Classification of simple Harish–Chandra modules over the Ovsienko–Roger superalgebra

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In this paper, we give a new method to classify all simple cuspidal modules for the
$\mathbb{Z}$-graded and $1/2\mathbb{Z}$-graded Ovsienko–Roger superalgebras. Using this result, we
classify all simple Harish–Chandra modules over some related Lie superalgebras,
including the $N = 1$ BMS$_3$ superalgebra, the super $W(2, 2)$, etc.

Keywords: Virasoro algebra; Ovsienko–Roger superalgebra; Harish–Chandra
modules; Cuspidal modules

2020 *Mathematics Subject Classification:* 17B10; 17B20; 17B65; 17B66; 17B68

1. Introduction

We denote by $\mathbb{Z}, \mathbb{N}, \mathbb{Z}_+, \mathbb{C}$ and $\mathbb{C}^*$ the sets of all integers, non-negative integers,
positive integers, complex numbers and nonzero complex numbers, respectively. All
vector spaces and algebras in this paper are over $\mathbb{C}$. Throughout the paper, we
always assume that all vector superspaces (resp. superalgebras, supermodules) are
defined over $\mathbb{C}$, and sometimes simply call them spaces (resp. algebras, modules).
We use $U(L)$ to denote the universal enveloping algebra for a Lie superalgebra L.

Superconformal algebras may be viewed as natural super-extensions of the Vira-
soro algebra and have been playing a fundamental role in string theory and
conformal field theory. In [**16**], Kac classified all physical superconformal algebras:
namely, the $N = 0$ (the Virasoro algebra Vir), $N = 1$ (the super Virasoro alge-
bras), $N = 2, 3, 4$ superconformal algebras, the superalgebra of all vector fields on

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the $N = 2$ supercircle, and a new superalgebra $CK(6)$. Representation theory for superconformal algebras has been intensively studied. It is a challenging problem to give complete classifications of simple Harish–Chandra modules for superconformal algebras. Based on the classification of simple jet modules introduced by Billig in [4] (see also [25]), Billig and Futorny gave a complete classification of simple Harish–Chandra modules for the Lie algebra of vector fields on a torus with the so-called A cover theory in [5]. Recently, with the study of jet modules, the classifications of such modules were completed for many Lie superalgebras: the $N = 1$ Ramond algebra in [8], the Witt superalgebra in [6, 30], the affine-Virasoro superalgebra in [10, 15], etc. The above Lie superalgebras are all $\mathbb{Z}$-graded. Compared with $\mathbb{Z}$-graded Lie superalgebras, it is more complicated to classify all simple jet modules for $1/2\mathbb{Z}$-graded Lie superalgebras (see [9] for the case of $N = 1$ Neveu–Schwarz algebra). Up to now, there are few papers to classify all simple Harish–Chandra modules over $1/2\mathbb{Z}$-graded Lie superalgebras.

The Ovsienko–Roger Lie algebra $\mathfrak{L}_\lambda := \mathrm{Vir} \ltimes \mathcal{F}_\lambda$ was introduced in [24] to study the extensions of the Virasoro algebra by the density module $\mathcal{F}_\lambda$. In particular, the algebra $\mathfrak{L}_0$, which is called the twisted Heisenberg–Virasoro algebra, plays an important role in moduli spaces of curves [1]. The algebra $\mathfrak{L}_{-1}$, named $W(2,2)$, is better known in the context of vertex operator algebras [31] and BMS/GCA correspondence [2]. Moreover, with the study of the Ovsienko–Roger Lie algebra $\mathfrak{L}_\lambda$, Harish–Chandra modules for many Lie algebras related to the Virasoro algebra have been classified (see [18, 20]).

Motivated by the above researches, we introduce the Ovsienko–Roger Lie superalgebra $\mathfrak{L}(\lambda, \epsilon) := \mathrm{Vir} \ltimes \mathcal{F}_\lambda, \epsilon = 0, 1/2$, where $\mathcal{F}_\lambda$ is viewed as the odd part of $\mathfrak{L}(\lambda, \epsilon)$. In the special case, $\mathfrak{L}(1/2, \epsilon)$, named Kuper algebra, is connected with the super Camassa–Holm-type systems (see [13]). Meanwhile, $\mathfrak{L}(-1/2, \epsilon)$ is a subalgebra or quotient algebra of many Lie superalgebras (see §5).

