive k-trial function that limit-determines it. This result is used below in the "if" direction. The proof is straightforward, and perhaps best presented by an example. Suppose $g(i, j, y)$ is a 2-trial function that limit-determines a relation $R(i, j)$. Then $R(i, j)$ can be specified thus: there are two numbers y at which $g(i, j, y)$ changes value, and at the greater of them the value is 1; or there aren't two numbers y such that $g(i, j, y)$ changes value at y but there is one number y at which $g(i, j, y)$ changes value, and then its value is 1, or there is no number y at which $g(i, j, y)$ changes value, and $g(i, j, 0)$ is 1. Each of these clauses expresses either an r.e. relation or a co-r.e. relation.

Let f be a recursive function that limit-determines a relation. For any i, an *i-modulus* for f is a number b such that

$$f(i_1, \ldots, i_n, b) = f(i_1, \ldots, i_n, y)$$

whenever $i_1, \ldots, i_n \leq i$ and $b \leq y$. For every i there are cofinitely many integers b that are i-moduli. The notion of an i-modulus is not, in general, recursive. The following is a recursive notion that approximates it and yields an i-modulus in the limit. An *r-seeming i-modulus* for f is a number $b \leq r$ such that

$$f(i_1, \ldots, i_n, b) = f(i_1, \ldots, i_n, y)$$

whenever $i_1, \ldots, i_n \leq i$ and $b \leq y \leq r$. Note that if b is an i-modulus for f,

https://doi.org/10.1017/bsl.2025.10125 Published online by Cambridge University Press

then b is also an r-seeming i-modulus for f for every $r \geq b$; and if b is not an i-modulus for f, then b fails to be an r-seeming i-modulus for f for cofinitely many r.

§2. Proof for sentences without identity. Let F be a satisfiable sentence of first-order predicate logic without identity. As noted above, there exists a model $\mathfrak{M}$ for it with universe $\mathbb{N}$ in which the predicates are interpreted as Δ^0_2 relations. By the Limit Lemma, for each predicate P of F there exists a recursive function f_P that limit-determines $P^{\mathfrak{M}}$. We call an integer b an i-modulus (*simpliciter*) iff it is an i-modulus for every function f_P, and an r-seeming i-modulus (*simpliciter*) iff it is an r-seeming i-modulus for every function f_P.

Let $\langle .,. \rangle$ be a standard primitive recursive bijective pairing function. Define $\varphi : \mathbb{N} \to \mathbb{N}$ by:

$$\varphi(p) = \begin{cases} i & \text{if } p = \langle i, b \rangle \text{ and } b \text{ is an } i\text{-modulus} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly φ is onto. Now let $\mathfrak{N}$ be the structure with universe $\mathbb{N}$ such that, for each n-adic predicate P of F and all $p_1, \ldots, p_n$,

$$\mathfrak{N} \models Pp_1 \ldots p_n \text{ iff } \mathfrak{M} \models P\varphi(p_1) \ldots \varphi(p_n)$$

Since F does not contain identity and φ is onto, it follows that $\mathfrak{N} \models F$. (This is a general model-theoretic fact: if a structure $\mathfrak{N}$ is the preimage of a relational structure $\mathfrak{M}$ under a function from the universe of $\mathfrak{N}$ onto the universe of $\mathfrak{M}$, then $\mathfrak{M}$ and $\mathfrak{N}$ model the same sentences of predicate logic without identity.)

We now show that the relations of the structure $\mathfrak{N}$ are Σ^*_1 by constructing k-trial recursive functions that limit-determine them. For each r define φ_r by:

$$\varphi_r(p) = \begin{cases} i & \text{if } p = \langle i, b \rangle \text{ and } b \text{ is an } r\text{-seeming } i\text{-modulus, or if } r < b \\ 0 & \text{otherwise.} \end{cases}$$

Let $p = \langle i, b \rangle$. If b is an i-modulus, then $\varphi_r(p) = i$ for all r. If b is not an i-modulus, then $\varphi_r(p) = i$ for r up to some point, but $= 0$ for all greater r. Thus for any fixed p, the value of $\varphi_r(p)$ can change at most once as r increases,

https://doi.org/10.1017/bsl.2025.10125 Published online by Cambridge University Press

and the eventual value of $\varphi_r(p)$ is $\varphi(p)$ for all sufficiently large r.

