

## A NOTE ON MARKOVIAN QUANTUM DYNAMICS

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(Received 12 August, 2016; accepted 15 October, 2016; first published online 15 May 2017)

### Abstract

Based on the definition of divisibility of Markovian quantum dynamics, we discuss the Markovianity of tensor products, multiplications and some convex combinations of Markovian quantum dynamics. We prove that the tensor product of two Markovian dynamics is also a Markovian dynamics and propose a new witness of non-Markovianity.

2010 *Mathematics subject classification*: primary 81S22; secondary 81P40, 52A37.

*Keywords and phrases*: quantum Markovianity, divisibility, witness.

### 1. Introduction

With the development of quantum information theory, Markovianity has attracted the attention of many researchers (see the article by Rivas et al. [19] and the references therein). This is not only due to the mysterious characteristic of Markovianity, but also due to the potential applications of non-Markovianity. Although in classical stochastic processes the theory of Markovianity and non-Markovianity is fully developed, the counterpart in quantum evolution still remains subtle and elusive. Various non-Markovian criteria have been introduced and several measures have been proposed, based on diverse considerations, in recent years. Owing to the ability of non-Markovian dynamics in regaining lost information and recovering coherence, non-Markovianity can be used in certain protocols and can be exploited for quantum key distribution and quantum metrology [5, 21].

Mathematically, a quantum Markovian process can be described by a master equation in the Lindblad form or equivalently by completely positive divisible maps [17]. Besides the above master equation or divisibility approach [18], there

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are many other ways to describe non-Markovianity, such as the breakdown of the semigroup property [22], the increasing of distinguishability between two evolving states [3], the nonmonotonic behaviours of mutual information items [13], the negative decay rate [10] or the inequality of the memoryless dynamical maps [11] (for more details, see the articles [1, 2, 4, 7, 12, 15, 16, 20]).

In analogy with the entanglement witness, Chruściński and Kossakowski [6] proposed a non-Markovianity witness and introduced the corresponding measure of non-Markovianity. This witness is defined under the settings that the quantum dynamics is invertible and satisfies a certain derivable condition. Based on the idea of using mutual information to quantify non-Markovianity proposed by Luo et al. [13], we introduce a new witness of non-Markovianity in a more general context.

The structure of this work is as follows. In Section 2, we review the divisibility definition of quantum non-Markovianity, discuss its basic properties and give several examples. A quantum dynamics which looks like a piecewise function is also constructed and investigated at the end of this section. In Section 3, we study some operations of Markovian quantum dynamics, such as tensor product, multiplication and convex combination. In Section 4, we expatiate on the new witness of quantum non-Markovianity and give a specific example. Finally, we conclude the paper with a discussion in Section 5.

## 2. Divisibility definition and examples

Let  $B(\mathcal{H})$  be the  $C^*$ -algebra of all bounded linear operators on a Hilbert space  $\mathcal{H}$ . For an operator  $A \in B(\mathcal{H})$ , we use  $A^\dagger$  and  $\text{tr}(A)$ , respectively, to denote the adjoint operator and the trace of  $A$ . An operator  $A$  is called a density operator if it is positive (that is,  $A = B^\dagger B$  for some  $B \in B(\mathcal{H})$ ) and of trace one. Let  $D(\mathcal{H})$  be the set of density operators in  $B(\mathcal{H})$ . A completely positive trace preserving (CPTP) linear mapping on  $B(\mathcal{H})$  is said to be a *quantum channel* of the quantum system described by  $\mathcal{H}$ . A family  $\Lambda = \{\Lambda_t\}_{t \geq 0}$  of quantum channels of  $\mathcal{H}$  is called a *quantum dynamics* of  $\mathcal{H}$ . Quantum dynamics is usually classified into Markovian or non-Markovian according to the absence or presence of memory effects, respectively.

**DEFINITION 2.1** [13]. Let  $\Lambda = \{\Lambda_t\}_{t \geq 0}$  be a quantum dynamics of  $\mathcal{H}$ . If, for all  $t > s \geq 0$ , there exists a quantum channel  $V_{t,s}$  of  $\mathcal{H}$  such that

$$\Lambda_t = V_{t,s}\Lambda_s,$$

then we say that  $\Lambda$  is *Markovian*, or *divisible*, and the quantum channels  $V_{t,s}$  are called *propagators* of  $\Lambda$ . The quantum dynamics  $\Lambda$  is said to be *non-Markovian*, or *indivisible*, if it is not Markovian.

