Ergod. Th. & Dynam. Sys. (1986), **6**, 241–257
Printed in Great Britain

Hölder continuous paths and hyperbolic toral automorphisms

M. C. IRWIN
Department of Pure Mathematics, Liverpool University, Liverpool, L69 3BX, England

(*Received* 8 *January* 1985 *and revised* 5 *November* 1985)

Abstract. Let $f: T^n \to T^n$ $(n \geq 3)$ be a hyperbolic toral automorphism. Let A be the set of $\alpha > 0$ such that there is a Hölder continuous path of index α in T^n with 1-dimensional orbit-closure under f. We prove that $\alpha_0 = \sup A$ can be expressed in terms of the eigenvalues of f, and that $\alpha_0 \in A$ if and only if $\alpha_0 < 1$.

1. *Introduction*

When Smale asked whether or not a hyperbolic toral automorphism can have a 1-dimensional (compact) invariant set, one natural response was to consider the orbit-closure of paths in the torus. Franks [**2**] and Mañé [**8**] proved that if a path is smooth enough then its orbit-closure is at least 2-dimensional. In fact, all rectifiable paths have this property [**8**]. In contrast, Hancock [**3**], [**4**] and Przytycki [**9**] answered Smales's question for tori of dimension ≥ 3 by showing how to construct C^0-paths with 1-dimensional orbit-closure. (Bowen [**1**] also obtained the answer, but by a different route. Smale himself had given a negative answer for T^2.)

There is a number that, to some extent, calibrates the gap between rectifiability and bare continuity, namely the Hölder index α of a Hölder continuous map. The purpose of the present paper is to locate the precise value of α at which the possibility of paths with 1-dimensional orbit-closure ceases. This is given by the following theorem:

THEOREM 1. *For* $n \geq 3$, *let* $f: T^n \to T^n$ *be a hyperbolic toral automorphism. Let* α_0 *be the maximum of* $\log|\mu|/\log|\lambda|$ *for all pairs* λ *and* μ *of* (*complex*) *eigenvalues of* f *for which the ratio is* ≤ 1. *Then*

(i) *if* $\alpha_0 < 1$ *there are* α-*Hölder paths in* T^n *with* 1-*dimensional orbit-closure for* $\alpha = \alpha_0$ *but for no greater* α;

(ii) *if* $\alpha_0 = 1$ *there are* α-*Hölder paths in* T^n *with* 1-*dimensional orbit-closure for every* $\alpha < 1$.

Since $\alpha = 1$ is essentially the case of rectifiable paths, Mañé's result completes the picture.

The paper is in two parts. In §§ 2–4, we construct α-Hölder paths with 1-dimensional orbit-closure for $\alpha = \alpha_0 < 1$ and $\alpha < \alpha_0 = 1$. In § 5 we prove that they

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do not exist for $\alpha > \alpha_0$. We have already made a start on the first part in [7], which dealt with the case of automorphisms of T^3 with real eigenvalues. (Since no power of such an automorphism can have repeated eigenvalues [7], this is precisely the case $\alpha = \alpha_0 < 1$.)

When I had completed writing this paper, I received a preprint of a paper by Mariusz Urbański, now published as [**10**], which performs a similar investigation in terms of the capacity of the image set rather than the Hölder index of the map. In his theorem 1, Urbański proves that if $f: T^n \to T^n$ is as above and if a curve in T^n has capacity $<2-\alpha_0$ then its orbit-closure under f has dimension ≥ 2. This immediately gives some restriction on the Hölder index with which a path can have 1-dimensional orbit-closure, and, in fact, it is quite easy to modify his proof to give the above result that the index can be at most α_0. (However, I am grateful to the referee for pointing out that I have actually proved rather more than this: see theorem 11 below.) Urbański's theorem 2 deals with the existence of paths with 1-dimensional orbit closure (and higher dimensional analogues). His theorem has more general statement but, in our terms, he proves that there exist such paths with capacity $\leq 2-\alpha_0+\varepsilon$, for any positive ε. The Hölder continuous paths in theorem 1 above have such a capacity, with $\varepsilon = 0$ in the case $\alpha_0 < 1$. Also, in the case $\alpha_0 < 1$, we prove that we can always find such paths that are simple (without self-intersections). In the case $\alpha_0 = 1$, one may prove that the paths that Urbański constructed are α-Hölder for α arbitrarily near 1. Both Urbański's paper and the present one make heavy use of techniques of Przytycki [9] to control the topological dimension of sets.

2. *Existence: strategy and preliminary results*

Let f be a hyperbolic automorphism of $T^n = \mathbb{R}^n/\mathbb{Z}^n$. That is to say, $f: T^n \to T^n$ lifts to a linear automorphism $L: \mathbb{R}^n \to \mathbb{R}^n$ with no eigenvalues of modulus 1. Thus $\mathbb{R}^n$ splits as a direct sum $E^u \oplus E^s$ of L-invariant subspaces such that the eigenvalues of $L|E^u$ and $L|E^s$ have modulus respectively >1 and <1. Let $\Pi: \mathbb{R}^n \to T^n$ be the quotient map. We make the standard identification $\mathbb{R}^n = \mathbb{R}^u \times \mathbb{R}^s$. We may assume that the product projection $\theta: \mathbb{R}^n \to \mathbb{R}^u$ restricts to an isomorphism ϕ of the unstable summand E^u onto $\mathbb{R}^u$. Finally, recall the definition of Hölder index: a map $g: X \to Y$ of metric spaces is *Hölder continuous of index* α $(0 < \alpha \leq 1)$ *at* $x \in X$, or *α-Hölder at* x, if, for some constant C,

$$d_Y(g(x), g(x')) \leq C d_X(x, x')^\alpha,$$

for all $x' \in X$. We say that g is α-Hölder if, for some C, the inequality holds for all $x, x' \in X$. Notice that if X is compact and g is α-Hölder then g is β-Hölder for all $\beta < \alpha$.

Proving the existence part of theorem 1 is easier for $n = 3$ than for $n > 3$. This is because, by a theorem of Hirsch and Williams [**5**], invariant sets of f cannot be $(n-1)$-dimensional. Also, since it is well known that there are dense orbits in T^n, the only n-dimensional invariant set of f is T^n. Thus to prove that the orbit-closure of a (non-constant) path in T^3 is 1-dimensional, we have merely to prove that it is not the whole of T^3, or, equivalently, that the orbit misses some non-empty open

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subset of T^3. We may assume that the unstable dimension u is 2 (otherwise replace f by f^{-1}). We take a small open neighbourhood N of 0 in $\mathbb{R}^u$. Thus $\Pi\theta^{-1}(N)$ is a small open neighbourhood of the circle $\Pi\theta^{-1}(0)$ in T^3. Now Π injectively immerses E^u densely in T^3, and the intersection of E^u with $\Pi^{-1}\Pi\theta^{-1}(N)$ is the neighbourhood $\phi^{-1}(N+\mathbb{Z}^2)$ of the lattice $G=\phi^{-1}(\mathbb{Z}^2)$ in E^u. We start with a map $\gamma_0: I\to E^u$, where $I=[0, 1]$ and make a sequence of perturbations to Lipschitz maps $\gamma_i: I\to E^u$, such that, for $i>0$, $L^i\gamma_i(I)$ misses $\phi^{-1}(N+\mathbb{Z}^2)$. If we are careful enough, we can ensure that (γ_i) converges to a Hölder continuous map $\gamma: I\to E^u$ whose index is related to the eigenvalues of $L|E^u$ (this is achieved by keeping a tight control on the Lipschitz constants of γ_i). Moreover, we can arrange that for all $i>0$, $L^i(\gamma)$ misses a rather smaller open neighbourhood $\phi^{-1}(N_1+\mathbb{Z}^2)$ of G, where $0\in N_1\subset N$. Now $L^i\gamma(I)\to 0$ as $i\to-\infty$. If we start with $\gamma_0(I)$ sufficiently near 0, the negative half-orbit $O^-(\gamma(I))$ (that is to say $\bigcup\{L^{-i}\gamma(I): i\geq 0\}$) misses all of $\phi^{-1}(N_1+\mathbb{Z}^2)$ except $\phi^{-1}(N_1)$. (In any case, it hits only finitely many components of the neighbourhood). Thus the orbit of $\Pi\gamma(I)$ in T^3 misses the open set $\Pi\theta^{-1}(N_1)\setminus\Pi\phi^{-1}(N_1)$, and hence has 1-dimensional closure.