Let P be any predicate of F, say P is n-place, let $p_1, \ldots, p_n$ be any numbers, with $p_k = \langle i_k, b_k \rangle$ for $1 \leq k \leq n$, and let b_0 be a 0-modulus. We define

$$g_P(p_1, \ldots p_n, r) = f_P(\varphi_r(p_1), \ldots, \varphi_r(p_n), b)$$

where b is the maximum of $b_0, b_1, \ldots, b_n$.

It is easy to see that, for each n-ary predicate P and all $p_1, \ldots, p_n$,

$$\lim_{y \to \infty} g_P(p_1, \ldots p_n, y) = \lim_{y \to \infty} f_P(\varphi(p_1), \ldots, \varphi(p_n), y)$$

since there will be a q such that $g_P(p_1, \ldots, p_n, r) = f_P(\varphi(p_1), \ldots, \varphi(p_n), b)$ for all $r \geq q$, and b is an i-modulus for i such that $\varphi(p_1), \ldots, \varphi(p_n) \leq i$. Hence each function g_P limit-determines $P^{\mathfrak{N}}$.

If P is n-adic, g_P is an n-trial function. For as noted above as r increases, a value $\varphi_r(p)$ can change only once, from a nonzero value to 0. This implies that as r increases $g_P(p_1, \ldots p_n, r)$ can change value at most n times.

This proof generalizes without difficulty to recursive sets of sentences without identity, provided that there are finitely many predicates in the sentences, since the basic Hilbert-Bernays Δ_2^0 result so generalizes.

§3. Proof with identity: specification of the model. The mapping φ in the proof just given is not injective, and hence the preimage does not preserve identity. This makes the proof inapplicable to sentences containing "=". To obtain a suitable mapping that is injective as well, Hensel and Putnam's strategy is to start with a Δ_2^0 model that contains infinitely many indiscernibles. These indiscernibles provide distinct objects to be the images of distinct arguments. In order to be injective the mapping φ can take $\langle i, b \rangle$ to i for at most one b. To insure this, a minimality condition needs to be put on b. This requires much more fine-grained information about the approximation procedure than was needed in §2.

Let T be a recursive set of sentences of first-order predicate logic with identity containing a finite number of predicates. Let T^S be the Skolemization of T. Let Eq be the (purely universal) axioms of identity (that is, reflexivity and

https://doi.org/10.1017/bsl.2025.10125 Published online by Cambridge University Press

substitutivity) for the language of T^S. Let $a_0, \ldots, a_k, \ldots$ be distinct constants foreign to T^S, and let In be the set of axioms asserting that these *a-constants* are order-indiscernible, with respect to the atomic formulas of T^S, i.e., axioms

$$F(a_{i_1}, \ldots, a_{i_n}) \leftrightarrow F(a_{j_1}, \ldots, a_{j_n})$$

whenever $F(x_1, .., x_n)$ is an atomic formula of $\mathcal{L}(T^S)$, $i_1 < \ldots < i_n$ and $j_1 < \ldots < j_n$.

Finally, let T^+ be the theory $T^S \cup \mathsf{Eq} \cup \mathsf{In} \cup \{a_1 \neq a_2\}$. An argument using Ramsey's Theorem shows that if T has an infinite model then T^+ has a model. (See Chang and Keisler [1973], pp. 147-148.) Note that T^+ implies $a_i \neq a_j$ whenever $i \neq j$.

Let D be the set of terms that can be built from the function signs and constants appearing in formulas in T^+. We assume a standard gödelization of D, and let $\tau(1), \tau(3), \ldots, \tau(2i+1), \ldots$ be a listing in order of increasing gödel numbers of the members of D that are not a-constants, while $\tau(2i) = a_i$ for each i. Thus τ is a bijection between $\mathbb{N}$ and D. We assume that if $\tau(i)$ is a proper subterm of $\tau(j)$ then $i < j$.