It is well known that the annihilation operator for cuspidal modules over the Virasoro algebra in [5] (lemma 2.2 below) plays an important role in the classification of simple cuspidal modules. With such annihilation operators for cuspidal modules over the super Virasoro algebras in [8, 9] (lemma 3.1 below), we get an annihilation operator for cuspidal modules over the Ovsienko–Roger superalgebra (lemma 3.2 below) and give a new method to classify all simple Harish–Chandra modules for the $\mathbb{Z}$-graded and $1/2\mathbb{Z}$-graded Ovsienko-Roger superalgebras (theorem 3.4 below). Moreover, this classification can be applied to studying various classes of Lie superalgebras uniformly (see §5). Note that we just do our research for $\lambda = -1/2$ in the whole paper although our calculations and proofs are all suitable for any $\lambda \in \mathbb{C}$.

The paper is organized as follows. In §2, we collect some basic results for our study. Simple cuspidal modules are classified in §3. In §4, we classify all simple Harish–Chandra modules for the Ovsienko–Roger superalgebra. Finally, using this classification, we can classify all simple Harish–Chandra modules over some related Lie superalgebras, including the $N = 1$ BMS$_3$ superalgebra, the super $W(2,2)$ algebra, etc., in §5.

2. Preliminaries

In this section, we collect some basic definitions and results for our study.

By definition, as a vector space, the Virasoro algebra Vir has a basis $\{L_m, C \mid m \in \mathbb{Z}\}$, subject to the following relations:

$$[L_m, L_n] = (n - m)L_{m+n} + \delta_{m+n,0}\frac{1}{12}(n^3 - n)C, \quad \forall m, n \in \mathbb{Z}. \qquad (2.1)$$

For any Vir-module V and $\lambda \in \mathbb{C}$, set $V^\lambda := \{v \in V \mid L_0 v = \lambda v\}$, which is generally called the weight space of V corresponding to the weight λ. A Vir-module V is called a weight module if V is the sum of all its weight spaces.

For a weight module V, we define

$$\mathrm{Supp}(V) := \{\lambda \in \mathbb{C} \mid V^\lambda \neq 0\}. \qquad (2.2)$$

Obviously, if V is a weight Vir-module, then there exists $\lambda \in \mathbb{C}$ such that $\mathrm{Supp}(V) \subset \lambda + \mathbb{Z}$. So $V = \sum_{i \in \mathbb{Z}} V_i$ is $\mathbb{Z}$-graded, where $V_i := V^{\lambda+i}$. A weight Vir-module $V = \sum V_i$ is called *Harish–Chandra* if all V_i are finite-dimensional. If, in addition, there exists a positive integer p such that

$$\dim V_i \leqslant p, \quad \forall i \in \mathbb{Z}, \qquad (2.3)$$

the module V is called *cuspidal*. A cuspidal module V with $p \leqslant 1$ is called *a module of the intermediate series*.

A Vir-module V is called a highest (resp. lowest) weight module, if there exists a nonzero $v \in V_\lambda$ such that

(1) V is generated by v as a Vir-module with $L_0 v = hv$ and $Cv = cv$ for some $h, c \in \mathbb{C}$;

(2) $\mathrm{Vir}_+ v = 0$ (resp. $\mathrm{Vir}_- v = 0$), where $\mathrm{Vir}_+ = \sum_{i>0} \mathrm{Vir}_i$, $\mathrm{Vir}_- = \sum_{i<0} \mathrm{Vir}_i$.

Clearly highest or lowest weight modules are Harish–Chandra modules. All simple Harish–Chandra modules over the Virasoro algebra were classified in [23].