The geometrical technique of constructing maps γ_i with the required properties depends on the eigenvalues of L. For $\alpha_0<1$ (as in [7]) we take $\phi^{-1}(N)$ to be a diamond with axes in the directions of the two eigenspaces (the 'strongly unstable' and 'weakly unstable' directions). Starting with γ_0 mapping onto a straight line segment in the strongly unstable direction, we are able to make all perturbations in the weakly unstable direction, and to finish with γ an embedding onto the graph of a function from one eigenspace to the other. In the case $\alpha_0=1$ the situation is not so straightforward, since we are forced to perturb in many different directions. This time $\phi^{-1}(N)$ is a Euclidean disc (after some linear change of coordinates in E^u) and we work with maps γ_i that map I onto polyhedral paths in E^u. We carefully control the minimum segment length of the paths, in order to control the Lipschitz constants of the maps.

For the general case $n\geq 3$, we again suppose $u\geq 2$ and construct a sequence of maps $\gamma_i: I\to E^u$, tending to a limit γ, such that the orbit-closure of $\Pi\gamma(I)$ is 1-dimensional In fact the paths γ_i and γ are all constructed in a 2-dimensional L-invariant subspace V of E^u, where the eigenvalues of $L|V$ are related by $\log|\mu|/\log|\lambda|=\alpha_0$. The construction of γ_i is made much more complicated by our efforts to control the dimension of the orbit-closure of $\Pi\gamma(I)$. The control-technique that we use is due to Przytycki [9]. As for $n=3$, the dimension of the closure of $O^-(\Pi\gamma(I))$ causes no problems, because $f^{-i}\Pi\gamma(I)\to\Pi(0)$. We try to ensure that locally the dimension of the closure of $O^+(\Pi\gamma(I))$ is at most 1 in the unstable direction and 0 in the stable direction.

We take a coordinate system in $\mathbb{R}^u$ such that its $(u-2)$-coordinate planes denoted X_j, say, are transverse to $\phi(V)$. We arrange that, for each X_j, $L^i\gamma(I)$ misses some small neighbourhood of $\Pi^{-1}\Pi\theta^{-1}(X_j)$ for all $i>0$. This implies that the orbit-closure of $\Pi\gamma(I)$ has dimension ≤ 1 in the unstable direction, basically because the complement of the neighbourhood in E^u has an open cover of bounded mesh and order 1 (for the notions of *mesh* and *order*, see [**6**]).

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To control dimension in the stable direction, we arrange that $L^i\gamma(I)$ misses a neighbourhood of $\Pi^{-1}\Pi(B^u(0, l)+\partial P)$, where P is a parallelepiped in E^s such that $\Pi(B^u(0, l)+\text{int } P)$ covers T^n. Here $B^u(0, l)$ is the ball with centre 0 and radius l in E^u. Roughly speaking, locally, in the stable direction, the complement of a number of copies of ∂P has an open cover of finite mesh and order 0, and this leads to the dimension of the orbit-closure of $\Pi\gamma(I)$ being locally 0 in the stable direction.

Recall that, in the $n=3$ case, we made $L^i\gamma_i(I)$ miss a neighbourhood of G, but that there was the possibility that later perturbations of γ_i into γ might affect this property. We got round this by making the total contribution of the later perturbations so small that the fact that $L^i\gamma_i(I)$ missed the original neighbourhood of G forced $L^i\gamma(I)$ to miss a rather smaller one. In the general case, the situation is more complicated because, instead of the single set G, we now have $2s+\frac{1}{2}u(u-1)$ sets that we would like to miss, namely $2s$ sets of the form $\Pi^{-1}\Pi(B^u(0, l)+Q_j)$, where Q_j are the $(s-1)$-dimensional faces of the parallelepiped P, and the $\frac{1}{2}u(u-1)$ sets $\Pi^{-1}\Pi\theta^{-1}(X_j)$. We make $2s+\frac{1}{2}u(u-1)$ perturbations in going from $L^i\gamma_{i-1}$ to $L^i\gamma_i$, each one to avoid a neighbourhood of one of the sets. Speaking very roughly, we can make the neighbourhoods as large as we like in the Q_j case, and as small as we like in the X_j case. We choose the neighbourhood-size to decrease so rapidly with j that, once we have missed the neighbourhood of a given set, all subsequent perturbations leading to $L^i\gamma_i$ *and* all subsequent perturbations leading from $L^i\gamma_i$ to $L^i\gamma$ are comparatively small in total, and so $L^i\gamma(I)$ misses a rather small neighbourhood of the given set. Of course there is a danger that, for example, perturbations of $L^{i+1}\gamma_i$ to miss large neighbourhoods in the course of obtaining $L^{i+1}\gamma_{i+1}$ may affect the way that $L^i\gamma_i(I)$ misses small neighbourhoods. However, this will not be the case if L is a sufficiently powerful expansion on V. We can always achieve this by replacing L by some large power of L. This is a useful trick, and we employ it on several occasions. Notice that replacing f by some power of f affects neither the ratio of logarithms of eigenvalues nor the dimension of the orbit-closure of a given subset of T^n.

We need a couple of preliminary results. The first shows how to obtain a Hölder continuous map as the limit of a sequence of Lipschitz maps.

LEMMA 2. *Let X and Y be metric spaces, with X compact, and let $\gamma_i: X\to Y$ $(i\geq 0)$ be a sequence of maps converging uniformly to $\gamma: X\to Y$. Given $\lambda>\mu>1$, $A>0$ and $B>0$, suppose that, for all $i\geq$ some i_0,*

(i) *γ_i is Lipschitz with constant $A\lambda^i/\mu^i$;*
(ii) *$d(\gamma, \gamma_i)\leq B/\mu^i$, where*

$$d(\gamma, \delta)=\sup\{d(\gamma(x), \delta(x)): x\in X\}.$$

Then γ is α-Hölder, with $\alpha=\log\mu/\log\lambda$.

Proof. Exercise, or see the penultimate paragraph of [7].

Recall that the norm $\|L\|$ of a linear automorphism $L:\mathbb{R}^n\to\mathbb{R}^n$ is the sup of $\{\|L(x)\|: x\in\mathbb{R}^n, \|x\|=1\}$ and the so-called 'minimum norm' $m(L)$ is its inf. Thus $\|L^{-1}\|=1/m(L)$.

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LEMMA 3. *Let L be a linear automorphism of $\mathbb{R}^2$ with eigenvalues λ and μ satisfying $|\lambda|=|\mu|=\rho$, say. Then*:

(i) *given $\sigma<\rho$, $m(L^r)\geq\sigma^r$ for sufficiently large r*;

(ii) *given $\tau>\rho$, $\|L^r\|\leq\tau^r$ for sufficiently large r.*

Proof. Immediate from the formula $\lim_{r\to\infty}\|L^r\|^{1/r}$ for the spectral radius.

3. *Existence*: *the case $\alpha_0<1$*

Recall that in explaining our strategy for proving the existence part of theorem 1 for general n, we said that we would need to make a sequence of perturbations to make $O^+(\gamma(I))$ miss neighbourhoods in V of a sequence of $2s+\frac{1}{2}u(u-1)$ sets (or, equivalently, $\gamma(I)$ miss the negative half-orbit of the neighbourhoods). In the case $\alpha_0<1$, each neighbourhood is a family of diamonds, and we shall isolate the construction of all perturbations leading to the map γ as a lemma. For the purposes of this lemma, let $L:\mathbb{R}^2\to\mathbb{R}^2$ be a linear map of the form

$$L(x, y)=(\lambda x, \mu y).$$

Let N be a positive integer. For $1\leq j\leq N$, let $\{D_{jk}: k\in\mathbb{N}\}$ be a family of disjoint open diamonds in $\mathbb{R}^2$ with centre $P_{jk}=(x_{jk}, y_{jk})$, width $b_j>0$ and height $2b_j$. That is to say,

$$D_{jk}=\{(x, y)\in\mathbb{R}^2: 2|x-x_{jk}|+|y-y_{jk}|<b_j\}.$$

Given $\tilde{b}_j$ with $0<\tilde{b}_j<b_j$, let $\tilde{D}_{jk}$ be the open diamond with centre P_{jk}, width $\tilde{b}_j$ and height $2\tilde{b}_j$.