Here are some observations from this definition.

**REMARK 2.2.** Let  $\Lambda = \{\Lambda_t\}_{t \geq 0}$  be a family of surjective CPTP maps. If  $\Lambda$  is Markovian, then the propagators  $V_{t,s}$  of  $\Lambda$  satisfy  $V_{t,s} = V_{t,u}V_{u,s}$  for all  $t > u > s \geq 0$ .

Indeed, for all  $t > u > s \geq 0$ , we have  $V_{t,s}\Lambda_s = \Lambda_t = V_{t,u}\Lambda_u = V_{t,u}V_{u,s}\Lambda_s$ . Since each mapping  $\Lambda_s : B(\mathcal{H}) \rightarrow B(\mathcal{H})$  is surjective,  $V_{t,s} = V_{t,u}V_{u,s}$ .

**REMARK 2.3.** If  $\{V_{t,s}\}_{t>s\geq 0}$  is a family of CPTP mappings such that  $V_{t,s} = V_{t,u}V_{u,s}$  for all  $t > u > s \geq 0$ , then  $\Lambda = \{V_{t,0}\}_{t\geq 0}$  is Markovian with  $\Lambda_t = V_{t,s}\Lambda_s$  for all  $t > s \geq 0$ .

**REMARK 2.4.** If  $\Lambda = \{\Lambda_t\}_{t\geq 0}$  is a family of CPTP mappings satisfying the semigroup property  $\Lambda_{t+s} = \Lambda_t\Lambda_s (\forall t, s \geq 0)$ , then  $\Lambda$  is Markovian.

In fact, when  $0 \leq s < t$ , we have  $\Lambda_t = \Lambda_{t-s}\Lambda_s = V_{t,s}\Lambda_s$ , where  $V_{t,s} = \Lambda_{t-s}$  is a CPTP mapping, so  $\Lambda$  is Markovian.

Next, we give several examples of Markovian and non-Markovian quantum dynamics. In the following, we use  $\dot{p}(t)$  to denote the derivative of  $p(t)$  with respect to  $t$  and  $M_n$  to denote the set of all complex matrices of size  $n \times n$ .

**EXAMPLE 2.5.** (i) For every  $\rho \in B(\mathcal{H})$  and any  $t \geq 0$ , let  $\Lambda_t(\rho) = U(t)\rho U(t)^\dagger$  with unitary  $U(t)$ ; then  $\Lambda = \{\Lambda_t\}_{t\geq 0}$  is Markovian with the propagators  $V_{t,s}(X) = U(t)U(s)^\dagger XU(s)U(t)^\dagger$ .

(ii) Let the Hamiltonian  $H \in B(\mathcal{H})$  be Hermitian, that is,  $H^\dagger = H$ . Denote the initial state by  $\rho$ . Then the dynamics  $\Lambda = \{\Lambda_t\}_{t\geq 0}$  with  $\rho(t) = \Lambda_t(\rho)$  satisfying the von Neumann equation  $\dot{\rho}(t) = -i[H, \rho(t)]$  with  $\rho(0) = \rho$  is Markovian, since  $\Lambda_t(\rho) = e^{-itH}\rho e^{itH}$ , where  $e^{-itH}$  is unitary. Generally, when  $f : [0, \infty) \rightarrow \mathbb{R}$  is continuous and  $H^\dagger = H$ , the equation  $\dot{\rho}(t) = -i[f(t)H, \rho(t)]$  describes a Markovian dynamics  $\Lambda = \{\Lambda_t\}_{t\geq 0}$ , where

$$\Lambda_t(\rho) = e^{-i \int_0^t f(\tau) d\tau H} \rho e^{i \int_0^t f(\tau) d\tau H}.$$

(iii) Let  $\Lambda_t(\rho) = (e^{-itH_1} \otimes e^{-itH_2})\rho(e^{itH_1} \otimes e^{itH_2})$  with Hermitian  $H_1$  and  $H_2$ . Then  $\Lambda = \{\Lambda_t\}_{t\geq 0}$  is Markovian, since  $e^{-itH_1} \otimes e^{-itH_2}$  is unitary.