THEOREM 2.1 [23]. *Let V be a simple Harish–Chandra modules over the Virasoro algebra* Vir. *Then V is a highest weight module, lowest weight module or a module of the intermediate series.*

It is well known that the intermediate series Vir-module is

$$\mathcal{A}_{a,\,b} := \sum_{i \in \mathbb{Z}} \mathbb{C}v_i \text{ with } L_m v_i = (a + i + bm)v_{m+i}, Cv_i = 0, \forall m, i \in \mathbb{Z},$$

where $a, b \in \mathbb{C}$. $\mathcal{A}_{a,\,b}$ is simple if and only if $a \notin \mathbb{Z}$ or $b \neq 0, 1$. As usual, we use $\mathcal{A}'_{a,\,b}$ to denote the irreducible sub-quotient module of $\mathcal{A}_{a,\,b}$ (see [17]). If $a \in \mathbb{Z}$, then $\mathcal{A}_{a,\,b} \cong \mathcal{A}_{0,\,b}$. So we always suppose that $a \notin \mathbb{Z}$ or $a = 0$ in $\mathcal{A}_{a,\,b}$. As in [12, 24], we set $\mathcal{F}_\lambda = \mathcal{A}_{0,\,\lambda}, \lambda \in \mathbb{C}$, which is called density module of the Virasoro algebra.

Motivated by [24], for $\epsilon = 0, 1/2$, we can define the Ovsienko–Roger superalgebra $\mathfrak{L}(\epsilon) := \mathrm{Vir} \ltimes \mathcal{F}_{-1/2}$. More precisely, $\mathfrak{L}(\epsilon)$ has a basis $\{L_n, G_r, C | n \in \mathbb{Z}, r \in \mathbb{Z} + \epsilon\}$

with the brackets

$$[L_m, L_n] = (n - m)L_{m+n} + \delta_{m+n,0}\frac{1}{12}(n^3 - n)C,$$

$$[L_m, G_r] = \left(r - \frac{1}{2}m\right)G_{r+m},$$

$$[G_r, G_s] = 0, \quad \forall m, n \in \mathbb{Z}, r, s \in \mathbb{Z} + \epsilon.$$

Here we shall note that the odd part of $\mathfrak{L}(\epsilon)$ is spanned by $\{G_n \mid n \in \mathbb{Z} + \epsilon\}$.

In the case of $\epsilon = 0$, $\mathfrak{L}(0)$ can be realized as the affine-Virasoro superalgebra $\mathbb{C}x \otimes \mathbb{C}[t, t^{-1}] \rtimes \text{Vir}$, where $\mathbb{C}x$ is the one-dimensional abelian Lie superalgebra.

For the Ovsienko–Roger superalgebra, all simple Harish–Chandra modules over $\mathfrak{L}(0)$ were classified in [**10**]. For the case of $\epsilon = 1/2$, it is more complicated to classify such modules as in [**9**]. This paper gives a uniform new method to consider both cases of $\epsilon = 0$ and $\epsilon = 1/2$. For convenience, we just write our research for the case of $\epsilon = 1/2$. So from now on we denote by $\mathfrak{L} = \mathfrak{L}(1/2)$ for short. Clearly, $\mathfrak{L}$ is a $\frac{1}{2}\mathbb{Z}$-graded Lie superalgebra with $\mathfrak{L}_i = \mathbb{C}L_i \oplus \delta_{0,i}\mathbb{C}C, \forall i \in \mathbb{Z}$ and $\mathfrak{L}_{i+1/2} = \mathbb{C}G_{i+\frac{1}{2}}, \forall i \in \mathbb{Z}$. The subalgebra of $\mathfrak{L}$ spanned by $\{L_i, C \mid i \in \mathbb{Z}\}$ is isomorphic to the Virasoro algebra Vir. The following annihilation operator $\Omega_{k,s}^{(m)}$ plays an important role in the classification of simple cuspidal modules over the Virasoro algebra.

LEMMA 2.2 [**5**, Corollary 3.7]. *Let* $\Omega_{k,s}^{(m)} = \sum_{i=0}^{m}(-1)^i\binom{m}{i}L_{k-i}L_{s+i}$. *For every* $\ell \in \mathbb{Z}_+$ *there exists* $m \in \mathbb{Z}_+$ *such that for all* $k, s \in \mathbb{Z}$, $\Omega_{k,s}^{(m)}$ *annihilate every cuspidal* Vir-*module with a composition series of length* ℓ.