LEMMA 4. *Suppose that, for all j,*

(i) $b_j-\tilde{b}_j>2\sum_{r=j+1}^{N} b_r$;

(ii) *$\lambda>2\mu$, and $\mu>1$ is sufficiently large for*

$$2\sum_{r=j+1}^{N} b_r+2\sum_{r=1}^{N} b_r(\mu-1)^{-1}<b_j-\tilde{b}_j;$$

(iii) *the distance between two distinct diamonds D_{jk} and $D_{jk'}$ is $>4\sqrt{5}\,Nb_j$.*

Then there exists a map $\gamma:I\to\mathbb{R}^2$, where $I=[0,1]$, which is α-Hölder, with $\alpha=\log\mu/\log\lambda$, and such that, for all j and k, $\gamma(I)$ does not intersect the negative half-orbit $O^-(\tilde{D}_{jk})$. Moreover we can always find γ of the form $\gamma(x)=(x, \delta(x))$, where $\delta:I\to\mathbb{R}$ is an α-Hölder map.

Proof. For all $i\geq 0$, let $D_{ijk}=L^{-i}(D_{jk})$ and $\tilde{D}_{ijk}=L^{-i}(\tilde{D}_{jk})$. Then D_{ijk} is an open diamond with centre

$$L^{-i}(P_{jk})=P_{ijk}=(x_{ijk}, y_{ijk}),$$

say, width b_j/λ^i and height $2b_j/\mu^i$. Similarly for $\tilde{D}_{ijk}$. Let $\rho:I\to\mathbb{R}^2$ be any map of the form

$$\rho(t)=(t, \sigma(t)),$$

where $\sigma:I\to\mathbb{R}$ is Lipschitz. Notice:

Remark 1. If $\operatorname{Lip}\sigma\leq 2\lambda^i/\mu^i$ and if $\rho(I)$ has non-empty intersection with D_{ijk} then either (a) $x_{ijk}\in I$ and $\rho(x_{ijk})\in D_{ijk}$, or (b) $x_{ijk}>1$ and $\rho(1)\in D_{ijk}$, or (c) $x_{ijk}<0$ and $\rho(0)\in D_{ijk}$.

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Remark 2. If Lip $\sigma \le 2\lambda^i/\mu^i$ and if $\rho(I)$ has non-empty intersection with two distinct diamonds D_{ijk} and $D_{ijk'}$ then

$$|x_{ijk} - x_{ijk'}| \ge 4Nb_j/\lambda^i.$$

Geometrically, since the edges of the diamond D_{ijk} have slope $2\lambda^i/\mu^i$, if $\rho(I)$ has secant slope $\le 2\lambda^i/\mu^i$ and enters D_{ijk}, it either crosses the vertical axis of D_{ijk} or terminates short of this axis. This gives remark 1. For $i=0$, remark 2 is clear from (iii), since ρ has Lipschitz constant $\sqrt{5}$. Mapping by L^{-i} gives the general statement.

We now prove the following:

SUBLEMMA 5. *For all $i \ge 0$ and $0 \le j \le N$, there exists a map $\gamma_{ij}: I \to \mathbb{R}^2$ (with $\gamma_{iN} = \gamma_{(i+1)0}$) of the form $\gamma_{ij}(t) = (t, \delta_{ij}(t))$ such that:*

(i) *δ_{ij} is Lipschitz with constant $(1+j/N)\lambda^i/\mu^i$ for all i, j;*

(ii) *$\gamma_{ij}(I) \cap D_{ijk}$ is empty for all i, k and j with $1 \le j \le N$;*

(iii) *$0 \le \delta_{ij}(t) - \delta_{i(j-1)}(t) \le 2b_j/\mu^i$ for all $t \in I$, i and j with $1 \le j \le N$.*

Proof. The proof is by induction on the pair (i, j), ordered lexicographically. To start the induction, define $\delta_{00}(t) = 0$. Now suppose that $\delta_{i(j-1)}$ is defined, for some $i \ge 0$, $1 \le j \le N$. Let

$$S = \{k \in \mathbb{N}: \gamma_{i(j-1)}(I) \cap D_{ijk} \ne \emptyset\}.$$

By remark 2, if k, $k' \in S$ with $k \ne k'$ then $|x_{ijk} - x_{ijk'}| \ge 4Nb_j/\lambda^i$. We obtain γ_{ij} from $\gamma_{i(j-1)}$ by a linear perturbation upwards on each interval

$$[x_{ijk} - 2Nb_j/\lambda^i,\ x_{ijk} + 2Nb_j/\lambda^i], \quad k \in S,$$

the perturbation being just large enough to avoid D_{ijk}. To be precise, let $\tau_k(t)$ denote the y-coordinate of the point at which the line $x = t$ intersects the top edge of D_{ijk}. We define the perturbation $\delta_{ij}(t) - \delta_{i(j-1)}(t)$ to be 0 if $|t - x_{ijk}| \ge 2Nb_j/\lambda^i$ for all $k \in S$, and to be

$$(1-w)(\tau_k(x_{ijk}) - \delta_{i(j-1)}(x_{ijk}))$$

if $|t - x_{ijk}| = 2wNb_j/\lambda^i$ for $w \le 1$, where $k \in S$ satisfies (a) of remark 1. We also have to allow for $\gamma_{i(j-1)}(I)$ terminating in D_{ijk} short of its vertical axis, as in (b) or (c) of remark 1. In these cases, we define $\delta_{ij}(t) - \delta_{i(j-1)}(t)$ to be respectively

$$(1-w)(\tau_k(1) - \delta_{i(j-1)}(1)),$$

where $1 - t = w(1 - x_{ijk} + 2Nb_j/\lambda^i)$ for $0 \le w \le 1$, and

$$(1-w)(\tau_k(0) - \delta_{i(j-1)}(0)),$$

where $t = w(x_{ijk} + 2Nb_j/\lambda^i)$ for $0 \le w \le 1$.

In the three cases (a), (b) and (c), $(1-w)$ multiplies a positive number which is less than the height $2b_j/\mu^i$ of D_{ijk}, and so property (iii) of the sublemma holds. Also on each linear portion of $\gamma_{ij} - \gamma_{i(j-1)}$ there is a vertical increment lying between $-2b_j/\mu^i$ and $2b_j/\mu^i$ in a horizontal distance of $2Nb_j/\lambda^i$, and so $\gamma_{ij} - \gamma_{i(j-1)}$ is Lipschitz with constant $\lambda^i/N\mu^i$. We deduce (i) for γ_{ij}, using (i) for $\gamma_{i(j-1)}$. (N.b. If $j = N$, this also gives (i) for $\delta_{(i+1)0}$, since $\lambda/\mu > 2$).

Finally, we have to check that our perturbation of $\gamma_{i(j-1)}(I)$ to avoid D_{ijk} does not produce an intersection of $\gamma_{ij}(I)$ with some other $D_{ijk'}$, $k' \notin S$. But, if this were

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the case, translating $\gamma_{ij}(I)$ slightly downwards would give it intersections with both D_{ijk} and $D_{ijk'}$ in a horizontal distance $<2Nb_j/\lambda^i$, which contradicts remark 2. This completes the proof of the sublemma.