(iv) Let the dynamics  $\Lambda = \{\Lambda_t\}$  be given by  $\Lambda_t(\rho) = \rho(t)$ , where  $\rho(t)$  is the solution with  $\rho(0) = \rho$  to the linear equation  $\dot{\rho}(t) = \mathcal{L}_t(\rho(t))$ . Rivas and Huelga [17] proved that  $\Lambda$  is Markovian if and only if the generators  $\mathcal{L}_t$  can be written in the form

$$\mathcal{L}_t(X) = -i[H(t), X] + \sum_{k=1}^N \gamma_k(t) \left[ V_k(t)XV_k^\dagger(t) - \frac{1}{2}\{V_k^\dagger(t)V_k(t), X\} \right], \tag{2.1}$$

where  $H(t)$  and  $V_k(t)$  are time-dependent operators,  $H(t)$  is self-adjoint and  $\gamma_k(t) \geq 0$  for all  $k = 1, 2, \dots, N$  and time  $t \geq 0$ .

**EXAMPLE 2.6** [6]. Consider the equation

$$\dot{\rho}(t) = \frac{1}{2}\gamma(t)[\sigma_z\rho(t)\sigma_z - \rho(t)],$$

which is a special case of equation (2.1), where  $H(t) = 0, N = 1, \gamma_1(t) = \gamma(t)/2$  and  $V_1(t) = \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ . The corresponding evolution of the state reads

$$\rho(t) = \begin{bmatrix} \rho_{11} & \rho_{12}e^{-\Gamma(t)} \\ \rho_{21}e^{-\Gamma(t)} & \rho_{22} \end{bmatrix},$$

where the initial state  $\rho = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}$  and  $\Gamma(t) = \int_0^t \gamma(\tau) d\tau$ . Let  $\Lambda_t(\rho) = \rho(t)$  and  $\Lambda = \{\Lambda_t\}_{t\geq 0}$ . Then it can be proved that:

- (1) for every  $t \geq 0$ ,  $\Lambda_t$  is a CP mapping if and only if  $\Gamma(t) \geq 0$ . Thus, when  $\Gamma(t)$  is nonnegative for all  $t \geq 0$ ,  $\Lambda$  is a quantum dynamics;
- (2) when the function  $\gamma$  is continuous on  $[0, \infty)$ ,  $\Lambda$  is Markovian if and only if  $\gamma(t) \geq 0 (\forall t \geq 0)$ .

Motivated by the concept of piecewise functions, we have the following proposition.

**PROPOSITION 2.7.** *Let  $\{\Phi_n\}$  be a sequence of quantum channels on  $B(\mathcal{H})$  and  $\{t_n\}$  be a sequence of nonnegative real numbers such that*

$$0 = t_0 < t_1 < t_2 < \dots < t_n < \dots, \quad [0, +\infty) = \bigcup_{n=1}^{\infty} [t_{n-1}, t_n)$$

and let  $\Lambda_t = \Phi_n$  for all  $t \in [t_{n-1}, t_n)$ ,  $n = 1, 2, \dots$ , and  $\Lambda = \{\Lambda_t\}_{t \geq 0}$ . Then  $\Lambda$  is Markovian if and only if for each  $n \in \mathbb{N}^+$ , there exists a quantum channel  $R_n$  such that  $\Phi_{n+1} = R_n \Phi_n$ .

**PROOF.** Suppose that  $\Lambda$  is Markovian. Then, for each  $(t, s)$  with  $0 \leq s < t$ , there exists a quantum channel  $V_{t,s}$  on  $B(\mathcal{H})$  such that  $\Lambda_t = V_{t,s} \Lambda_s$ . Especially,  $\Phi_{n+1} = \Lambda_{t_n} = V_{t_n, t_{n-1}} \Lambda_{t_{n-1}} = R_n \Phi_n$ , where  $R_n = V_{t_n, t_{n-1}}$  for all  $n \in \mathbb{N}^+$ .

