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Partition Algebras are Cellular

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Abstract. The partition algebra $P_n(q)$ is a generalization both of the Brauer algebra and the Temperley–Lieb algebra for q-state n-site Potts models, underpinning their transfer matrix formulation on the arbitrary transverse lattices. We prove that for arbitrary field k and any element $q \in k$ the partition algebra $P_n(q)$ is always cellular in the sense of Graham and Lehrer. Thus the representation theory of $P_n(q)$ can be determined by applying the developed general representation theory on cellular algebras and symmetric groups. Our result also provides an explicit structure of $P_n(q)$ for arbitrary field and implies the well-known fact that the Brauer algebra $D_n(q)$ and the Temperley–Lieb algebra $TL_n(q)$ are cellular.

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1. Introduction

Let k be a field of arbitrary characteristic. For each element $q \in k$ and a natural number n the partition algebra $P_n(q)$, defined in [11], is a generalization both of the Brauer algebra $D_n(q)$ in [2], and also of the Temperley–Lieb algebra $TL_n(q)$ in [15]. It is of interest in statistic mechanics and appears in high physical dimensions (see [13] and [14]).

In the partition algebra $P_n(q)$, the linear basis is by definition the set of all partitions of $\{1, 2, \ldots, 2n\}$. The multiplication of two basis elements is the 'natural concatenation' depending on the parameter q. If one takes k to be the complex number field and $q \neq 0$, then Martin proved in [12] that the partition algebra $P_n(q)$ is always quasi-hereditary in the sense of Cline, Parshall and Scott [3]. Moreover, he also determined the structure of indecomposable projective modules in this case. Clearly, the partition algebra always has a homomorphic image isomorphic to the group algebra of the symmetric group Σ_n on the letters $\{1, 2, \ldots, n\}$. If the characteristic of k is positive, then the group algebra $k\Sigma_n$ usually is not quasi-hereditary. This might suggest that in general the partition algebra $P_n(q)$ is no longer quasi-hereditary (see Section 5). Thus, one may ask which structure the partition algebra could have and how to determine its irreducible representations.

In this paper, we shall consider these questions. We show that the partition algebra shares a structure similar to that of Brauer algebra developed in [7]. More precisely, we prove that for any field k and $q \in k$, the partition algebra $P_n(q)$ is cellular in the sense of Graham and Lehrer [6], where cellular algebras, motivated by the properties of the Kazhdan–Lusztig basis of Hecke algebras, were introduced to handle a class of important algebras including the Ariki–Koike Hecke algebras [1], Brauer algebras and many others. Thus, applying the general representation theory of cellular algebras to partition algebras, we can get a description of the irreducible modules of $P_n(q)$ for any field of arbitrary characteristic. As a consequence, we get also some results of P. Martin.

The paper is organized as follows: In the second section we recall some relevant definitions and facts on partition algebras from [11]. The third section deals with cellular algebras, where we recall an equivalent definition of cellular algebras in [9] and establish our main lemma. The fourth section is the proof of the main result of this paper. The last section contains a simple example to explain the main result.

2. Partition Algebras

In this section we recall the definition of partition algebras and some basic facts from [11] which are needed in this paper.

Let M be a finite set. We denote by E_M the set of all equivalent relations on, or equivalently all partitions of the set M:

$$\begin{aligned} E_M := \{&\rho = \big((M_1)(M_2)\cdots(M_i)\cdots\big) \mid \emptyset \neq M_i \subset M, \cup_i M_i = M,\\ &M_i \cap M_j = \emptyset\, (i \neq j)\} \end{aligned}$$

For example, we take $M = \{1, 2, 3\}$, then

$$E_M = \{(123),\ (1)(23),\ (12)(3),\ (13)(2),\ (1)(2)(3)\}.$$

If $\rho = ((M_1)\cdots(M_s))$, we define $|\rho|$ to be the number of equivalence classes of ρ. If we call each M_j in ρ a part of ρ, then $|\rho|$ is the number of parts of $|\rho|$.

Note that there is a partial order on E_M: if ρ_1 and ρ_2 are two elements in E_M, we say by definition that ρ_1 is smaller than or equal to ρ_2 if and only if each part of ρ_1 is a subset of a part of ρ_2. With this partial order, E_M is a lattice.

If $\mu \in E_M$ and $\nu \in E_N$, then we define $\mu \cdot \nu \in E_{M\cup N}$ as being the smallest ρ in $E_{M\cup N}$ such that $\mu \cup \nu \subset \rho$.