To prove lemma 4, we show that (γ_{ij}) converges to a limit γ which has the required properties. By (iii) of the sublemma, for all $(i',j')\geq(i,j)$,

$$0\leq\delta_{i'j'}(t)-\delta_{ij}(t)\leq\left(\sum_{r=j+1}^{N}b_r+\left(\sum_{r=1}^{N}b_r\right)\sum_{m=1}^{\infty}1/\mu^m\right)2/\mu^i$$
$$\leq\left(\sum_{r=1}^{N}b_r\right)\left(\sum_{m=0}^{\infty}1/\mu^m\right)2/\mu^i,$$

which $\to 0$ as $i\to\infty$. Thus the limit δ of (δ_{ij}) exists and satisfies

$$0\leq\delta(t)-\delta_{ij}(t)\leq\left(\sum_{r=j+1}^{N}b_r+\sum_{r=1}^{N}b_r(\mu-1)^{-1}\right)2/\mu^i. \tag{1}$$

We write $\gamma(t)=(t,\delta(t))$. By (ii) of the sublemma, $\gamma_{ij}(I)\cap D_{ijk}$ is empty. The minimum vertical distance from $\tilde{D}_{ijk}$ to the complement of D_{ijk} is $(b_j-\tilde{b}_j)/\mu^i$. By (1) and by hypothesis (ii) of the lemma, this is greater than the maximum vertical distance from $\gamma_{ij}(I)$ to $\gamma(I)$. Hence $\gamma(I)\cap\tilde{D}_{ijk}$ is empty. Finally, if we write $\delta_i=\delta_{iN}$ then δ is the limit of (δ_i), δ_i is Lipschitz with constant $2\lambda^i/\mu^i$ and

$$\|\delta_i-\delta\|\leq 2\sum_{r=1}^{N}b_r(\mu-1)^{-1}/\mu^i.$$

Thus, by lemma 2, δ is α-Hölder. Hence γ is α-Hölder. This completes the proof of lemma 4.

We now prove the existence part of theorem 1 in the $\alpha_0<1$ case.

THEOREM 6. *Let $f: T^n\to T^n$ and α_0 be as in theorem* 1. *If $\alpha_0<1$, then there is an embedded path that is α_0-Hölder and has* 1*-dimensional orbit-closure.*

Proof. We use the notation of the first paragraph of § 2. Since $\alpha_0<1$, the eigenvalues of f are all real. Let λ and μ be eigenvalues with $\alpha_0=\log|\mu|/\log|\lambda|$. By taking a suitable (possibly negative) power of f, we may assume that all eigenvalues of f are positive and that $\lambda>\mu>1$. Let V be the 2-dimensional eigenspace corresponding to λ and μ, and let $W=\phi(V)$. We may assume, after conjugating with an automorphism of $\mathbb{R}^u$ with matrix in $GL_u(\mathbb{Z})$, that W is transverse to each coordinate $(u-2)$-plane of $\mathbb{R}^u$. (After possibly reordering the original coordinates of $\mathbb{R}^u$, we can choose generators of W of the form $(1,0,a_3,\ldots,a_u)$ and $(0,1,b_3,\ldots,b_u)$. If $e_1,\ldots,e_u$ is the standard basis of $\mathbb{R}^u$, we can pick a new basis $(1,0,m_3,\ldots,m_u)$, $(0,1,n_3,\ldots,n_u)$, $e_3,\ldots,e_u$, where m_r and n_r are integers such that $m_r\neq a_r$, $n_r\neq b_r$, and

$$(m_r-a_r)(n_s-b_s)\neq(m_s-a_s)(n_r-b_r)$$

for all r, s with $3\leq r\leq u$, $3\leq s\leq u$ and $r\neq s$. The coordinate planes with respect to this new basis have the required transversality.)

Consider $\mathbb{R}^u$ as a cubical complex C with the integer lattice $\mathbb{Z}^u$ as vertices. Let K be the complement of the $(u-2)$-skeleton of this complex, and let $\mathcal{U}$ be the

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open cover of K consisting of all open star neighbourhoods of barycentres of $(u-1)$- and u-dimensional cubes of C in the barycentric first-derived simplicial complex of C. Thus $\mathcal{U}$ is a cover of mesh $\sqrt{u}$ and order 1.

It is convenient to identify the subspace V with the plane $\mathbb{R}^2$ by an isomorphism such that the induced map $L:\mathbb{R}^2\to\mathbb{R}^2$ has the form

$$L(x, y) = (\lambda x, \mu y),$$

and, renorming $\mathbb{R}^u$ if necessary, we can assume that the identification is an isometry.

Following Przytycki [9], we construct an s-dimensional parallelepiped P in E^s, with a vertex at 0 and $(s-1)$-dimensional faces Q_j, $1\le j\le 2s$, such that the faces through 0 (given by $j=1,\dots,s$) generate subspaces Y_j of E^s with the property that

$$\Pi^{-1}\Pi(E^u)\cap Y_j=\{0\}. \tag{2}$$

(We can make such a construction because $\Pi^{-1}\Pi(E^u)\cap E^s$ is countable.) Let P have diameter d. Let $B^u(0, l)$ be the open ball in E^u with centre 0 and radius l, where l is large enough for Π to map the vector sum $B^u(0, l/2)+\text{int}\,P$ onto T^n. (This is possible since $\Pi(E^u)$ is dense in T^n.) For $1\le j\le 2s$, let

$$b_j=3^{2s-j}\cdot 2\sqrt{5}\,l \quad\text{and}\quad \tilde{b}_j=3^{2s-j}\sqrt{5}\,l.$$

For $\varepsilon_j>0$, let N_j^s be the open ε_j-neighbourhood of Q_j in E^s. For each point $q\in E^u$ such that $\Pi(q)\in\Pi(N_j^s)$ and $(q+B^u(0, l))\cap V\ne\emptyset$, consider the centroid p of the open disc $(q+B^u(0, l))\cap V$. Enumerate all such q and p as (q_{jk}) and (p_{jk}), $k\in\mathbb{N}$. Notice that the diamond $\tilde{D}_{jk}$ in $\mathbb{R}^2$ $(=V)$, defined earlier in this section, contains the ball $(q_{jk}+B^u(0, l))\cap V$, since the latter has radius $\le l$. We choose ε_j so small that, for all $k\ne k'$, the distance from q_{jk} to $q_{jk'}$ is greater than $4(1+\sqrt{5}\,N)b_j$, where $N=2s+\frac{1}{2}u(u-1)$. (It follows from (2) that we can do this.) The inequality guarantees that the diamonds D_{jk} and $D_{jk'}$ are disjoint, and that the distance between them is greater than $4\sqrt{5}\,Nb_j$.

Let X_j $(j=2s+1,\dots,N)$ be the coordinate $(u-2)$-planes in $\mathbb{R}^u$. Then $\Pi^{-1}\Pi\theta^{-1}(X_j)\cap V$ is, by the transversality assumption, a subgroup of V isomorphic to $\mathbb{Z}^2$, and we enumerate its points as (p_{jk}), $k\in\mathbb{N}$. We pick, successively, positive numbers $b_{2s+1},\dots,b_N$ satisfying $b_j\le b_{j-1}/3$ and so small that the distance between distinct diamonds D_{jk} and $D_{jk'}$ is at least $4\sqrt{5}\,Nb_j$. Let $\tilde{b}_j=b_j/2$.

We have now arranged that conditions (i) and (iii) of lemma 4 hold for all j, k with $1\le j\le N$, $k\in\mathbb{N}$. Replacing L by some positive power if necessary, we can assume that (ii) holds as well. (Since the power has the same E^s, E^u and V as L, this does not invalidate any of our previous work.) Thus, by lemma 4, there exists an α_0-Hölder map $\gamma: I\to V$ such that $\gamma(I)\cap O^-(\tilde{D}_{jk})$ is empty for all j, k. Moreover γ embeds I as the graph of an α_0-Hölder map $\delta: I\to\mathbb{R}$.

We must now prove that the orbit-closure of $\Pi\gamma(I)$ is 1-dimensional. First note that, since γ(I) is clearly 1-dimensional, so is the orbit of $\Pi\gamma(I)$. Since $f^n\Pi\gamma(t)\to\Pi(0)$ uniformly on I as $n\to-\infty$, the negative half-orbit $O^-(\Pi\gamma(I))$ has 1-dimensional closure. Thus it suffices to prove that $O^+(\Pi\gamma(I))$ has 1-dimensional closure.

Take an open ε-neighbourhood N^u of $X=\bigcup\{X_j: 2s+1\le j\le N\}$ in $\mathbb{R}^u$, where $\varepsilon>0$ is small enough for $\Pi^{-1}\Pi\theta^{-1}(N^u)\cap V$ to be contained in

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$\bigcup\{\tilde{D}_{jk}: 2s+1\le j\le N, k\in\mathbb{N}\}$. Thus $O^+(\gamma(I))$ is disjoint from $\Pi^{-1}\Pi\theta^{-1}(N^u)$. Take also an open ε_1-neighbourhood N_1^u of X in $\mathbb{R}^u$, with $0<\varepsilon_1<\varepsilon$, and choose $\eta>0$ so small that

$$\theta^{-1}(N_1^u)+B^s\subset\theta^{-1}(N^u),$$

where $B^s=B^s(0,\eta)$.