Conversely, assume that for each  $n \in \mathbb{N}^+$ , there exists a quantum channel  $R_n$  such that  $\Phi_{n+1} = R_n \Phi_n$ . For each  $(t, s)$  with  $0 \leq s < t$ , define  $V_{t,s} = \mathbf{1}$  (the identity mapping) if  $s, t \in [t_{n-1}, t_n)$ ;  $V_{t,s} = R_{n-1} R_{n-2} \dots R_{n-k}$  if  $t_{n-k-1} \leq s < t_{n-k} < \dots < t_{n-1} \leq t < t_n$ . Then  $V_{t,s}$  is a quantum channel on  $B(\mathcal{H})$  such that  $\Lambda_t = V_{t,s} \Lambda_s$ . Thus,  $\Lambda$  is Markovian.  $\square$

### 3. Operations of Markovian quantum dynamics

We discuss the closure of the operations of Markovian quantum dynamics in this section. Let us consider the tensor product of two quantum dynamics at first.

**PROPOSITION 3.1.** *Let  $\mathcal{H}$  and  $\mathcal{K}$  be two Hilbert spaces. Suppose that  $\Lambda = \{\Lambda_t\}_{t \geq 0}$  and  $\Phi = \{\Phi_t\}_{t \geq 0}$  are Markovian dynamics of  $\mathcal{H}$  and  $\mathcal{K}$ , respectively. Put  $\Lambda \otimes \Phi = \{\Lambda_t \otimes \Phi_t\}_{t \geq 0}$ . Then  $\Lambda \otimes \Phi$  is a Markovian dynamics of  $\mathcal{H} \otimes \mathcal{K}$ .*

**PROOF.** Let  $V_{t,s}$  and  $\mathcal{E}_{t,s}$  be the propagators of  $\Lambda$  and  $\Phi$ , respectively. Then, by using the fact that for all  $t > u > s \geq 0$ ,  $\Lambda_t \otimes \Phi_t = V_{t,s} \Lambda_s \otimes \mathcal{E}_{t,s} \Phi_s = (V_{t,s} \otimes \mathcal{E}_{t,s})(\Lambda_s \otimes \Phi_s)$ , we establish that  $\Lambda \otimes \Phi$  is a Markovian dynamics.  $\square$

For pointwise multiplication and convex combination of two quantum dynamics, we obtain the following basic results.

**PROPOSITION 3.2.** *Suppose that  $\Lambda = \{\Lambda_t\}_{t \geq 0}$  and  $\Phi = \{\Phi_t\}_{t \geq 0}$  are Markovian dynamics of  $\mathcal{H}$  with the propagators  $V_{t,s}$  and  $\mathcal{E}_{t,s}$ , respectively. Then:*

- (1) for any quantum channel  $\Psi$  of  $\mathcal{H}$ , the quantum dynamics  $\Lambda \Psi = \{\Lambda_t \Psi\}_{t \geq 0}$  is Markovian and, when  $\Psi V_{t,s} = V_{t,s} \Psi (\forall t > s \geq 0)$ ,  $\Psi \Lambda = \{\Psi \Lambda_t\}_{t \geq 0}$  is also Markovian;

- (2) when  $\Lambda_s \mathcal{E}_{t,s} = \mathcal{E}_{t,s} \Lambda_s$  (for all  $t > s \geq 0$ ), the quantum dynamics  $\Lambda * \Phi = \{\Lambda_t \Phi_t\}_{t \geq 0}$  is Markovian;
- (3) when  $V_{t,s} = \mathcal{E}_{t,s}$  (for all  $t > s \geq 0$ ) for all  $0 < p < 1$ , the convex combination  $(1 - p)\Lambda + p\Phi = \{(1 - p)\Lambda_t + p\Phi_t\}_{t \geq 0}$  is Markovian.

**REMARK 3.3.** The condition that  $V_{t,s} = \mathcal{E}_{t,s}$  (for all  $t > s \geq 0$ ) does not imply that  $\Lambda = \Phi$ . For example, choose two different quantum channels  $\Delta$  and  $\Gamma$  of  $\mathcal{H}$  and a family  $\{V_{t,s}\}_{t > s \geq 0}$  of quantum channels with the property that  $V_{t,s} = V_{t,u} V_{u,s}$  for all  $t > u > s \geq 0$ . Then  $\{V_{t,0}\Delta\}_{t \geq 0}$  and  $\{V_{t,0}\Gamma\}_{t \geq 0}$  are Markovian dynamics of  $\mathcal{H}$  with the same propagators  $V_{t,s}$ , but  $\Lambda \neq \Phi$ .