We are mainly interested in the case $M = \{1, 2, \ldots, n, 1', 2', \ldots, n'\}$. Note that E_M depends only upon the cardinality $|M|$ of M. So we sometimes write E_{2n} for E_M. To formulate our definitions, we denote by M' the set $\{1', 2', \ldots, n', 1'', 2'', \ldots, n''\}$.

DEFINITION 2.1. Let $f: E_M \times E_M \to \mathbf{Z}$ be such that $f(\mu, \nu)$ is the number of parts of $\mu \cdot \nu \in E_{M\cup M'}$ (note that $|M \cup M'| = 3n$) containing exclusively elements with a single prime.

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For example, in the case $n = 3$, $((123)(1'2')(3')) \cdot ((1')(2'3')(1'')(2'')(3'')) = ((123)(1'2'3')(1'')(2'')(3''))$ and $f(\mu, \nu) = 1$.

DEFINITION 2.2. Let $C: E_M \times E_M \to E_M$ be such that $C(\mu, \nu)$ is obtained by deleting all single primed elements of $\mu \cdot \nu$ (discarding the $f(\mu, \nu)$ empty brackets so produced), and replacing all double primed elements with single primed ones.

The partition algebra $P_n(q)$ is defined in the following way.

DEFINITION 2.3 ((see [11])). Let k be a field and $q \in k$. We define a product on

$$E_M : E_M \times E_M \to E_M, \qquad (\mu, \nu) \longmapsto \mu\nu = q^{f(\mu,\nu)} C(\mu, \nu).$$

This product is associative. Let $P_n(q)$ denote the vector space over k with the basis E_M. Then, by linear extention of the product on E_M, the vector space $P_n(q)$ becomes a finite-dimensional algebra over k with the above product. We call this algebra $P_n(q)$ the *partition algebra*.

If we take $B_M = \{\rho \in E_M \mid \text{ each part of } \rho \text{ has exactly two elements of } M\}$ and define the product of two elements in B_M in the same way as in 2, then the subspace $D_n(q)$ of $P_n(q)$ with the basis B_M becomes a finite-dimensional algebra. This is just the Brauer algebra. Similarly, if we take $P_M = \{\rho \in B_M \mid \rho \text{ is planar}\}$, then we get the Temperley–Lieb algebra $TL_n(q)$ with the basis P_M and the product 2.3. The word ‘planar’ means that if we think of the basis elements diagrammatically, then there are no edges crossing each other in the diagram (see [6]).

For an element $\mu \in P_n(q)$, we define $\#^P(\mu)$ to be the maximal number of distinct parts of μ containing both primed and unprimed elements of M, over the E_M basis elements with nonzero coefficients in μ.

The following fact is true in $P_n(q)$.

LEMMA 2.4. *For* $\mu, \nu \in P_n(q)$, *we have* $\#^P(\mu\nu) \leqslant \min\{\#^P(\mu), \#^P(\nu)\}$.

Given a partition $\rho \in E_M$, if we interchange the primed element j' with unprimed element j, then we get a new partition of M, let us denote this new partition by $i(\rho)$. Then i extends by linearity to $P_n(q)$.

For example, if $n = 4$ and $\rho = ((12)(341'2')(3'4'))$ then $i(\rho) = ((1'2')(3'4'12)(34))$.

LEMMA 2.5. *The linear map* i *is an anti-automorphism of* $P_n(q)$ *with* $i^2 = \mathrm{id}$.

Proof. Clearly, the map i is k-linear with $i^2 = \mathrm{id}$. It remain to check that $i(\mu\nu) = i(\nu)i(\mu)$ holds true for all $\mu, \nu \in E_M$. However, this follows immediately from the graphical realization of the product in $P_n(q)$ (see [12]), or from a verification of the above equation for the products of two generators of $P_n(q)$ displayed in [11].

In the following, a k-linear anti-automorphism i of a k-algebra A with $i^2 = \mathrm{id}$ will be called an *involution*.

3. Cellular Algebras

First, we recall the original definition of cellular algebras introduced by Graham and Lehrer. Then we give an equivalent definition which is convenient to use for looking at the structure of cellular algebras, and establish our technical lemma.

DEFINITION 3.1. (Graham and Lehrer, [6]). Let R be a commutative Noetherian integral domain. An associative R-algebra A is called a *cellular algebra* with cell datum (I, M, C, i) if the following conditions are satisfied:

(C1) The finite set I is partially ordered. Associated with each $\lambda \in I$ there is a finite set $M(\lambda)$. The algebra A has an R-basis $C^{\lambda}_{S,T}$ where (S, T) runs through all elements of $M(\lambda) \times M(\lambda)$ for all $\lambda \in I$.