Since T^+ has a model, every conjunction of substitution instances of members of T^+ over D is consistent. Compactness then yields the consistency of the set of all such instances, and by König's Lemma (as in Gödel's completeness proof) this immediately yields a quasi-model $\mathfrak{M}_o$ of T^+ with universe D. "Quasi-model" here means a structure in which all the axioms of T^+ hold, but the interpretation of the identity sign "=" may not be true identity. Due to the inclusion of the axioms Eq, however, "=" will be interpreted in $\mathfrak{M}_o$ as a congruence relation relative to the other predicates and the function signs of T^+.

Let $\mathfrak{M}$ be the isomorphic copy with universe $\mathbb{N}$ of $\mathfrak{M}_o$, via the bijection τ. The interpretations in $\mathfrak{M}$ of the predicates of T^+ can be taken to be Δ^0_2. Hence for each predicate P of T^+ there is a recursive function f_P that limit-determines $P^{\mathfrak{M}}$. As before, we say an integer b is an i-modulus iff it is an i-modulus for each function f_P, and similarly for "r-seeming i-modulus".

By the definition of τ, the even numbers are the indiscernibles of $\mathfrak{M}$. Indeed,

https://doi.org/10.1017/bsl.2025.10125 Published online by Cambridge University Press

the reason we started with the Skolemization of T and built up the quasi-model $\mathfrak{M}_\circ$ over universe D was to insure not only that the predicates were Δ^0_2 but also that the indiscernibles comprised a recursive set.

Let $(i_1, \ldots, i_n)$ be an n-tuple of integers, and let i be the largest odd integer among the i_k, if there is one, and let $i = 0$ if all the i_k are even. An n-tuple $(j_1, \ldots, j_n)$ is *similar* to $(i_1, \ldots, i_n)$ iff:

1) for each k if $i_k \leq i$ then $j_k = i_k$;

2) the tuples obtained from the two n-tuples by eliminating all entries $\leq i$ are order-isomorphic.

Note that if i_k is an even integer $> i$, then the a-constant $\tau(i_k)$ does not occur in any term $\tau(i_j)$ for $i_j \leq i$. Consequently, the axioms In imply the following: if P is an n-place predicate and $(i_1, \ldots, i_n)$ and $(j_1, \ldots, j_n)$ are similar, then

$$\mathfrak{M} \models Pi_1 \ldots i_n \leftrightarrow Pj_1 \ldots j_n$$

We now introduce a different pairing function. Let $\pi(i, b) = i + \frac{1}{2}b(b+1)$. π is a bijection between $\mathbb{N}$ and pairs (i, b) such that $i \leq b$. π is monotonic in each argument Moreover, if $\pi(j, c) < \pi(i, b)$, where $j \leq c$ and $i \leq b$, then $c \leq b$. This "ordering property" will be crucial for the proof of Lemma 3(iii) in §4.

A pair (i, b) is *ideal* iff i is odd and b is the smallest number $\geq i$ such that

(i) b is an $(i + 2n_0)$-modulus, where n_0 is the largest adicity of any predicate letter of T; and

(ii) for all $j < i$, $f_=(j, i, b) = 0$.

Note that if (i, b) is ideal, then i will be the smallest member of its $=^{\mathfrak{M}}$-equivalence class.

Now define $\varphi : \mathbb{N} \to \mathbb{N}$ by $\varphi(p) = i$ if $p = \pi(i, b)$ for an ideal pair (i, b); and otherwise $\varphi(p)$ is the least even number not among $\varphi(0), \ldots, \varphi(p-1)$. Since for every i there is at most one b such that (i, b) is ideal, it follows that φ is injective and its range includes all even numbers.

Lemma 1. The range of φ includes exactly one element from each $=^{\mathfrak{M}}$-equivalence class.

Proof. First we show that the range of φ contains at least one element from each $=^{\mathfrak{M}}$-equivalence class. Since the range contains every even number, we

https://doi.org/10.1017/bsl.2025.10125 Published online by Cambridge University Press

need consider only those $=^{\mathfrak{M}}$-equivalence classes all of whose members are odd integers. Suppose i is the least member of such a class. Thus $\mathfrak{M} \models \neg j = i$ for each $j < i$. Hence if b is the smallest $(i + 2n_0)$-modulus $\geq i$, we have $f_=(j, i, b) = 0$ for each $j < i$. Consequently (i, b) is ideal, so that $\varphi(\pi(i, b)) = i$.