As B^u ranges over all open balls of radius $l/4$ in E^u, $\{\Pi(B): B=B^u+B^s\}$ gives an open cover of T^n. Thus, to show that $O^+(\Pi\gamma(I))$ has 1-dimensional closure in T^n, it suffices to show that the closure H of $O^+(\Pi^{-1}\Pi\gamma(I))\cap B$ has dimension ≤ 1 for all B. We do this by showing that H is contained in F^u+F^s, where F^u and F^s are closed subsets of B^u and B^s of dimension at most 1 and 0 respectively.

By construction, $O^+(\Pi^{-1}\Pi\gamma(I))$ is disjoint from $\Pi^{-1}\Pi\theta^{-1}(N_1^u)+B^s$ and hence from the negative half-orbit of this set. This negative half-orbit contains $O^-(\Pi^{-1}\Pi\theta^{-1}(N_1^u)\cap E^u)+B^s$. Now the complement of $\Pi^{-1}\Pi\theta^{-1}(N_1^u)\cap E^u$ is a closed set contained in $\phi^{-1}(K)$. Hence the complement of its negative half-orbit in E^u is a closed set contained in $\bigcap\{L^{-i}\phi^{-1}(K): i\ge 0\}$, and hence covered by $L^{-i}\phi^{-1}(\mathcal{U})$ for all $i\ge 0$. Thus its intersection with B^u is a subset F^u of dimension ≤ 1. We have $H\subset F^u+B^s$.

For each non-empty component of $\Pi^{-1}\Pi L^i(B^u(0,l/2)+N_j^s)\cap B$, where $i\ge 0$ and $1\le j\le 2s$, remove the corresponding components of $\Pi^{-1}\Pi L^i(B^u(0,l)+N_j^s)\cap B$ from B. This leaves a set $B^u+F_i^s$, where F_i^s is closed in B^s. The connected components of F_i^s give an open cover of F_i^s of order 0. The cover has mesh $\nu^i d$, where ν is the largest eigenvalue of $L|E^s$, because each member of the cover is contained in a set isometric to $L^i(\operatorname{int} P)$. Since the mesh tends to zero as $i\to\infty$, the closed set

$$F^s=\bigcap\{F_i^s: i\ge 0\}$$

has dimension ≤ 0. Moreover $O^+(\Pi^{-1}\Pi\gamma(I))$ is contained in B^u+F^s, and hence so is H. Finally, since E^u+E^s is a direct sum, $H\subset F^u+F^s$, as asserted. This completes the proof of theorem 6.

4. *Existence*: *the case* $\alpha_0=1$

We begin with a description of the type of perturbation that we employ in the $\alpha_0=1$ case. We call a continuous map $\delta: S\to\mathbb{R}^2$, where S is a straight line segment, a *piecewise linear piecewise embedding* (PLPE), if, for some subdivision of S, each subsegment is mapped linearly by δ onto a straight line segment in $\mathbb{R}^2$. The *minimum segment length* of δ is the minimum of the lengths $\delta(S_k)$ for all subsegments S_k of the subdivision.

LEMMA 7. *Let* (p_k), $k\in\mathbb{N}$, *be a sequence of distinct points in* $\mathbb{R}^2$, *such that the minimum distance h between points of the sequence is positive. Let* a, b, ω *and* $m=\min\{a,h/3\}$ *be positive numbers satisfying* $4b\le m$, $6b\le m\omega$. *For any* PLPE $\delta_0: I\to\mathbb{R}^2$ *with minimum segment length* $\ge a$, *there is a* PLPE $\delta_1: I\to\mathbb{R}^2$ *with minimum segment length* $\ge b$, *such that*:

(i) $\operatorname{Lip}\delta_1\le(1+\omega)\operatorname{Lip}\delta_0$;

(ii) $\|\delta_1-\delta_0\|\le 6b$;

(iii) $\delta_1(I)\cap B(p_k,b)$ *is empty for all* $k\in\mathbb{N}$.

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Proof. The map δ_1 is δ_0 composed with a PLPE $\sigma:\delta_0(I)\to\mathbb{R}^2$. It is enough to describe σ on one segment S of $\delta_0(I)$. For each non-empty set $S\cap B(p_k, b)$, choose any subsegment S_k of S of length m that contains the set. Two such subsegments S_k and $S_{k'}$, $k\neq k'$, do not overlap, for this would imply

$$\|p_k - p_{k'}\|\le 2m+2b\le 5m/2\le 5h/6.$$

We let σ be the identity off the subsegments S_k, and show how to define σ on S_k. Let $S_k=[q_1, q_2]$. We distinguish three cases:

(1) one end-point, say q_1, in $B(p_k, 2b)$ but not in $B(p_k, b)$;
(2) one end-point, say q_1, in $B(p_k, b)$;
(3) neither end-point in $B(p_k, 2b)$.

(Notice that we cannot have both end-points in $B(p_k, 2b)$, since $m\ge 4b$.)

Case (1). Let S_k intersect the boundary $\partial B(p_k, b)$ in points q_3 and q_4, with q_3 nearer to q_1. Let q_5 be the mid-point of the shorter arc of $\partial B(p_k, b)$ joining q_3 to q_4. Let q_6 be a point distant b from q_1 and q_3 such that $[q_1, q_6]$ and $[q_6, q_3]$ do not intersect $B(p_k, b)$. Let q_7 be similarly related to q_3 and q_5, and let q_8 be similarly related to q_5 and q_4 (see figure 1). Define σ on S_k to be the PLPE which increases length by

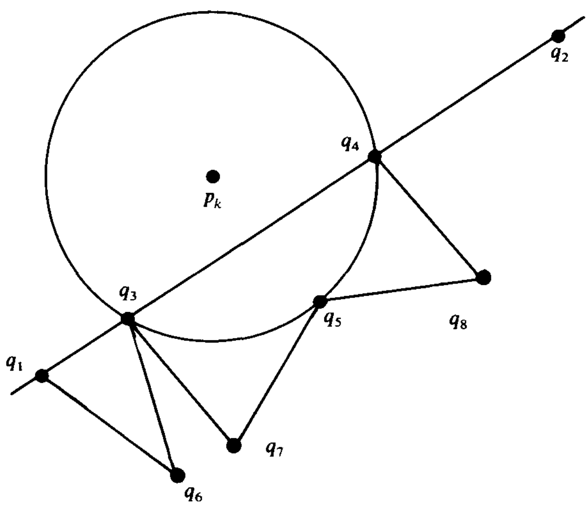

FIGURE 1

a constant factor and maps S_k onto the polyhedral path $q_1q_6q_3q_7q_5q_8q_4q_2$. Clearly σ has Lipschitz constant at most $(m+6b)/m$, which is at most $1+\omega$. If $q\in[q_4, q_2]$ then the polygonal path distance from q to $\sigma(q)$ along $\sigma(S_k)$ is at most $6b$. On the other hand, it is easy to see that the convex hull of $\{q_1, q_6, q_7, q_8, q_4\}$ has diameter at most $6b$, so

$$\|q-\sigma(q)\|\le 6b \qquad \text{for } q\notin[q_4, q_2].$$

Thus all three conditions of the lemma are satisfied in this case.

Case (3). We define points as in case 1, except that we no longer need a point q_6. This time σ maps S onto $q_1q_3q_7q_5q_8q_4q_2$.

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Case (2). *Let* S intersect $\partial B(p_k, b)$ in q_3. If q_1, which is an end-point of S, is also an end-point of an adjacent segment T of $\gamma(I)$, let T intersect $\partial B(p_k, b)$ in q_4, and let q_5 be the mid-point of the shorter arc of $\partial B(p_k, b)$ joining q_3 to q_4. If q_1 is an endpoint of $\gamma(I)$, let q_5 be an arbitrary point of $\partial B(p_k, b)$ near q_3. Let q_6 be the point distant b from q_3 and q_5 such that $[q_3, q_6]$ and $[q_6, q_5]$ do not intersect $B(p_k, b)$. (See figure 2). We define σ to map S_k onto the polygonal path $q_5q_6q_3q_2$. Of course σ will also map T onto a polygonal path ending at q_5. Similar estimates to those in case 1 again give conditions (i) and (ii).