(C2) The map i is an R-linear anti-automorphism of A with $i^2 = \mathrm{id}$ which sends $C^{\lambda}_{S,T}$ to $C^{\lambda}_{T,S}$.

(C3) For each $\lambda \in I$ and $S, T \in M(\lambda)$ and each $a \in A$ the product $aC^{\lambda}_{S,T}$ can be written as $(\sum_{U\in M(\lambda)} r_a(U, S)C^{\lambda}_{U,T}) + r'$ where r' is a linear combination of basis elements with upper index μ strictly smaller than λ, and where the coefficients $r_a(U, S) \in R$ do not depend on T.

The basis $\{C^{\lambda}_{S,T}\}$ of a cellular algebra A is called a cellular basis. With this basis there is a bilinear form Φ_{λ}, for each $\lambda \in I$, which is defined by $C^{\lambda}_{S,T}C^{\lambda}_{U,V} = \Phi_{\lambda}(T, U)C^{\lambda}_{S,V}$ modulo the ideal generated by all basis elements with upper index μ strictly smaller than λ. Graham and Lehrer proved that the isomorphism classes of simple modules are parametrized by the set $\Lambda_0 = \{\lambda \in I \mid \Phi_{\lambda} \neq 0\}$.

Typical examples of cellular algebras are Brauer algebras, Hecke algebras of type A and B, Temperley–Lieb algebras and many others. We shall prove that partition algebras are cellular.

The following is the equivalent definition of cellular algebras:

DEFINITION 3.2 [(see [9])]. Let A be an R-algebra where R is a commutative Noetherian integral domain. Assume there is an involution i on A. A two-sided ideal J in A is called a *cell ideal* if and only if $i(J) = J$ and there exists a left ideal $\Delta \subset J$ such that Δ is finitely generated and free over R and that there is an isomorphism of A-bimodules $\alpha : J \simeq \Delta \otimes_R i(\Delta)$ (where $i(\Delta) \subset J$ is the i-image of Δ) making the following diagram commutative:

$$\begin{array}{ccc} J & \xrightarrow{\alpha} & \Delta \otimes_R i(\Delta) \\ {\scriptstyle i}\downarrow & & \downarrow {\scriptstyle x \otimes y \mapsto i(y) \otimes i(x)} \\ J & \xrightarrow{\alpha} & \Delta \otimes_R i(\Delta) \end{array}$$

The algebra A (with the involution i) is called *cellular* if and only if there is an R-module decomposition $A = J'_1 \oplus J'_2 \oplus \cdots \oplus J'_n$ (for some n) with $i(J'_j) = i(J'_j)$ for each j and such that setting $J_j = \oplus^{j}_{l=1} J'_l$ gives a chain of two-sided ideals of

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$A: 0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$ (each of them fixed by i) and for each j ($j = 1, \ldots, n$) the quotient $J'_j = J_j/J_{j-1}$ is a cell ideal (with respect to the involution induced by i on the quotient) of A/J_{j-1}.

The $\Delta's$ obtained from each section J_j/J_{j-1} are called *standard modules* of the cellular algebra A, and the above chain of ideals in A is called a cell chain of A. Note that all simple A-modules can be obtained from standard modules [6]. (Standard modules are called Weyl modules in [6]).

To construct cellular algebras, we have the following lemma which is essentially implied in [10].

LEMMA 3.3. *Let A be an algebra with an involution i. Suppose there is a decomposition*

$$A = \bigoplus_{j=1}^{m} V_j \otimes_k V_j \otimes_k B_j \quad \textit{(direct sum of vector space)},$$

where V_j is a vector space and B_j is a cellular algebra with respect to an involution σ_j and a cell chain $J_1^{(j)} \subset \cdots \subset J_{s_j}^{(j)} = B_j$ for each j. Define $J_t = \bigoplus_{j=1}^{t} V_j \otimes_k V_j \otimes_k B_j$. Assume that the restriction of i on $V_j \otimes_k V_j \otimes_k B_j$ is given by $w \otimes v \otimes b \mapsto v \otimes w \otimes \sigma_j(b)$. If for each j there is a bilinear form $\phi_j: V_j \otimes_k V_j \to B_j$ such that $\sigma_j(\phi_j(w, v)) = \phi_j(v, w)$ for all $w, v \in V_j$ and that the multiplication of two elements in $V_j \otimes V_j \otimes B_j$ is governed by ϕ_j modulo J_{j-1}, that is, for $x, y, u, v \in V_j$ and $b, c \in B_j$, we have

$$(x \otimes y \otimes b)(u \otimes v \otimes c) = x \otimes v \otimes b\phi_j(y, u)c$$

modulo the ideal J_{j-1}, and if $V_j \otimes V_j \otimes J_l^{(j)} + J_{j-1}$ is an ideal in A for all l and j, then A is a cellular algebra.