Furthermore, the following example shows that the set of all Markovian quantum dynamics is not convex.

**EXAMPLE 3.4.** Let  $\rho \in D(\mathbb{C}^2)$  be an initial state and  $\sigma_i$  ( $i = 1, 2, 3$ ) be Pauli matrices, that is,

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{3.1}$$

Consider the master equation

$$\dot{\rho}(t) = \sum_{i=1}^3 \gamma_i(t) \{\sigma_i \rho(t) \sigma_i - \rho(t)\}, \quad t \geq 0, \tag{3.2}$$

where  $\gamma_i(t)$  are suitable real functions. Note that equation (3.2) is a special case of equation (2.1), where  $H(t) = 0, N = 3$  and  $V_i(t) = \sigma_i$  ( $i = 1, 2, 3$ ). Denote by  $\Lambda = \{\Lambda_t\}_{t \geq 0}$  the quantum dynamics described by equation (3.2) with  $\Lambda_0 = \mathbf{1}$ , the identity channel. Chruściński and Wudarski [8] proved that  $\Lambda$  is Markovian if and only if  $\gamma_i \geq 0$  for all  $i = 1, 2, 3$ .

Following Chruściński and Wudarski [9], we take  $\gamma_1(t) = \gamma_2(t) = 1/2, \gamma_3(t) = -(1/2) \tanh t$ ; then the dynamics  $\Lambda$  is non-Markovian. By computation,

$$\Lambda_t(\rho) = \frac{1 + e^{-2t}}{2} \rho + \frac{1 - e^{-2t}}{4} (\sigma_1 \rho \sigma_1 + \sigma_2 \rho \sigma_2) = \frac{1}{2} [\Lambda_t^{(1)}(\rho) + \Lambda_t^{(2)}(\rho)],$$

where

$$\Lambda_t^{(1)}(\rho) = \frac{1 + e^{-2t}}{2} \rho + \frac{1 - e^{-2t}}{2} \sigma_1 \rho \sigma_1, \quad \Lambda_t^{(2)}(\rho) = \frac{1 + e^{-2t}}{2} \rho + \frac{1 - e^{-2t}}{2} \sigma_2 \rho \sigma_2.$$

The master equations corresponding to  $\Lambda_t^{(1)}$  and  $\Lambda_t^{(2)}$  are

$$\dot{\rho}(t) = \sigma_1 \rho(t) \sigma_1 - \rho(t), \tag{3.3}$$

$$\dot{\rho}(t) = \sigma_2 \rho(t) \sigma_2 - \rho(t), \tag{3.4}$$

respectively. Since equations (3.3) and (3.4) are two special cases of equation (2.1) with nonnegative  $\gamma_k$ , we have that  $\Lambda^{(1)} = \{\Lambda_t^{(1)}\}_{t \geq 0}$  and  $\Lambda^{(2)} = \{\Lambda_t^{(2)}\}_{t \geq 0}$  are Markovian. But, as a convex combination of them,  $\Lambda$  is non-Markovian.

Now we can conclude that a convex combination of two Markovian quantum dynamics may be non-Markovian and so the set of all Markovian quantum dynamics of  $\mathcal{H}$  is not convex. However, a convex combination of two Markovian quantum dynamics may be Markovian.

**EXAMPLE 3.5.** Suppose that  $\gamma(t)$  and  $\alpha(t)$  are nonnegative continuous functions on  $[0, \infty)$ . For every  $X = [x_{ij}] \in M_2$ , define

$$\Lambda_t(X) = \begin{bmatrix} x_{11} & x_{12}e^{-\Gamma(t)} \\ x_{21}e^{-\Gamma(t)} & x_{22} \end{bmatrix} \quad \text{and} \quad \Phi_t(X) = \begin{bmatrix} x_{11} & x_{12}e^{-A(t)} \\ x_{21}e^{-A(t)} & x_{22} \end{bmatrix},$$

where  $\Gamma(t) = \int_0^t \gamma(\tau) d\tau$  and  $A(t) = \int_0^t \alpha(\tau) d\tau$ . Denote  $\Lambda = \{\Lambda_t\}_{t \geq 0}$  and  $\Phi = \{\Phi_t\}_{t \geq 0}$ . Note that from Example 2.6(2),  $\Lambda$  and  $\Phi$  are Markovian.