Next we show uniqueness. If i is an odd number in the range of φ, then i is the smallest member of its $=^{\mathfrak{M}}$-equivalence class. So the only other number that could be in this $=^{\mathfrak{M}}$-equivalence class and in the range of φ is an even integer $j > i$. But then the pair (i, j) is similar to the pair $(i, j + 2)$, so by indiscernibility, $\mathfrak{M} \models i = j \leftrightarrow i = j + 2$. Thus if $\mathfrak{M} \models i = j$, by the transitivity of identity we would also have $\mathfrak{M} \models j = j + 2$, a contradiction. □

Now let $\mathfrak{N}$ be the structure with universe $\mathbb{N}$ such that, for each n-place predicate P of $\mathcal{L}(T)$ and all $p_1, \ldots, p_n$,

$$\mathfrak{N} \models Pp_1 \ldots p_n \leftrightarrow \mathfrak{M} \models P\varphi(p_1) \ldots \varphi(p_n)$$

Thus φ is an isomorphism between $\mathfrak{N}$ and the substructure of $\mathfrak{M}$ whose universe is the range of φ. From Lemma 1 it follows that $=^{\mathfrak{N}}$ is the identity relation, and that $\mathfrak{N}$ is a model of T.

§4. The predicates of $\mathfrak{N}$ are Σ_1^*.

As in §2, we define recursive functions that approximate those used in the specification of $\mathfrak{N}$. A pair (i, b) is *r-ideal* iff i is odd and b is the smallest number $\geq i$ such that

(i) b is an r-seeming $(i + 2n_0)$-modulus; and

(ii) for all $j < i$, $f_=(j, i, b) = 0$.

For each r, if $p = \pi(i, b)$ for an r-ideal pair (i, b), let $\varphi_r(p) = i$; otherwise let $\varphi_r(p)$ be the earliest even number not among $\varphi_r(0), \ldots, \varphi_r(p - 1)$. Note that $\varphi_r(p)$ is recursive in both arguments.

Because r-seeming moduli are either moduli for cofinitely many r or non-moduli for cofinitely many r, for every m there is a number $\rho(m)$ with the following property: for every $r \geq \rho(m)$ and all pairs (i, b) with $i \leq b$ such that $\pi(i, b) \leq m$, b is an r-seeming $(i + 2n_0)$-modulus iff b is an $(i + 2n_0)$-modulus. Hence for every $r \geq \rho(m)$ and every $p \leq m$, $\varphi_r(p) = \varphi(p)$.

For any n-tuple $(i_1, \ldots, i_n)$ let $[\![i_1, \ldots, i_n]\!]$ be the earliest n-tuple similar to

https://doi.org/10.1017/bsl.2025.10125 Published online by Cambridge University Press

$(i_1, \ldots, i_n)$. Let P be any predicate of T, say P is n-adic, let $p_1, \ldots, p_n$ be any numbers, with $p_k = \pi(i_k, b_k)$ with $i_k \leq b_k$ for $1 \leq k \leq n$, and let b_0 be a $2n_0$-modulus. We define

$$g_P(p_1, \ldots, p_n, r) = f_P([\![\varphi_r(p_1), \ldots, \varphi_r(p_n)]\!], b)$$

where b is the maximum of $b_0, b_1, \ldots b_n$. Note that each g_P is recursive.

Lemma 2. For each predicate P, g_P limit-determines $P^{\mathfrak{N}}$.

Proof. It is enough to show, for each predicate P and all $p_1, ..., p_n$, that

$$\lim_{y\to\infty} g_P(p_1, \ldots, p_n, y) = \lim_{y\to\infty} f_P([\![\varphi(p_1), \ldots, \varphi(p_n)]\!], y).$$

Let m be the largest of $p_1, \ldots, p_n$. Then for any $r \geq \rho(m)$, $g_P(p_1, \ldots, p_n, r) = f_P([\![\varphi(p_1), \ldots, \varphi(p_n)]\!], b)$ where b is chosen as above. Hence

$$\lim_{y\to\infty} g_P(p_1, \ldots, p_n, y) = f_P([\![\varphi(p_1), \ldots, \varphi(p_n)]\!], b)$$

for this b.