Proof. Define $C = \bigoplus_{j=2}^{m} V_j \otimes V_j \otimes B_j$ and $J = J_1$. Then $C = A/J$ is an algebra. Now by [10] it is easy to see that the algebra A is an inflation of C along J. Note that J is, in fact, an iterated inflation since B_1 is a cellular algebra. Thus A is an iterated inflation and a cellular algebra by [10].

A direct proof of this lemma reads as follows. Since

$$J_1^{(j)} \subset \cdots \subset J_{s_j}^{(j)} = B_j, \quad j = 1, \ldots, m$$

is a cell chain for the given cellular algebras B_j, we can check that the following chain of ideals in A satisfies all conditions in Definition 3.2:

$$\begin{aligned} V_1 \otimes V_1 \otimes J_1^{(1)} &\subset \cdots \subset V_1 \otimes V_1 \otimes J_{s_1}^{(1)} \subset V_1 \otimes V_1 \otimes B_1 \oplus V_2 \otimes V_2 \otimes J_1^{(2)} \\ &\subset V_1 \otimes V_1 \otimes B_1 \oplus V_2 \otimes V_2 \otimes J_2^{(2)} \\ &\subset \cdots \subset V_1 \otimes V_1 \otimes B_1 \oplus V_2 \otimes V_2 \otimes B_2 \end{aligned}$$

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$$\subset \cdots \subset \bigoplus_{j=1}^{m-1} V_j \otimes V_j \otimes B_j \oplus V_m \otimes V_m \otimes J_1^{(m)} \subset \cdots$$
$$\subset \bigoplus_{j=1}^{m-1} V_j \otimes V_j \otimes B_j \oplus V_m \otimes V_m \otimes J_{s_m}^{(m)} = A.$$

Now we take a fixed nonzero element $v_j \in V_j$ and suppose that $\alpha\colon J_t^{(j)} \to \Delta_t^{(j)} \otimes i(\Delta_t^{(j)})$ is the bimodule isomorphism in the definition of the cell ideal $J_t^{(j)}$. Define

$$\beta\colon V_j \otimes V_j \otimes J_t^{(j)} \to (V_j \otimes v_j \otimes \Delta_t^{(j)}) \otimes (v_j \otimes V_j \otimes \otimes i(\Delta_t^{(j)}))$$
$$u \otimes v \otimes x \longmapsto \textstyle\sum_l (u \otimes v_j \otimes x_l) \otimes (v_j \otimes v \otimes y_l),$$

where $u, v \in V_j, x \in J_t^{(j)}$ and $\alpha(x) = \sum_l x_l \otimes y_l$. Then one can verify that β makes the corresponding diagram in the definition of cell ideals commutative. Hence $V_j \otimes v_j \otimes \Delta_t^{(j)}$ is a standard module for A, and $V_j \otimes V_j \otimes J_t^{(j)}$ is a cell ideal in the corresponding quotient of A. Thus A is a cellular algebra.

4. Proof of the Main Result

In this section we prove the main result of this paper and give some corollaries.

THEOREM 4.1. *The partition algebra $P_n(q)$ is a cellular algebra.*

The proof of this theorem is based on a series of lemmas. We keep the notation introduced in the previous sections. Recall that E_n denotes the set of all partitions of $\{1, 2, \ldots, n\}$.

For each $l \in \{0, 1, \ldots, n\}$, we define a vector space V_l which has as a basis the set

$$\mathcal{S}_l = \{(\rho, S) \mid \rho \in E_n, |\rho| \geqslant l, \quad S \text{ is a subset of the set of all parts of } \rho \quad \text{with } |S| = l\}.$$

(Note that in [11] this set is denoted by $S_n(l)$.)