3. Simple cuspidal $\mathfrak{L}$-module

In this section, we shall consider cuspidal $\mathfrak{L}$-modules. Note that C acts trivially on any cuspidal module by [**17**]. The following annihilation operator $\overline{\Omega}_{r,s}^{(m)}$ was given in [**9**] (also see [**8**]).

LEMMA 3.1 [**9**]. *Let* V *be a cuspidal* $\mathfrak{L}$-*module. Then there exists* $m \in \mathbb{Z}_+$ *such that for all* $r \in \mathbb{Z} + \frac{1}{2}, s \in \mathbb{Z}$, $\overline{\Omega}_{r,s}^{(m)}$ *annihilate* V, *where* $\overline{\Omega}_{r,s}^{(m)} = \sum_{i=0}^{m}(-1)^i\binom{m}{i}G_{r-i}L_{s+i}$.

By lemma 3.1, we can easily get the following annihilation operator $\underline{\Omega}_{r,s}^{(m)}$ on cuspidal modules over $\mathfrak{L}$.

LEMMA 3.2. *Let* V *be a cuspidal* $\mathfrak{L}$-*module. Then there exists* $m \in \mathbb{Z}_+$ *such that*

$$\underline{\Omega}_{r,s}^{(m)}V = 0, \tag{3.1}$$

where $\underline{\Omega}_{r,s}^{(m)} := \sum_{i=0}^{m}(-1)^i\binom{m}{i}G_{r-i}G_{s+i}$ *for any* $r, s \in \mathbb{Z} + \frac{1}{2}$.

Proof. For the cuspidal module V, by lemma 3.1, there exists $m \in \mathbb{Z}_+$ such that for all $r \in \mathbb{Z} + \frac{1}{2}, s \in \mathbb{Z}, \overline{\Omega}_{r,s}^{(m)} V = 0$, that is

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} G_{r-i} L_{s+i} V = 0, \quad \forall r \in \mathbb{Z} + \frac{1}{2}, s \in \mathbb{Z}. \tag{3.2}$$

By the action of G_t on (3.2), we get

$$\frac{1}{2} \sum_{i=0}^{m} (-1)^i \binom{m}{i} (s + i - 2t) G_{r-i} G_{s+t+i} V = 0, \quad \forall r \in \mathbb{Z} + \frac{1}{2}, s \in \mathbb{Z}. \tag{3.3}$$

Choosing $t = t_1, t_2, t_1 \neq t_2$ in (3.3), we get the lemma. $\qquad\square$

Now we use lemma 3.2 to classify all simple cuspidal modules over $\mathfrak{L}$ without complicated calculations.

LEMMA 3.3. *Let V be a simple cuspidal $\mathfrak{L}$-module. Then there exists $N \in \mathbb{Z}_+$ such that $\mathfrak{L}_{\overline{1}}^N V = 0$.*

Proof. By lemma 3.2, we can get (3.1). Multiply both sides of (3.1) by $G_{s+1} G_{s+2} \cdots G_{s+m}$, $G_{r-j+1} \cdots G_{r-1} G_r G_{s+j+1} \cdots G_{s+m}, 1 \leqslant j \leqslant m$ to get

$$G_r G_s G_{s+1} \cdots G_{s+m} V = 0, \tag{3.4}$$

$$G_{r-j} \cdots G_{r-1} G_r G_{s+j} G_{s+j+1} \cdots G_{s+m} V = 0, \quad \forall 1 \leqslant j \leqslant m. \tag{3.5}$$

Fix some $s \in \mathbb{Z} + 1/2$ and set $\mathcal{O}_n = \{s, s+1, s+2, \cdots, s+n\}$. By (3.4) the following identity

$$G_{r_0} G_{r_1} \cdots G_{r_{m+1}} V = 0 \tag{3.6}$$

holds for all $r_0, r_1, \cdots, r_{m+1} \in \mathcal{O}_{m+1}$.

By (3.4) and (3.5) we see that (3.6) holds for all $r_0, r_1, \cdots, r_{m+1} \in \mathcal{O}_{m+2}$.