For  $0 \leq p \leq 1$ , let  $\Theta_t = p\Lambda_t + (1 - p)\Phi_t$  and  $\Theta = \{\Theta_t\}_{t \geq 0}$ . For each pair  $(s, t)$  with  $t > s \geq 0$ , we define

$$\mathcal{E}_{t,s}(X) = \begin{bmatrix} x_{11} & x_{12}f(t, s) \\ x_{21}f(t, s) & x_{22} \end{bmatrix} \quad \text{for all } X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \in M_2,$$

where

$$f(t, s) = \frac{pe^{-\Gamma(t)} + (1 - p)e^{-A(t)}}{pe^{-\Gamma(s)} + (1 - p)e^{-A(s)}}.$$

Then  $\Theta_t = \mathcal{E}_{t,s}\Theta_s$ , where  $\mathcal{E}_{t,s}$  is a quantum channel, since  $0 < f(t, s) \leq 1$ . So, the convex combination  $\Theta$  of  $\Lambda$  and  $\Phi$  is also Markovian.

The following example shows that the set of all non-Markovian quantum dynamics is not convex.

**EXAMPLE 3.6.** For any initial state  $\rho \in D(\mathbb{C}^2)$ , define the quantum dynamics  $\Lambda = \{\Lambda_t\}_{t \geq 0}$  as follows:

$$\Lambda_t(\rho) = p(t)\rho + \{1 - p(t)\} \frac{I}{2}, \tag{3.5}$$

where  $0 < p(t) \leq 1$  and  $I$  is the  $2 \times 2$  identity matrix. The master equation corresponding to equation (3.5) is

$$\dot{\rho}(t) = \sum_{i=1}^3 -\frac{\dot{p}(t)}{4p(t)} \{\sigma_i \rho(t) \sigma_i - \rho(t)\}, \quad t \geq 0,$$

where  $\sigma_i$  are Pauli matrices (see equation (3.1)). From Example 2.5(4), we know that the quantum dynamics  $\Lambda$  is Markovian if and only if for any  $t \geq 0$ ,  $\dot{p}(t) \leq 0$  [8].

Consider

$$p^{(1)}(t) = \begin{cases} e^{-t/20}, & 0 \leq t \leq 2.5, \\ \left(t - \frac{5}{2} - \frac{e^{-1/8}}{40}\right)^2 - \frac{e^{-1/4}}{1600} + e^{-1/8}, & 2.5 < t \leq 2.5 + \frac{e^{-1/8}}{20}, \\ -\left(t - \frac{5}{2} - \frac{3e^{-1/8}}{40}\right)^2 + \frac{e^{-1/4}}{1600} + e^{-1/8}, & 2.5 + \frac{e^{-1/8}}{20} < t \leq 2.5 + \frac{e^{-1/8}}{10}, \\ e^{-(1/20)(t - (e^{-1/8}/10))}, & t > 2.5 + \frac{e^{-1/8}}{10} \end{cases}$$

and

$$p^{(2)}(t) = \begin{cases} e^{-t}, & 0 \leq t \leq 5, \\ \left(t - 5 - \frac{e^{-5}}{2}\right)^2 + e^{-5} - \frac{e^{-10}}{4}, & 5 < t \leq 5 + e^{-5}, \\ -\left(t - \frac{3}{2}e^{-5} - 5\right)^2 + \frac{e^{-10}}{4} + e^{-5}, & 5 + e^{-5} < t \leq 5 + 2e^{-5}, \\ e^{-(t-2e^{-5})}, & t > 5 + 2e^{-5}. \end{cases}$$