If $\rho \in E_n$, we may write ρ in a standard way: Suppose $\rho = ((M_1) \cdots (M_s))$, we write each M_i in such a way that $M_i = (a_1^{(i)} a_2^{(i)} \cdots a_{t_i}^{(i)})$ with $a_1^{(i)} < a_2^{(i)} < \cdots < a_{t_i}^{(i)}$. If $a_1^{(1)} < a_1^{(2)} < \cdots < a_1^{(s)}$, then we say that ρ is written in standard form. It is clear that there is only one standard form for each ρ. We may also introduce an order on the set of all parts of ρ by saying that $M_j < M_k$ if and only if $a_1^{(j)} < a_1^{(k)}$.

If $N \subset M$ and $\rho \in E_M$, we denote by $r_N(\rho)$ the partition of $M \backslash N$ obtained from ρ by deleting all elements in N from the parts of ρ, and by $d_N(\rho)$ the set of parts of ρ which do not contain any element in N. Finally, we denote by Σ_r the symmetric group of all permutations on $\{1, 2, \ldots, r\}$ and by $k\Sigma_r$ the corresponding group algebra over the field k.

LEMMA 4.2. *Each element* $\rho \in E_M$ *can be written uniquely as an element of* $V_l \otimes V_l \otimes k\Sigma_l$ *for a natural number* $l \in \{0, 1, \dots, n\}$.

Proof. Take a partition $\rho \in E_M$, we define $x := r_{\{1',2',\dots,n'\}}(\rho) \in E_n$. If we identify the set $\{1', 2', \dots, n'\}$ with $\{1, 2, \dots, n\}$ by sending j' to j, then $y := r_{\{1,2,\dots,n\}}(\rho)$ lies in E_n. Let S_ρ be the set of parts of ρ containing both primed and unprimed elements. Then $|S_\rho| = \#^P(\rho)$. Now let S be the set of those parts of x which are obtained from elements of S_ρ by deleting the numbers contained in $\{1', 2', \dots, n'\}$. Similarly, we get a subset T of the set of all parts of y. It is clear that both S and T contain $l(= |S_\rho|)$ elements. Now if we write $S = \{S_1, \dots, S_l\}$ and $T = \{T_1, T_2, \dots, T_l\}$ such that $S_1 < S_2 < \cdots < S_l$ and $T_1 < T_2 < \cdots < T_l$, we may define a permutation $b \in \Sigma_l$ by sending j to k if there is a part $Y \in S_\rho$ containing both S_j and T_k', where $T_k' = \{a' \mid a \in T_k\}$. Since x, y and b are uniquely determined by ρ in a standard form, we can associate with the given ρ a unique element $(x, S) \otimes (y, T) \otimes b$. Obviously, (x, S) and (y, T) belong to V_l and $b \in \Sigma_l$. Conversely, each element $(x, S) \otimes (y, T) \otimes b$ with $(x, S), (y, T) \in \mathcal{S}_l$ and $b \in \Sigma_l$ corresponds to a unique partition $\rho \in E_M$. This finishes the proof of the lemma.

For example, for $\rho = ((1232'3')(41')(54')(5'))$, we have $x = ((123)(4)(5))$, $y = ((1)(23)(4)(5))$, $S = \{(123), (4), (5)\}$, $T = \{(1), (23), (4)\}$ and $b = (12) \in \Sigma_3$.

Now we want to define a bilinear form $\phi_l : V_l \otimes V_l \to k\Sigma_l$. Let (ρ, S) be in $\mathcal{S}_l$. We may assume that $S = \{S_1, \dots, S_l\}$ with $S_1 < S_2 < \cdots < S_l$. We define

$$\phi_l : V_l \otimes V_l \to k\Sigma_l$$

by sending $(x, S) \otimes (y, T)$ to zero if there are i and j with $1 \leqslant i, j \leqslant l$ and $i \neq j$ and there is a part of $x \cdot y \in E_n$ containing both S_i and S_j, or dually there are i and j with $1 \leqslant i, j \leqslant l$ and $i \neq j$ and there is a part of $x \cdot y \in E_n$ containing both T_i and T_j, or there is a number $1 \leqslant i \leqslant l$ and a part of $x \cdot y$ containing only S_i, or dually there is a number $1 \leqslant i \leqslant l$ and a part of $x \cdot y$ containing only T_i, and to $q^{|d_{S \vee T}(x \cdot y)|} b \in k\Sigma_l$ in other case, where $S \vee T$ stands for the union of all parts of S and T, and b is defined as follows: Since for each i there is a unique part of $x \cdot y$ containing both S_i and a unique part T_j, we define b to be the permutation taking i to j. Thus $b \in \Sigma_l$. We denote this b by $p_l(x, S; y, T)$. If we extend ϕ_l by linearity to the whole space $V_l \otimes V_l$, then we have the following lemma.