What remains to be established is that

$$f_P([\![\varphi(p_1), \ldots, \varphi(p_n)]\!], b) = \lim_{y\to\infty} f_P([\![\varphi(p_1), \ldots, \varphi(p_n)]\!], y)$$

Let q be the largest integer in the n-tuple $[\![\varphi(p_1), \ldots, \varphi(p_n)]\!]$; it suffices to show that b is a q-modulus. If all the $\varphi(p_j)$ are even, then $[\![\varphi(p_1), \ldots, \varphi(p_n)]\!]$ will contain only even numbers $\leq 2n_0$. Since b is a $2n_0$-modulus, it is indeed a q-modulus. Otherwise, let $\varphi(p_j)$ be the largest odd number among $\varphi(p_1), \ldots, \varphi(p_n)$. Then $[\![\varphi(p_1), \ldots, \varphi(p_n)]\!]$ will contain only numbers $\leq \varphi(p_j) + 2n_0$, that is, $q \leq \varphi(p_j) + 2n_0$. But $(\varphi(p_j), b')$ is ideal for some $b' \leq b$, so that b is a $(\varphi(p_j) + 2n_0)$-modulus. □

Lemma 3. Suppose b is an r-seeming i-modulus but not an $(r+1)$-seeming i-modulus. Then for any $c \leq r$ and any $j \geq i$, c is not an $(r+1)$-seeming j-modulus.

Proof. The supposition on b implies that for some predicate P and some $i_1, \ldots, i_n \leq i$,

$$f_P(i_1, \ldots, i_n, r) \neq f_P(i_1, \ldots, i_n, r+1).$$

https://doi.org/10.1017/bsl.2025.10125 Published online by Cambridge University Press

The existence of such $i_1, \ldots, i_n$ imply that c is not an $(r+1)$-seeming j-modulus, so long as $c \leq r$ and $j \geq i$. □

Lemma 4. **(i)** Suppose $i < j$ and both (i, b) and (j, c) are r-ideal. Then $b \leq c$.

(ii) Suppose $c \leq r$ and (j, c) is $(r+1)$-ideal. Then (j, c) is r-ideal.

(iii) Suppose (j, c) is r-ideal and $(r+1)$-ideal. Then for every $q < \pi(j, c)$, $\varphi_r(q) = \varphi_{r+1}(q)$.

Proof. **(i)** By definition, b is the least number $\geq i$ that is an r-seeming $(i + 2n_0)$-modulus and such that $f_=(k, i, b) = 0$ for each $k < i$. We show that c also has these properties; consequently $b \leq c$. Since $i < j$ and c is an r-seeming $(j + 2n_0)$-modulus, c is also an r-seeming $(i + 2n_0)$-modulus. Since $c \geq j$ also $c \geq i$. Since (j, c) is r-ideal, $c \leq r$, so that for every $k < i$, $f_=(k, i, c) = f_=(k, i, b)$. Hence, for every $k < i$, $f_=(k, i, c) = 0$.

(ii) If (j, c) is $(r+1)$-ideal but not r-ideal, there must be a number $b < c$ such that (j, b) is r-ideal but not $(r+1)$-ideal. And this, in turn, can happen only if b is not an $(r+1)$-seeming $(j + 2n_0)$-modulus. But then, by Lemma 3, c is not an $(r+1)$-seeming $(j + 2n_0)$-modulus, contradicting the $(r+1)$-ideality of (j, c).