Then, for all  $t \geq 0$ ,  $p^{(1)}(t) \in (0, 1]$  and  $p^{(2)}(t) \in (0, 1]$ . Let

$$\Lambda_t^{(1)}(\rho) = p^{(1)}(t)\rho + \{1 - p^{(1)}(t)\} \frac{I}{2}, \quad \Lambda_t^{(2)}(\rho) = p^{(2)}(t)\rho + \{1 - p^{(2)}(t)\} \frac{I}{2}.$$

Note that when  $t \in (5/2 + e^{-1/8}/40, 5/2 + 3e^{-1/8}/40)$ ,  $\dot{p}^{(1)}(t) > 0$  and, when  $t \in (5 + e^{-5}/2, 5 + 3e^{-5}/2)$ ,  $\dot{p}^{(2)}(t) > 0$ . Thus, both  $\Lambda^{(1)} = \{\Lambda_t^{(1)}\}_{t \geq 0}$  and  $\Lambda^{(2)} = \{\Lambda_t^{(2)}\}_{t \geq 0}$  are non-Markovian.

Let  $\Phi_t = (1/2)[\Lambda_t^{(1)} + \Lambda_t^{(2)}]$  and denote  $\Phi = \{\Phi_t\}_{t \geq 0}$ ; then

$$\Phi_t(\rho) = \frac{1}{2}[p^{(1)}(t) + p^{(2)}(t)](\rho) + \frac{1}{2}[\{1 - p^{(1)}(t)\} + \{1 - p^{(2)}(t)\}] \frac{I}{2}.$$

Observe that for all  $t \geq 0$ ,  $\dot{p}^{(1)}(t) + \dot{p}^{(2)}(t) < 0$ , so the convex combination  $\Phi$  of the two non-Markovian quantum dynamics  $\Lambda^{(1)}$  and  $\Lambda^{(2)}$  is Markovian.

#### 4. Witness of non-Markovian quantum dynamics

In this section, we introduce a witness of non-Markovian quantum dynamics by exploiting the correlation flow between a system and an arbitrary ancillary. Recall that quantum mutual information of a bipartite state  $\sigma$  of the system  $AB$  described by  $\mathcal{H} \otimes \mathcal{H}$  is

$$I(\sigma) = S(\sigma^a) + S(\sigma^b) - S(\sigma),$$

where  $\sigma^a = \text{tr}_B \sigma$ ,  $\sigma^b = \text{tr}_A \sigma$  are the reduced states of  $\sigma$  and  $S(X) = -\text{tr} X \log_2 X$  denotes the von Neumann entropy of a density operator  $X$  [14].

Let  $\Lambda = \{\Lambda_t\}_{t \geq 0}$  be a Markovian quantum dynamics of  $\mathcal{H}$ . Then, for each  $(t, s)$  with  $0 \leq s < t$ , there exists a quantum channel  $V_{t,s}$  on  $B(\mathcal{H})$  such that  $\Lambda_t = V_{t,s} \Lambda_s$ . From the monotonicity of the quantum mutual information under local operations, we have for each bipartite state  $\sigma$  of the system  $AB$  and  $0 \leq s < t < +\infty$ ,

$$\begin{aligned} F_\sigma(s, t) &= I((\Lambda_t \otimes \Lambda_s)\sigma) \\ &= I((V_{t,s} \Lambda_s \otimes \Lambda_s)\sigma) \\ &= I((V_{t,s} \otimes 1)(\Lambda_s \otimes \Lambda_s)\sigma) \\ &\leq I((\Lambda_s \otimes \Lambda_s)\sigma) \\ &= F_\sigma(s, s). \end{aligned}$$

This observation leads to the following result.

**PROPOSITION 4.1.** For a quantum dynamics  $\Lambda = \{\Lambda_t\}_{t \geq 0}$  of  $\mathcal{H}$ , if there exists a bipartite state  $\sigma$  of the system  $AB$  such that  $F_\sigma(s, t) > F_\sigma(s, s)$  for some parameters  $s$  and  $t$  with  $0 \leq s < t < +\infty$ , then  $\Lambda$  is non-Markovian.

We call such a state  $\sigma$  in Proposition 4.1 a witness of non-Markovianity of  $\Lambda$ .