LEMMA 4.3. *The map* $\phi_l : V_l \otimes_k V_l \to k\Sigma_l$ *is a bilinear form.*

The multiplication of two elements in $P_n(q)$ is given by the following two lemmas:

LEMMA 4.4. *Let* μ, ν *be in* E_M. *If* $\mu = (u, R) \otimes (x, S) \otimes b_1 \in V_l \otimes V_l \otimes k\Sigma_l$ *and* $\nu = (y, T) \otimes (v, Q) \otimes b_2 \in V_l \otimes V_l \otimes k\Sigma_l$, *then*

$$\mu\nu = (u, R) \otimes (v, Q) \otimes b_1 \phi_l((x, S), (y, T)) b_2$$

modulo $J_{l-1} = \bigoplus_{j=0}^{l-1} V_j \otimes V_j \otimes k\Sigma_j$.

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Proof. By the definition of the multiplication in $P_n(q)$ and the definition of $d_{S\vee T}(x \cdot y)$, we know that $f(\mu, \nu) = |d_{S\vee T}(x \cdot y)|$. Hence, it is sufficient to show that the element $(u, R) \otimes (v, Q) \otimes b_1\phi_l((x, S), (y, T))b_2$ just presents the element $q^{f(\mu,\nu)}C(\mu, \nu)$ in $P_n(q)$ modulo J_{l-1}.

If $\phi_l((x, S), (y, T)) = 0$, then, by the definiton of ϕ_l, we see that $\#^P(\mu\nu) < l$. This implies that $C(\mu, \nu) \in J_{l-1}$. Now assume that $\phi_l((x, S), (y, T)) = q^{|d_{S\vee T}(x\cdot y)|}b$, where b is defined as above. Now we have to show that $(u, R) \otimes (v, Q) \otimes b_1bb_2$ presents the element $C(\mu, \nu)$. Indeed, by the definition of ϕ_l, we have obviously that $r_{\{1',2',\ldots,n'\}}(C(\mu, \nu)) = u \in E_n$ and that $r_{\{1,2,\ldots,n\}}(C(\mu, \nu)) = v \in E_n$ if we identify j' with j for $1 \leqslant j \leqslant n$. Note that there is only l distinct parts of $x \cdot y$, sayiny $P_1, P_2, \ldots, P_l$, containing a single S_i and a single T_{ib}. Hence, there is a part in $C(\mu, \nu)$ which contains both $R_{ib^{-1}}$ and S_i. Since T_i and Q_{ib_2} are contained in the same part of ν, we see finally that $R_{ib_1^{-1}}$ and Q_{ibb_2} are contained in the same part of $C(\mu, \nu)$. Hence $C(\mu, \nu)$ is presented by $(u, R) \otimes (v, Q) \otimes b_1bb_2$. This finishes the proof.

LEMMA 4.5. *Let l and m be two natural numbers with $l < m$. Take $\alpha = (u, R) \otimes (x, S) \otimes b \in V_m \otimes V_m \otimes k\Sigma_m$ with $b \in \Sigma_m$ and $\beta = (y, T) \otimes (v, Q) \otimes c \in V_l \otimes V_l \otimes k\Sigma_l$ with $c \in \Sigma_l$. If $\alpha\beta = q^{|d_{S\vee T}(x\cdot y)|}(w, F) \otimes (z, G) \otimes d$, then*

(1) *if $|F| = l$, then $(z, G) = (v, Q)$, $d = d'c$, and (w, F) and $d' \in \Sigma_l$ do not depend on c.*

(2) *if $|F| < l$, then for any $c_1 \in \Sigma_l$ there holds $\alpha((y, T) \otimes (v, Q) \otimes c_1) \in J_{l-1}$.*

Proof. (1) If $|F| = l$, then $|G| = l$. Since G is always obtained from Q, we infer that (z, G) must be (v, Q). Hence d is also of the desired form. The other assertions follow immediately from the definition of the multiplication of two basis elements in $P_n(q)$.

(2) This is trivial since c and c_1 can be considered as two bijections from T to Q. If there is a part of $x \cdot y$ containing more than one elements of T, then we always have $\alpha((y, T) \otimes (v, Q) \otimes c_1) \in J_{l-1}$ for any $c_1 \in \Sigma_l$. The proof is finished.