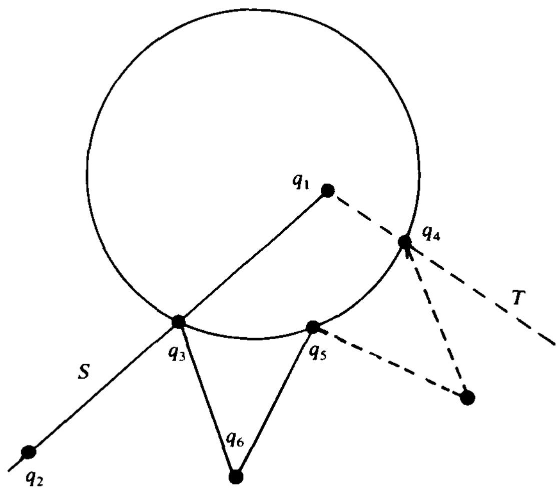

FIGURE 2

We now prove an analogue of lemma 4. As before, (p_{jk}), $k \in \mathbb{N}$, $1 \le j \le N$, are points of $\mathbb{R}^2$. For fixed j, the minimum distance between points of the sequence (p_{jk}) is the positive number h_j. This time, D_{jk} and $\tilde{D}_{jk}$ are balls $B(p_{jk}, b_j)$ and $B(p_{jk}, \tilde{b}_j)$ respectively, for some $b_j > \tilde{b}_j > 0$. For the purposes of the next lemma L is a linear automorphism of $\mathbb{R}^2$.

LEMMA 8. *Let* $L: \mathbb{R}^2 \to \mathbb{R}^2$ *have eigenvalues* λ, μ *with* $|\lambda| = |\mu| > 1$. *Let* τ *be a number with* $|\lambda| \le \|L\| < \tau < |\lambda|^2$ *and let* $\omega > 0$ *satisfy* $(1+\omega)^N < \tau/\|L\|$ *and* $\omega < 1$. *Suppose that, for all* j:

(i) $b_j \le \omega b_{j-1}/6$;

(ii) $m(L)$ *is large enough for* $b_1 \le m(L)\omega b_N/6$;

(iii) ω *is small enough for* $\omega b_j(1-\omega/6)^{-1} < b_j - \tilde{b}_j$;

(iv) $m(L)$ *is large enough for*

$$\omega b_j(1-\omega/6)^{-1} + 6b_1(1-\omega/6)^{-1}(m(L)-1)^{-1} < b_j - \tilde{b}_j;$$

(v) $h_j \ge 18 b_j/\omega$.

Then there is a map $\gamma: I \to \mathbb{R}^2$, *where* $I = [0, 1]$, *which is Hölder constant with index* $2\log|\lambda|/\log\tau - 1$ *and such that, for all* j, k, $\gamma(I)$ *does not intersect the negative half-orbit* $O^-(\tilde{D}_{jk})$.

Proof. As before, we prove a sublemma by induction on the pair (i, j).

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SUBLEMMA 9. *For all* $i \geq 0$ *and* $0 \leq j \leq N$, *there exists a map* $\gamma_{ij} : I \to \mathbb{R}^2$ (*with* $\gamma_{iN} = \gamma_{(i+1)0}$) *such that*:

(i) $L^i\gamma_{ij}$ *is Lipschitz with constant* $(6b_1/\omega)\tau^i(1+\omega)^j$ *for all i, j*;

(ii) $L^i\gamma_{ij}(I) \cap D_{jk}$ *is empty for all i, j, k with* $1 \leq j \leq N$;

(iii) $\|L^i\gamma_{ij} - L^i\gamma_{i(j-1)}\| \leq 6b_j$ *for all i, j with* $1 \leq j \leq N$;

(iv) $L^i\gamma_{ij}$ *has minimum segment length* $\geq b_j$ *for all i, j with* $1 \leq j \leq N$.

Proof. Notice that (i) for $L^i\gamma_{iN}$ implies (i) for $L^{i+1}\gamma_{(i+1)0}$, by the inequality $\|L\|(1+\omega)^N < \tau$. We start with γ_{00} mapping I linearly onto a straight line segment of length $6b_1/\omega$.

Suppose that $\gamma_{i(j-1)}$ is defined and satisfies (i)-(iv) of the sublemma. We apply lemma 7 with $\delta_0 = L^i\gamma_{i(j-1)}$, $b = b_j$, $h = h_j$ and $p_k = p_{jk}$. For $j > 1$, the minimum segment length a of δ_0 is at least $6b/\omega$ by property (iv) of the sublemma and (i) of lemma 8. For $j = 1$, a is the minimum segment length of $L^i\gamma_{(i-1)N}$, and so is at least $m(L)b_N$ by (iv) of the sublemma. This is at least $6b_1/\omega$ by (ii) of lemma 8. Combining these estimates with (v) of lemma 8 gives the inequality $6b \leq m\omega$ for lemma 7, which in turn gives $4b \leq m$, since $\omega \leq 1$. We define γ_{ij} by putting $\delta_1 = L^i\gamma_{ij}$. Properties (i)-(iv) of the sublemma for γ_{ij} are immediate from property (i) for $\gamma_{i(j-1)}$ and the conclusions of lemma 7.

We now show that (γ_{ij}) converges to a limit γ having the properties stated in lemma 9. Summing contributions, using (iii) of the sublemma, we have that, for all $(i', j') > (i, j)$,

$$\|L^i\gamma_{i'j'} - L^i\gamma_{ij}\| \leq 6\left(\sum_{r=j+1}^{N} b_r + \left(\sum_{s=1}^{i'-i-1} \|L^{-s}\|\right) \sum_{r=1}^{N} b_r + \|L^{i-i'}\| \sum_{r=1}^{j'-1} b_r\right),$$

whence, using (i) of lemma 8 and the fact that $\|L^{-1}\| = 1/m(L) = \|L\|/|\lambda|^2 < 1$,

$$\|L^i\gamma_{i'j'} - L^i\gamma_{ij}\| \leq b_j\omega(1-\omega/6)^{-1} + 6b_1(1-\omega/6)^{-1}(m(L)-1)^{-1}$$

and

$$\|\gamma_{i'j'} - \gamma_{ij}\| \leq (b_j\omega(1-\omega/6)^{-1} + 6b_1(1-\omega/6)^{-1}(m(L)-1)^{-1})m(L)^{-i}.$$

Thus (γ_{ij}) converges, to a limit γ, say. Note that

$$\|L^i\gamma - L^i\gamma_{ij}\| \leq b_j\omega(1-\omega/6)^{-1} + 6b_1(1-\omega/6)^{-1}(m(L)-1)^{-1}, \qquad (3)$$

which, together with (iv) of lemma 9 and (ii) of the sublemma, implies that $L^i\gamma(I) \cap \tilde{D}_{jk}$ is empty for all i, j, k with $i \geq 0$, $1 \leq j \leq N$ and $k \in \mathbb{N}$.

Now γ is the limit of (γ_i) where $\gamma_i = \gamma_{iN}$. By (i) of the sublemma, γ_i is Lipschitz with constant $(6b_1/\omega)(1+\omega)^N\tau^i m(L)^{-i}$. Also, by (3),

$$\|\gamma - \gamma_i\| \leq (b_N\omega(1-\omega/6)^{-1} + 6b_1(1-\omega/6)^{-1}(m(L)-1)^{-1})m(L)^{-i}.$$

Thus, by lemma 2, γ is Hölder continuous with index $\log m(L)/\log \tau$. But $m(L) > |\lambda|^2/\tau > 1$, and so γ has index $\log(|\lambda|^2/\tau)/\log \tau$, as required. This completes the proof of lemma 8.