We shall use the induction on k to prove that

$$G_{r_0} G_{r_1} \cdots G_{r_{m+1}} V = 0$$

for all $r_0, r_1, \cdots, r_{m+1} \in \mathcal{O}_{m+k}$ for all $k \geqslant 1$. Then, according to the arbitrariness of s, we get the lemma by choosing $N = m + 2$.

Suppose that (3.6) holds for all $r_0, r_1, \cdots, r_{m+1} \in \mathcal{O}_n$ and some $n > m + 1$. Now we shall prove that

$$G_{r_0} G_{r_1} \cdots G_{r_m} G_{s+n+1} V = 0 \tag{3.7}$$

holds for all $r_0 < r_1 < \cdots < r_m \in \mathcal{O}_n$.

Case 1. $r_0 = s + n - m$.

In this case $r_i = s + n - m + i$ for any $i = 1, 2, \ldots, m$. So (3.7) follows from (3.4) directly.

Case 2. $r_0 = s + n - m - k$ for some $1 \leqslant k \leqslant n - m$.

Replacing s, r by $s + n - m + 1, s + n - k$ in (3.1), respectively, we get

$$\left(G_{s+n-k}G_{s+n-m+1} - \binom{m}{1} G_{s+n-k-1}G_{s+n-m+2} \right.$$

$$\left. + \cdots + (-1)^{m-1}\binom{m}{m-1} G_{s+n-k-m+1}G_{s+n} + (-1)^m G_{r_0}G_{s+n+1} \right) V = 0.$$

So we get

$$G_{r_0}G_{s+n+1}V \subset \left(\sum_{r_i, r_j \in \mathcal{O}_n} G_{r_i}G_{r_j} \right) V. \tag{3.8}$$

In this case (3.7) follows by inductive hypothesis. $\qquad\square$

THEOREM 3.4. *Let V be a simple cuspidal $\mathfrak{L}$-module. Then V is isomorphic to the Harish-Chandra module of the intermediate series: $V = \sum v_i \cong \mathcal{A}'_{a,b}$ for some $a, b \in \mathbb{C}$ with $L_m v_i = (a + i + bm)v_{m+i}, G_r v_i = 0$ for all $m, i \in \mathbb{Z}, r \in \mathbb{Z} + \frac{1}{2}$.*

Proof. Clearly $\dim V_i \leqslant p$ for some positive integer p holds for almost $i \in \mathbb{Z}$ and C acts on V as zero (see [**17**]). Now $\mathfrak{L}_{\bar{1}}^i V$ is $\mathfrak{L}$-submodule since $\mathfrak{L}_{\bar{1}}^{i+1} V \subset \mathfrak{L}_{\bar{1}}^i V$ for all $i \in \mathbb{N}$. So $\mathfrak{L}_{\bar{1}} V = V$ or $\mathfrak{L}_{\bar{1}} V = 0$.

By lemma 3.3, we get

$$\mathfrak{L}_{\bar{1}}^N V = 0. \tag{3.9}$$

If $\mathfrak{L}_{\bar{1}} V = V$ then $\mathfrak{L}_{\bar{1}}^N V = V = 0$, which is a contradiction. So $\mathfrak{L}_{\bar{1}} V = 0$ and the proposition follows from theorem 2.1. $\qquad\square$

4. Simple Harish–Chandra module

Now we can classify all simple Harish–Chandra modules over $\mathfrak{L}$. The following result is well-known.

LEMMA 4.1. *Let M be a Harish–Chandra module over the Virasoro algebra with $\mathrm{supp}(M) \subseteq \lambda + \mathbb{Z}$. If for any $v \in M$, there exists $N(v) \in \mathbb{N}$ such that $L_i v = 0, \forall i \geqslant N(v)$, then $\mathrm{supp}(M)$ is upper bounded.*

LEMMA 4.2. *Suppose M is a simple Harish–Chandra module which is not cuspidal over $\mathfrak{L}$, then M is a highest (or lowest) weight module.*

Proof. It is essentially the same as that of lemma 4.2 (1) in [**9**].