There is, of course, a dual version of the above lemma, in which the case of $\beta\alpha$ is considered.

By Lemma 4, we may identify E_M with $\bigcup_{l=0}^n \mathcal{S}_l$. Then we have the following fact.

LEMMA 4.6. *$J_t := \sum_{j=0}^t V_j \otimes V_j \otimes k\Sigma_j$ is an ideal of $P_n(q)$.*

This follows from Lemma 2 and Lemma 4. The following lemma is a consequence of definitions and Lemma 4.

LEMMA 4.7. *If $\mu = (x, S) \otimes (y, T) \otimes b$ with $(x, S), (y, T) \in \mathcal{S}_l$ and $b \in \Sigma_l$, then $i(\mu) = (y, T) \otimes (x, S) \otimes b^{-1}$.*

Note that the bilinear form ϕ_l is not symmetric, but we have the following fact.

LEMMA 4.8. *Let* $i: k\Sigma_l \to k\Sigma_l$ *be the involution on* $k\Sigma_l$ *defined by* $\sigma \mapsto \sigma^{-1}$ *for all* $\sigma \in \Sigma_l$. *Then* $i\phi_l(v_1, v_2) = \phi_l(v_2, v_1)$ *for all* $v_1, v_2 \in V_l$.

Proof. We may assume that $v_1 = (x, S)$ and $v_2 = (y, T)$. If $\phi_l(v_1, v_2) = 0$, then it follows from the definition of ϕ_l and $x \cdot y = y \cdot x$ that $\phi_l(v_2, v_1) = 0$. Hence, we assume now that $\phi_l(v_1, v_2) \neq 0$. In this case, if S_i and T_{ib} with $b = p_l(x, S; y, T)$ are contained in the same part of $x \cdot y$, then T_i and $S_{ib^{-1}}$ are contained also in the same part of $y \cdot x$. Thus $p_l(y, T; x, S) = b^{-1}$. This shows that $i\phi_l(v_1, v_2) = \phi_l(v_2, v_1)$. The proof is finished.

Now we are in the position to prove our main result.

Proof of the Theorem. Put $J_{-1} = 0$, $\Sigma_0 = \{1\}$ and $B_l = k\Sigma_l$. Then the partition algebra has a decomposition

$$P_n(q) = V_0 \otimes_k V_0 \otimes_k B_0 \oplus \cdots \oplus V_l \otimes_k V_l \otimes_k \otimes B_l \oplus \cdots \oplus V_n \otimes_k V_n \otimes_k B_n.$$

Note that B_r is a cellular algebra with respect to the involution $\sigma \longmapsto \sigma^{-1}$ for $\sigma \in \Sigma_r$ (see [6]). According to Lemma 4.4 and Lemma 4.5, the chain displayed in the proof of Lemma 3.1 is a chain of ideals in $P_n(q)$. Hence, by the lemmas in this section, the above decomposition satisfies all conditions in Lemma 3.1. Thus, the algebra $P_n(q)$ is a cellular algebra.

We have the following corollary which is proved in [6] by a complicated computation for each algebra; respectively. Here we have a simpler unified proof.

COROLLARY 4.9. (1) *The Brauer algebra* $D_n(q)$ *is cellular*.

(2) *The Temperley–Lieb algbera* $TL_n(q)$ *is cellular*.

Proof. Since the Brauer algebra and the Temperley–Lieb algebra are special subalgebras of $P_n(q)$, we can get a similar cellular structure if we restrict us to the special basis that defines the particular algebra respectively.

From the proof of the theorem, we have the following fact.

COROLLARY 4.10. *The standard modules of* $P_n(q)$ *are* $\Delta_l(\lambda) := V_l \otimes v_l \otimes \Delta(\lambda)$, *where* $l \in \{0, 1, \ldots, n\}$ *and* λ *is a partition of* l, v_l *is a fixed nonzero element of* V_l, *and* $\Delta(\lambda)$ *is a standard module of* $k\Sigma_l$. *For* $l = 0$, *we take* $\lambda = (0)$ *and* $\Delta(0) = k$.

Moreover, we have complete information about the set of simple modules.

COROLLARY 4.11. *Let* $P_n(q)$, $(n > 1)$ *be the partition algebra over a field* k *of characteristic* p *(possibly* $p = 0$*). If* $q \neq 0$ *then the nonisomorphic simple modules are parametrized by* $\{(m, \lambda) \mid 0 \leqslant m \leqslant n, \lambda$ *is a* p*-regular partition of* $m\}$.