(iii) For reductio, suppose q is the least number $< \pi(j, c)$ such that $\varphi_r(q) \neq \varphi_{r+1}(q)$. Let (i, b) be the unique pair with $i \leq b$ and $q = \pi(i, b)$. Note that, by the ordering property of π, $b \leq c$, so that $b \leq r$. Now (i, b) cannot be r-ideal and $(r+1)$-ideal, since then $\varphi_r(q) = \varphi_{r+1}(q) = i$. If (i, b) were neither neither r-ideal nor $(r+1)$-ideal, then $\varphi_r(q) = \varphi_{r+1}(q) =$ the least even number not among $\varphi_r(0), \ldots, \varphi_r(q-1)$, which by hypothesis are equal to $\varphi_{r+1}(0), \ldots, \varphi_{r+1}(q-1)$. By (ii), (i, b) cannot be $(r+1)$-ideal but not r-ideal.

The only possibility remaining is that (i, b) is r-ideal but not $(r+1)$-ideal. If $i = j$, then by minimality $b = c$, so that $\pi(i, b) = q$, contrary to the choice of q. If $j < i$ then by (i) we have $c \leq b$, so that $\pi(j, c) < \pi(i, b)$, that is, $\pi(j, c) < q$, also contrary to the choice of q. Hence $i < j$. Since (i, b) is not $(r+1)$-ideal, b is not an $(r+1)$-seeming $(i + 2n_0)$-modulus, so, by Lemma 3, c is not an $(r+1)$-seeming $(j + 2n_0)$-modulus, contradicting the $(r+1)$-ideality of (j, c). □

Lemma 5. Suppose P is an n-adic predicate. Then g_P is a $2n$-trial function.

https://doi.org/10.1017/bsl.2025.10125 Published online by Cambridge University Press

Proof. Let $p = \pi(i, b)$ where $i \leq b$ and i is odd. Call p *r-unready* iff $b > r$ and *r-ideal* if (i, b) is r-ideal. The *r-index* of an n-tuple of numbers is $2\alpha + \beta$, where α is the number of entries in the n-tuple that are r-unready, and β is the number of entries in the n-tuple that are r-ideal.

First we note that if p is neither r-unready nor r-ideal, then p is neither $(r+1)$-unready nor $(r+1)$-ideal. This follows immediately from Lemma 4(ii). Hence the $(r+1)$-index of an n-tuple is never more than its r-index.

We shall now show the following: If $[\![\varphi_{r+1}(p_1), \ldots, \varphi_{r+1}(p_n)]\!] \neq [\![\varphi_r(p_1), \ldots, \varphi_r(p_n)]\!]$ then the $(r+1)$-index of $(p_1, \ldots, p_n)$ is lower than its r-index. Since the maximal r-index of an n-tuple is $2n$, this will prove the lemma, since $g_P(p_1, \ldots, p_n, r) \neq g_P(p_1, \ldots, p_n, r+1)$ only if $[\![\varphi_{r+1}(p_1), \ldots, \varphi_{r+1}(p_n)]\!] \neq [\![\varphi_r(p_1), \ldots, \varphi_r(p_n)]\!]$.

Suppose the indices are equal. Then none of $p_1, \ldots, p_n$ is r-unready but not $(r+1)$-unready, as this would lower the index by 1 or 2.

If none of $p_1, \ldots, p_n$ is r-ideal, then both φ_r and φ_{r+1} take them all to even numbers, although possibly not the same ones. Nonetheless, their order will be the same as the order of $p_1, \ldots, p_n$ so that $[\![\varphi_{r+1}(p_1), \ldots, \varphi_{r+1}(p_n)]\!] = [\![\varphi_r(p_1), \ldots, \varphi_r(p_n)]\!]$.

Now suppose that at least one of $p_1, \ldots, p_n$ is r-ideal. Let $\varphi_r(p_j)$ be the largest odd number among $\varphi_r(p_1), \ldots, \varphi_r(p_n)$. Then p_j is also $(r+1)$-ideal: if it were not then the index would decrease. By Lemma 3(i) and (iii), φ_{r+1} agrees with φ_r on all the other numbers among $p_1, \ldots, p_n$ except possibly some numbers p_k larger than p_j, which are not r-ideal, and so are taken to even numbers larger than $\varphi_r(0), \ldots, \varphi_r(p_k - 1)$. (Since at least half of the numbers $\leq p_k$ are not r-ideal, $\varphi_r(p_k) \geq p_k$, whereas, since p_j is r-ideal, $\varphi_r(p_j) < p_j$.). Similarly, φ_{r+1} also takes these numbers to even numbers larger than $\varphi_r(p_j)$, else the index would increase; the values of φ_{r+1} on these numbers may be different from those of φ_r, but the ordering of those values will be the same. That is, $(\varphi_r(p_1), \ldots, \varphi_r(p_n))$ will be similar to $(\varphi_{r+1}(p_1), \ldots, \varphi_{r+1}(p_n))$, so that $[\![\varphi_{r+1}(p_1), \ldots, \varphi_{r+1}(p_n)]\!] = [\![\varphi_r(p_1), \ldots, \varphi_r(p_n)]\!]$. □