**EXAMPLE 4.2.** Consider the quantum dynamics  $\Lambda = \{\Lambda_t\}_{t \geq 0}$  with  $\rho(t) = \Lambda_t(\rho)$  satisfying

$$\frac{d}{dt}\rho(t) = \frac{1}{2}\gamma(t)\{\sigma_z\rho(t)\sigma_z - \rho(t)\}.$$

It has been shown in Example 2.6 that when  $\gamma(t)$  is continuous on  $[0, +\infty)$ ,  $\Lambda$  is Markovian if and only if  $\gamma(t) \geq 0$  for all  $t \geq 0$ .

Assume that  $\gamma(t)$  is continuous on  $[0, +\infty)$  and  $\gamma(t_0) < 0$  for some  $t_0 \in [0, +\infty)$ . We prove that the state

$$\sigma = \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|)$$

is a witness of non-Markovianity of  $\Lambda$ . From the continuity of  $\gamma$ , we see that there exists a positive number  $\delta$  such that  $\gamma(t) < 0$  for any  $t \in [t_0, t_0 + \delta]$  and so  $\Gamma(t_0 + \delta) < \Gamma(t_0)$ . By computation,

$$\begin{aligned} \sigma_{t_0+\delta, t_0} &= (\Lambda_{t_0+\delta} \otimes \Lambda_{t_0})\sigma = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & e^{-\Gamma(t_0+\delta)-\Gamma(t_0)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{-\Gamma(t_0+\delta)-\Gamma(t_0)} & 0 & 0 & 1 \end{bmatrix}, \\ \sigma_{t_0, t_0} &= (\Lambda_{t_0} \otimes \Lambda_{t_0})\sigma = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & e^{-2\Gamma(t_0)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{-2\Gamma(t_0)} & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

and

$$\sigma_{t_0+\delta, t_0}^a = \sigma_{t_0+\delta, t_0}^b = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_{t_0, t_0}^a = \sigma_{t_0, t_0}^b = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore,

$$F_\sigma(t_0, t_0 + \delta) - F_\sigma(t_0, t_0) = I((\Lambda_{t_0+\delta} \otimes \Lambda_{t_0})\sigma) - I((\Lambda_{t_0} \otimes \Lambda_{t_0})\sigma) = S(\sigma_{t_0, t_0}) - S(\sigma_{t_0+\delta, t_0}).$$

Let

$$p = \frac{1}{2} - \frac{1}{2}e^{-\Gamma(t_0)-\Gamma(t_0+\delta)} \quad \text{and} \quad q = \frac{1}{2} - \frac{1}{2}e^{-2\Gamma(t_0)}.$$

Then  $0 < p < q < 0.5$  and

$$\begin{aligned} S(\sigma_{t_0+\delta, t_0}) &= -p \log_2 p - (1 - p) \log_2(1 - p), \\ S(\sigma_{t_0, t_0}) &= -q \log_2 q - (1 - q) \log_2(1 - q). \end{aligned}$$

Since  $f(x) = -x \log_2 x - (1 - x) \log_2(1 - x)$  is strictly increasing on  $[0, 0.5]$ ,

$$S(\sigma_{t_0+\delta, t_0}) = f(p) < f(q) = S(\sigma_{t_0, t_0})$$

and so  $F_\sigma(t_0, t_0 + \delta) - F_\sigma(t_0, t_0) > 0$ . Thus, in this case,  $\sigma$  is a witness of non-Markovianity of  $\Lambda$ .



## 5. Summary

Although the set of all Markovian quantum dynamics is not convex, there exist nontrivial examples (such as Proposition 3.2(3) and Example 3.6) which show that the convex combination of two Markovian quantum dynamics may be still Markovian. Therefore, finding the convex subset which consists of certain Markovian quantum dynamics is interesting and challenging. Besides the witness we have proposed, which is based on a necessary condition of Markovianity, other witnesses can be introduced according to different understandings of non-Markovianity, so as to enrich the criteria for non-Markovianity in quantum dynamics.

## Acknowledgements

This research was partially supported by the National Natural Science Foundation of China (Nos. 11371012, 11401359 and 11471200), the Fundamental Research Funds for the Central Universities (No. GK201604001) and the Innovation Fund Project for the Graduate Program of Shaanxi Normal University (No. 2016CBY005).

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