Fix $\lambda \in \mathrm{supp}(M)$. Since M is not cuspidal, there exists $k \in \frac{1}{2}\mathbb{Z}$ such that $\dim M_{-k+\lambda} > 2(\dim M_\lambda + M_{\lambda+\frac{1}{2}} + \dim M_{\lambda+1})$. Without loss of generality, we may assume that $k \in \mathbb{N}$. Then there exists a nonzero element $w \in M_{-k+\lambda}$ such that $L_k w = L_{k+1} w = G_{k+\frac{1}{2}} w = 0$. Therefore, $L_i w = G_{i-\frac{1}{2}} w = 0$ for all $i \geqslant k^2$, since $[\mathfrak{L}_i, \mathfrak{L}_j] = \mathfrak{L}_{i+j}$.

It is easy to see that $M' = \{v \in M \,|\, \dim \mathfrak{L}^+ v < \infty\}$ is a nonzero submodule of M, where $\mathfrak{L}^+ = \sum_{n \in \mathbb{Z}_+}(\mathbb{C}L_n + \mathbb{C}G_{n-\frac{1}{2}})$. Hence $M = M'$. So, lemma 4.1 tells us that $\mathrm{supp}(M)$ is upper bounded, that is M is a highest weight module. $\qquad\square$

Combining lemma 4.2 and theorem 3.4, we can get the following result.

THEOREM 4.3. *Any simple Harish–Chandra module over $\mathfrak{L}$ is a highest weight module, lowest weight module or is isomorphic to $\mathcal{A}'_{a,b}$ for some $a, b \in \mathbb{C}$.*

5. Applications

Some Lie superalgebras were constructed in [**28**] as an application of the classification of Balinsky–Novikov super-algebras with dimension 2|2. As applications of the above results, we can classify all Harish–Chandra modules over many Lie superalgebras listed in table 7 in [**28**].

5.1. The Lie superalgebra $\mathfrak{q}$

By definition the Lie superalgebra $\mathfrak{q} = \mathfrak{q}_{\bar{0}} + \mathfrak{q}_{\bar{1}}$, where $\mathfrak{q}_{\bar{0}} := \mathbb{C}\{L_m, H_m, C \mid m \in \mathbb{Z}\}$ and $\mathfrak{q}_{\bar{1}} = \mathbb{C}\{G_p \mid p \in \mathbb{Z} + \frac{1}{2}\}$, is a subalgebra of the $N = 2$ Neveu–Schwarz superconformal algebra, with the following relations:

$$[L_m, L_n] = (n - m)L_{n+m} + \frac{1}{12}(n^3 - n)C,$$

$$[H_m, H_n] = \frac{1}{3}m\delta_{m+n,0}C, \quad [L_m, H_n] = nH_{m+n},$$

$$[L_m, G_p] = \left(p - \frac{m}{2}\right)G_{p+m}, \quad [H_m, G_p] = G_{m+p}, \tag{5.1}$$

$$[G_p, G_q] = 0,$$

for $m, n \in \mathbb{Z}, p, q \in \mathbb{Z} + \frac{1}{2}$.

Clearly $\mathcal{A}_{a,b,c} = \sum_{i \in \mathbb{Z}} \mathbb{C}v_i$ is a $\mathfrak{q}$-module with

$$L_m v_i = (a + bm + i)v_{m+i}, \quad H_m v_i = cv_{m+i}, \ G_r v_i = 0, \forall m, i \in \mathbb{Z}, r \in \mathbb{Z} + \frac{1}{2}.$$

Moreover $\mathcal{A}_{a,b,c}$ is simple if and only if $a \notin \mathbb{Z}$, or $b \neq 0, 1$ or $c \neq 0$. We also use $\mathcal{A}'_{a,b,c}$ to denote the simple sub-quotient of $\mathcal{A}_{a,b,c}$.