In the case of $q = 0$, *the above assertion is also valid except* $m = 0$.

Recall that a partition is p-regular if it does not have p-equal parts ($p \neq 0$); if $p = 0$, then all partitions are p-regular.

Proof. It follows from the above corollary that the simple $P_n(q)$-modules are parametrized by $\{(l, \lambda) \mid \Phi_{(l,\lambda)} \neq 0\}$. If $l \neq 0$, then it follows from 4 and an easy

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computation that $\Phi_{(l,\lambda)} \neq 0$ if and only if the corresponding linear form Φ_λ for the cellular algebra $k\Sigma_l$ is not zero. (Here we use the fact that $\phi_l((x, S), (x, S)) = q^{|x|-l}\,\mathrm{id} \in k\Sigma_l$.) Now it follows from [4] (7.6) that $\Phi_\lambda \neq 0$ if and only if λ is a p-regular partition of l. If $m = 0$, then $\Phi_{(l,\lambda)} \neq 0$ if and only if $q \neq 0$. Hence, the statements follow.

Recall that an ideal J of an algebra A is called a *heredity ideal* if J is idempotent, $J(\mathrm{rad}(A))J = 0$ and J is a projective left (or, right) A-module. An algebra A is called *quasi-hereditary* if there is a finite chain $0 = J_0 \subset J_1 \subset \cdots \subset J_n = A$ of ideals in A such that J_j/J_{j-1} is a heredity ideal in A/J_{j-1} for $1 \leqslant j \leqslant n$. Quasi-hereditary algebras were introduced by Cline, Parshall and Scott to study the highest-weight categories in the representation theory of Lie algebras and algebraic groups (see [3]).

In [12], Martin proved that over a field of characteristc 0 the partition algebra $P_n(q)$ is quasi-hereditary if the parameter q is not zero. More generally, we have the following fact which comes from the above corollary and [6] 3.10.

COROLLARY 4.12. *Suppose that the base field k is of characteristic p, and $q \neq 0$. Then the partition algebra $P_n(q)$ is quasi-hereditary if $p = 0$ or p is bigger than n.*

5. An Example

Let us consider a simple example to illustrate the main result and meantimes to show that in general the partition algebra is not quasi-hereditary.

We take $n = 2$ and $q \in k$. Then the partition algebra $P_2(q)$ is a 15-dimensional algebra over k. The corresponding vector spaces V_j and the bilinear forms ϕ_j can be described as follows

$$V_0 = kv_1 + kv_2, \qquad V_1 = ku_1 + ku_2 + ku_3, \qquad V_2 = k,$$

$$\phi_0 = \begin{pmatrix} q^2 & q \\ q & q \end{pmatrix}, \qquad \phi_1 = \begin{pmatrix} q & 0 & 1 \\ 0 & q & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad \phi_2 = (1)\,.$$

Then $P_2(q) = V_0 \otimes V_0 \otimes k \oplus V_1 \otimes V_1 \otimes k \oplus V_2 \otimes V_2 \otimes k\Sigma_2$ and $\dim_k P_2(q) = 2^2 + 3^2 + 2! = 15$.

If the characteristic of the field is two and $q = 1$, then $J_0 = V_0 \otimes V_0 \otimes k$ is an idempotent ideal of $P_2(1)$. Hence, J_0 is a heredity ideal (see [9]), and the global dimension of $P_2(1)$ is finite if and only if so is the global dimension of the algebra $P_2(1)/J_0$ by [5]. Since ϕ_1 is not singular, we can deduce that J_1/J_0 is also a heredity ideal in $P_2(1)/J_0$. Thus the global dimension of $P_2(1)/J_0$ is finite if and only if the global dimension of $P_2(1)/J_1$ is finite. But we know that $P_2(1)/J_1 \cong k\Sigma_2$ and that the global dimension of $k\Sigma_2$ is infinite. Thus the global dimension of $P_2(1)$ is infinite. Hence $P_2(1)$ is not quasi-hereditary since quasi-hereditary algebras always have finite global dimension.

The Brauer algebra $D_2(q)$ is three-dimensional, and the corresponding datum are:

$$V_0 = kv_2, \quad V_1 = 0, \quad V_2 = k, \quad \phi_0 = (q) \text{ and } \phi_2 = (1).$$

Hence the decomposition of $D_2(q)$ is $D_2(q) = V_0 \otimes V_0 \otimes k \oplus V_2 \otimes V_2 \otimes k\Sigma_2$.

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