Lemmas 2 and 5 show that each $P^{\mathfrak{N}}$ is Σ_1^*, and so conclude the proof.

The proof shows that, for each predicate P, the complexity of $P^{\mathfrak{N}}$ depends

https://doi.org/10.1017/bsl.2025.10125 Published online by Cambridge University Press

only on the adicity P, and not on any quantificational structure of T, where by complexity we mean the number of r.e. and co-r.e. sets whose Boolean combination defines $P^{\mathfrak{N}}$. No upper bound is put on this complexity. However, there are many different techniques for encoding polyadic predicates of high adicity by dyadic or triadic predicates. Most of these involve additional quantifiers, so their applicability in this context would have to be very delicate. But the question remains: is there an upper bound on the Boolean complexity needed even for predicates of arbitrarily large adicity.[2]

References

Chang, C.C. and H. Jerome Keisler [1973]. *Model Theory*, Elsevier.

Goldfarb, Warren [2018], Putnam's theorem on the complexity of models, in G. Hellman and R.T. Cook, eds., *Hilary Putnam on Logic and Mathematics*, Springer.

Hájek, Petr and Pavel Pudlák, [1993] *Metamathematics of First-Order Arithmetic*. Springer.

Hensel, Gustav and Hilary Putnam [1969], Normal models and the field Σ_1^*, *Fundamenta Mathematicae*, vol. 64, 231-240

Hilbert, David, and Paul Bernays [1939], *Grundlagen der Mathematik*, Vol. 2. Berlin: Springer.

Kreisel, Georg [1953], Note on arithmetic models for consistent formulae of the predicate calculus II, *Proceedings of the XIth International Congress of Philosophy, vol. XIV*, Amsterdam: North-Holland, 39–49.

Mostowski, Andrzej [1953], On a system of axioms which has no recursively enumerable model, *Fundamenta Mathematicae 40*, 56–61.

Odifreddi, Piergiorgio [1989], *Classical Recursion Theory*, Elsevier

Putnam, Hilary [1957], Arithmetic models for consistent formulae of quantification theory, *Journal of Symbolic Logic 22*, 110–111.

[2]I am overwhelmingly grateful to Stephen Mackereth for discovering the Hensel-Putnam paper, as well as for valuable suggestions on an earlier draft. I am also grateful to Albert Visser for many helpful suggestions and corrections, and to Carl Jockusch for useful corrections. The question voiced in the last paragraph emerged from comments of both Visser and Jockusch. My debt to the anonymous referee is profound, as his or her patient and careful scrutiny provided not just more corrections but an observation that enabled me to fix a serious blunder, as well as simplifying the proof in several ways.

https://doi.org/10.1017/bsl.2025.10125 Published online by Cambridge University Press

Putnam, Hilary [1965], Trial and error predicates and the solution to a problem of Mostowski, *Journal of Symbolic Logic*, vol. 30, 49–57.

Shoenfield, Joseph R. [1959], On degrees of unsolvability, *Annals of Mathematics*, vol. 69, 644–653.

Visser, Albert [2017], The interpretation existence lemma, in G. Jäger and Wilfried Sieg, eds., *Feferman on Foundations*, Springer, 101-144.

DEPARTMENT OF PHILOSOPHY, HARVARD UNIVERSITY, CAMBRIDGE, MASSACHUSETTS
E-mail: goldfarb@fas.harvard.edu

https://doi.org/10.1017/bsl.2025.10125 Published online by Cambridge University Press