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The existence part of theorem 1 in the $\alpha_0 = 1$ case is as follows:

THEOREM 10. *Let $f: T^n \to T^n$ and α_0 be as in theorem 1. If $\alpha_0 = 1$, then, for any $\alpha < 1$, there is an α-Hölder path in T^n with 1-dimensional orbit-closure.*

Proof. Let λ and μ be eigenvalues of f with $|\lambda| = |\mu|$. Replacing f by f^{-1} if necessary, we can assume $|\lambda| > 1$. Given $\alpha < 1$, choose a number τ, with $|\lambda| < \tau < |\lambda|^2$, near enough to $|\lambda|$ for $2 \log |\lambda| / \log \tau - 1$ to be greater than α. As in the proof of theorem 6, we let V be a two-dimensional subspace of $\mathbb{R}^u$ such that $L|V$ has λ and μ as eigenvalues. We identify V with $\mathbb{R}^2$ by some linear isomorphism. By lemma 3, if we replace f (and τ) by some sufficiently large power we can assume that $\|L|V\| < \tau$. Now choose $\omega > 0$ satisfying $(1+\omega)^N < \tau / \|L|V\|$, where $N = 2s + \frac{1}{2}u(u-1)$. We also take $\omega < \frac{6}{13}$, noting that for such values of ω, condition (iii) of lemma 8 holds for all j, when $b_j = \tilde{b}_j/2$.

We construct points p_{jk}, $1 \le j \le N$, $k \in \mathbb{N}$ in V, exactly as in the proof of theorem 6. For $1 \le j \le 2s$, we define

$$b_j = (6/\omega)^{2s-j} 2l \quad \text{and} \quad \tilde{b}_j = (6/\omega)^{2s-j} l.$$

As in theorem 6, we have numbers ε_j at our disposal, and we choose them so small that the distance from p_{jk} to $p_{jk'}$ for $k \neq k'$ is at least $18 b_j/\omega$. Next we successively choose $b_{2s+1}, \ldots, b_N$ positive but so small that, for $2s+1 \le j \le N$, $b_j \le \omega b_{j-1}/6$ and the distance from p_{jk} to $p_{jk'}$, $k \neq k'$, is at least $18 b_j/\omega$. Let $\tilde{b}_j = b_j/2$. For $1 \le j \le N$, let D_{jk} and $\tilde{D}_{jk}$ be the open balls in V with centre p_{jk} and radius b_j and $\tilde{b}_j$ respectively. Finally, after replacing f by some power if necessary, we can assume, writing $L|V$ as $L: \mathbb{R}^2 \to \mathbb{R}^2$, that $m(L)$ is large enough for conditions (ii) and (iv) of lemma 8 to hold. Then the map $\gamma: I \to \mathbb{R}^2 = V$ given by lemma 8 is α-Hölder, and the orbit of $\Pi\gamma$ is 1-dimensional by the argument of the proof of theorem 6.

5. *Non-existence*

The following result implies the non-existence part of theorem 1(i).

THEOREM 11. *Let $f: T^n \to T^n$ be a hyperbolic automorphism with $\alpha_0 < 1$. Let $\delta: I \to T^n$ be a non-constant path where $I = [0, 1]$. If, for some $\alpha > \alpha_0$, δ is α-Hölder at every point of I, then the orbit-closure of $\delta(I)$ contains a coset of an f-invariant toral subgroup of T^n (and so is at least 2-dimensional).*

Thus if δ is nowhere locally constant, and if $\delta(I)$ has 1-dimensional orbit closure, then for any given $\alpha > \alpha_0$, the set of points at which δ is not α-Hölder is dense in I.

We begin with a piece of undergraduate analysis. The reason that α-Hölder maps $g: I \to \mathbb{R}$ are not important for $\alpha > 1$ is that they are trivially constant, being differentiable with zero derivative. We generalise this remark, as follows.

LEMMA 12. *Let $k > 1$, and let $g: I \to \mathbb{R}$ be continuous. Suppose that, for all $t_0 \in [0, 1)$, there is some sequence (t_n), decreasing to t_0, such that:*

$$|g(t_n) - g(t_0)| \le (t_n - t_0)^k.$$

Then g is constant.

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Proof. As we have remarked,

$$|g(t')-g(t)|\le|t'-t|^k$$

for all t, $t'\in I$ implies that g is constant. Suppose that g is not constant, so that, for some a, $b\in I$, with $a<b$ say,

$$|g(b)-g(a)|>(b-a)^k.$$

By continuity of g at b, there exists $d>0$ such that

$$|g(t)-g(a)|>(t-a)^k$$

for all $t\in I$ with $|t-b|\le d$. Now consider the set X of $t\in I$ with $t\ge a$ such that, for some $r\ge 0$ there exists a chain

$$a=\tau_0<\tau_1<\cdots<\tau_r=t$$

with

$$\tau_i-\tau_{i-1}<d \quad\text{and}\quad |g(\tau_i)-g(a)|\le(\tau_i-a)^k$$

for $1\le i\le r$. Then X is non-empty (it contains a), and it is easy to see that, if $\sup X=\xi$, then $\xi\in X$ and hence that $\xi=1$ (using the sequence decreasing to ξ to obtain a contradiction if $\xi<1$). But some point of the chain (τ_i) joining a to 1 lies in the d-neighbourhood of b, which gives a contradiction. Hence g is constant.

For simplicity, we first give the proof of theorem 1 in the case $n=3$. Let $\Pi:\mathbb{R}^3\to T^3=\mathbb{R}^3/\mathbb{Z}^3$ be the quotient map, and let $L:\mathbb{R}^3\to\mathbb{R}^3$ be the hyperbolic linear automorphism covering f. The condition $\alpha_0<1$ implies that the eigenvalues λ, μ, ν of f are real and we may assume, replacing f by some power if necessary, that they satisfy

$$\lambda>\mu>1>\nu>0,$$

so that $\alpha_0=\log\mu/\log\lambda$. Let U be open in T^3, and let $V=\Pi^{-1}(U)$. Let δ lift to $\gamma:I\to\mathbb{R}^3$, where γ is α-Hölder at every point of I. We must show that, for some $r\in\mathbb{Z}$, $f^r\delta(I)$ intersects U, or, equivalently, $\delta(I)$ intersects $f^{-r}(U)$, or, equivalently, $\gamma(I)$ intersects $L^{-r}(V)$.

For convenience we work with coordinates $x=(x_1,x_2,x_3)$ in $\mathbb{R}^3$ with respect to a basis v_1, v_2, v_3 of eigenvectors of L corresponding to the eigenvalues λ, μ, ν in that order. Given $p\in\mathbb{R}^3$, let $C_i(p,l,a)$ be the solid circular cylinder with centre p, radius a, and axis in the v_i-direction of length $2l$. That is to say,

$$C_i(p,l,a)=\{x\in\mathbb{R}^3\colon |x_i-p_i|\le l,\ (x_j-p_j)^2+(x_k-p_k)^2\le a^2,\ j\ne i\ne k\ne j\}.$$

LEMMA 13. *There exists $a>0$ and $l>0$ such that, for all i and for all $p\in\mathbb{R}^3$, any path in $C_i(p,l,a)$ running its length (i.e. passing through a point of the boundary disc $x_i=p_i-l$ and a point of $x_i=p_i+l$) intersects V.*

Proof. Let $p\in\mathbb{R}^3$. The x_i-axis maps under Π to a subgroup of T^3 that is dense in T^3. (Its closure is an f-invariant total subgroup G_i. Since f is hyperbolic, $\dim G_i>1$ and so by theorem 9 of [1], $G_i=T^3$.) Hence so does the parallel line through p. Thus, for some $l>0$, the segment $|x_i-p_i|\le l$ of this line intersects V. Since V is open, it contains, for some small $a>0$, some interior cross-sectional disc $x_i=c_i$ of the cylinder $C_i(p,l,a)$. Thus the property holds for $C_i(p,l,a)$, and also for $C_i(q,l,a)$,

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for $q \in \mathbb{R}^3$ sufficiently near p, by openness of V. Using compactness of the unit cube (with respect to the original coordinates in $\mathbb{R}^3$), we obtain a and l for which the property holds uniformly for $C_i(p, l, a)$ for all p in the unit cube, and hence, by covering translations, for all $p \in \mathbb{R}^3$.

COROLLARY 14. *Let l and a be as in lemma 3. For all $p \in \mathbb{R}^3$ and for all $r > 0$, any path in $C_1(p, l\lambda^{-r}, a\mu^{-r})$ running its length intersects $L^{-r}(V)$.*

Proof. Under L^r, $C_1(p, l\lambda^{-r}, a\mu^{-r})$ maps onto an elliptic cylinder contained in, and with the same axis as, $C_1(L^r(p), l, a)$. Thus a curve running the length of $C_1(p, l\lambda^{-r}, a\mu^{-r})$ maps to a curve running the length of $C_1(L^r(p), l, a)$. By lemma 13, the latter curve intersects V.

Returning to the proof of theorem 11, choose $\beta \in \mathbb{R}$ with $\alpha > \beta > \log \mu/\log \lambda$, and put $\varepsilon = \beta - (\log \mu/\log \lambda)$. Write $\gamma = (\gamma_1, \gamma_2, \gamma_3)$. We first suppose that γ_1 is not the constant function. Then, by lemma 12, for some subinterval $J = [t_0, t_0 + c]$, $c > 0$, of I,

$$|\gamma_1(t) - \gamma_1(t_0)| \geq (t - t_0)^{\alpha/\beta},$$

for all $t \in J$. We may assume without loss of generality that $\gamma_1(t) > \gamma_1(t_0)$ for $t > t_0$ in J. Choose $r > 0$ large enough for

$$2l\lambda^{-r} \leq c^{\alpha/\beta} \quad \text{and} \quad A(2l)^\beta \lambda^{-r\varepsilon} \leq a,$$

where A is the Hölder constant of γ at t_0. (That is to say,

$$\|\gamma(t) - \gamma(t_0)\| \leq A|t - t_0|^\alpha,$$

where $\| \quad \|$ is the Euclidean norm in the (x_1, x_2, x_3)-coordinate system.) Thus

$$\gamma_1(t_0 + c) - \gamma_1(t_0) \geq 2l\lambda^{-r},$$

and by making c smaller if necessary, we can assume that $t_0 + c$ is the smallest t for which equality holds, for this value of r. Thus, for all $t \in J$,

$$\begin{aligned}((\gamma_2(t) - \gamma_2(t_0))^2 + (\gamma_3(t) - \gamma_3(t_0))^2)^{\frac{1}{2}} &\leq \|\gamma(t) - \gamma(t_0)\| \\ &\leq A(t - t_0)^\alpha \\ &\leq A(\gamma_1(t) - \gamma_1(t_0))^\beta \\ &\leq A(2l\lambda^{-r})^\beta \\ &= A(2l)^\beta \lambda^{-r\varepsilon} \lambda^{-r \log \mu/\log \lambda} \\ &\leq a\mu^{-r}.\end{aligned}$$

Hence $\gamma|J$ is a curve in $C_1(\gamma(t_0) + l\lambda^{-r}v_1, l\lambda^{-r}, a\mu^{-r})$. It runs the length of the cylinder, since it joins $\gamma(t_0)$ to $\gamma(t_0 + c)$. Thus, by corollary 14, $\gamma(J)$ intersects $L^{-r}(V)$.

Now suppose that γ_1 is constant, with $\gamma_1(t) = p_1$, say, for all $t \in I$. Suppose that γ_2 is not constant. Let d be the length of the interval $\gamma_2(I)$, and let e be a bound for $|\gamma_3(I)|$. Then, for $r > 0$, $L^r\gamma(I)$ is contained in, and runs the length of, the cylinder $C_2(p(r), d\mu^r, e\nu^r)$, where $p(r) = (p_1, p_2(r), 0)$ for some $p_2(r) \in \mathbb{R}$. Thus, for r sufficiently large, $L^r\gamma(I)$ intersects V. A similar argument shows that if γ_3 is not constant, $L^{-r}\gamma(I)$ intersects V for sufficiently large r. This completes the proof of theorem 11 in the case $n = 3$.

https://doi.org/10.1017/S0143385700003424 Published online by Cambridge University Press

In the general case, we may assume that the hyperbolic linear automorphism $L:\mathbb{R}^n \to \mathbb{R}^n$ covering f has eigenvalues λ_i, $1 \le i \le n$ satisfying

$$\lambda_1 > \lambda_2 > \cdots > \lambda_u > 1 > \lambda_{u+1} > \cdots > \lambda_n > 0.$$

Take coordinates in $\mathbb{R}^n$ with respect to a basis $v_1, \ldots, v_n$ of corresponding eigenvectors. The closure of the image of the x_i-axis under the quotient map $\Pi:\mathbb{R}^n \to T^n$ is an f-invariant toral subgroup which we denote by G_i. As before, $C_i(p, l, a)$ is the solid spherical cylinder in $\mathbb{R}^n$ with centre p, radius l and axis in the v_i-direction of length $2l$. Lemma 13 generalises as follows:

LEMMA 13′. *Let U be an open neighbourhood in T^n of a point of G_i, let $q \in T^n$, let $V = \Pi^{-1}(q+U)$ and let $H_i = \Pi^{-1}(q+G_i)$. There exist positive numbers l, a and b such that, for all $p \in \mathbb{R}^n$ with distance $<b$ from H_i, any path in $C_i(p, l, a)$ running its length intersects V.*

To prove the lemma, note that, given $p \in H_i$, the x_i-axis $0x_i$ intersects the set $V-p$, since $\Pi(0x_i)$ is dense in G_i, and hence $0x_i + p$ intersects V. The proof now follows that of lemma 13, using openness of V to get the result for cylinders with centre near p, and compactness of G_i and covering translations to get it for cylinders with centre in the lift of some open neighbourhood of $q + G_i$. The number b is the distance from the boundary of the neighbourhood to $q + G_i$. Now let δ lift to $\gamma: I \to \mathbb{R}^n$, where γ is α-Hölder at every point of I. We first prove the theorem when $\gamma_1, \gamma_2, \ldots, \gamma_{i-1}$ are constant but γ_i is non-constant, where $1 \le i < u$. We choose $\beta \in \mathbb{R}$ such that $\alpha > \beta > \log \lambda_{i+1}/\log \lambda_i$. By lemma 1, for some subinterval $J = [t_0, t_0 + c]$, $c > 0$ of I,

$$|\gamma_i(t) - \gamma_i(t_0)| \ge (t - t_0)^{\alpha/\beta}$$

for $t \in J$. Let q be some ω-limit point of $\delta(t_0)$ under f, and let (r_j) be an increasing sequence such that $f^{r_j}\delta(t_0) \to q$ as $j \to \infty$. We prove that the orbit-closure of $\delta(I)$ contains $q + G_i$ by showing that the orbit of $\delta(I)$ intersects $q + U$ for each neighbourhood U (in T^n) of each point of G_i. Let U be such a neighbourhood of such a point, and let l, a and b be the corresponding positive numbers given by lemma 13′. As in the proofof the $n = 3$ case, there is, for large enough j, some restriction of γ running the length of the cylinder $C_i(\gamma(t_0) + l\lambda_i^{-r_j}v_i, l\lambda_i^{-r_j}, a\lambda_{i+1}^{-r_j})$. Moreover, we can take j large enough for the distance from $f^{r_j}\delta(t_0)$ to q to be less than b. Since $\gamma_1, \ldots, \gamma_{i-1}$ are constant, L^{r_j} maps the restriction of γ into a path running the length of the cylinder $C_i(p, l, a)$, where $p = L^{r_j}\gamma(t_0) + lv_i$. Since the distance from p to H_i is less than b, lemma 13′ tells us that $L^{r_j}l(I)$ intersects V.

A similar argument with f replaced by f^{-1} proves the theorem for $\gamma_n, \gamma_{n-1}, \ldots, \gamma_{i+1}$ constant but γ_i non-constant, where $u+1 < i \le n$. This leaves the case where all coordinate functions but γ_u and γ_{u+1} are constant. In this case, the orbit-closure of $\delta(I)$ contains $q + G_u$ (resp. $q + G_{u+1}$) when γ_u (resp. γ_{u+1}) is non-constant, q being any ω-limit (resp. α-limit) point of $\delta(0)$. The proof is an obvious modification of the corresponding part of the $n = 3$ proof.

https://doi.org/10.1017/S0143385700003424 Published online by Cambridge University Press

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