PROPOSITION 5.1. *Any simple cuspidal $\mathfrak{q}$-module V is isomorphic to the module $\mathcal{A}'_{a,b,c}$ of the intermediate series for some $a, b, c \in \mathbb{C}$.*

Proof. Clearly, the subalgebra $\mathfrak{q}' := \operatorname{span}\{L_m, G_r, C \mid m \in \mathbb{Z}, r \in \mathbb{Z} + \frac{1}{2}\}$ is isomorphic to $\mathfrak{L}$. By theorem 3.4, we can choose an irreducible $\mathfrak{q}'$-module V' with $G_r V' = 0$ for all $r \in \mathbb{Z} + \frac{1}{2}$. In this case we have $V = \operatorname{Ind}_{\mathfrak{q}'}^{\mathfrak{q}} V'$. Moreover we have $G_r V = 0$ for all $r \in \mathbb{Z} + \frac{1}{2}$ by (5.1). In this case the $\mathfrak{q}$-module V is simple if and only if V is a simple $\mathfrak{q}_{\bar{0}}$-module. So the proposition follows from the main theorem in [**22**]. $\square$

REMARK 5.2. Proposition 5.1 palys a key role in the classification of all simple cuspidal weight modules for the $N = 2$ Neveu–Schwarz superconformal algebra, see [**21**].

5.2. The $N = 1$ BMS$_3$ superalgebra

The Bondi–Metzner–Sachs (BMS$_3$) algebra is the symmetry algebra of asymptotically flat three-dimensional spacetimes [**7**]. It is the semi-direct product of the Virasoro algebra with its adjoint module. The $N = 1$ super-BMS$_3$ is a minimal supersymmetric extension of the BMS$_3$ algebra, which has been introduced to describe the asymptotic structure of the $N = 1$ supergravity in [**3**, **11**].

DEFINITION 5.3. *The N=1 BMS$_3$ superalgebra $\mathcal{B}$ is a Lie superalgebra with a basis $\{L_m, I_m, Q_r, C_1, C_2 \mid m \in \mathbb{Z}, r \in \mathbb{Z} + \frac{1}{2}\}$, with the following commutation relations:*

$$[L_m, L_n] = (n - m)L_{m+n} + \frac{1}{12}\delta_{m+n,0}(n^3 - n)C_1,$$

$$[L_m, I_n] = (n - m)I_{m+n} + \frac{1}{12}\delta_{m+n,0}(n^3 - n)C_2,$$

$$[Q_r, Q_s] = 2I_{r+s} + \frac{1}{3}\delta_{r+s,0}\left(r^2 - \frac{1}{4}\right)C_2,$$

$$[L_m, Q_r] = -\left(\frac{m}{2} - r\right)Q_{m+r},$$

$$[I_m, I_n] = [M_n, Q_r] = 0, \quad [C_1, \mathfrak{g}] = [C_2, \mathfrak{g}] = 0,$$

for any $m, n \in \mathbb{Z}, r, s \in \mathbb{Z} + \frac{1}{2}$.

Note that $\mathcal{B} = \mathcal{B}_{\bar{0}} + \mathcal{B}_{\bar{1}}$, where $\mathcal{B}_{\bar{0}} := \mathbb{C}\{L_m, I_m, C_1, C_2 \mid m \in \mathbb{Z}\}$ and $\mathcal{B}_{\bar{1}} = \mathbb{C}\{Q_p \mid p \in \mathbb{Z} + \frac{1}{2}\}$. The quotient algebra $\mathcal{B}/J$ is isomorphic to $\mathfrak{L}$, where $J = \mathbb{C}\{I_m, C_2 \mid m \in \mathbb{Z}\}$.

Clearly the Vir-module $\mathcal{A}_{a,b}$ can become a $\mathcal{B}$-module with the trivial actions of I_m, Q_r for any $m \in \mathbb{Z}, r \in \mathbb{Z} + \frac{1}{2}$.

PROPOSITION 5.4. *Any simple cuspidal $\mathcal{B}$-module V is isomorphic to the module $\mathcal{A}'_{a,b}$ of the intermediate series for some $a, b \in \mathbb{C}$.*

Proof. Clearly, the subalgebra $\mathcal{B}_{\bar{0}}$ is isomorphic to $W(2,2)$. By theorem 4.6 in [**14**], we can choose an irreducible $\mathcal{B}_{\bar{0}}$-module V' with $I_m V' = C_1 V' = C_2 V' = 0$ for all $m \in \mathbb{Z}$. In this case we have $V = \mathrm{Ind}_{\mathcal{B}_{\bar{0}}}^{\mathcal{B}} V'$. Moreover we have $I_m V = 0$ and $[G_r, G_s]V = 0$ for all $m \in \mathbb{Z}, r, s \in \mathbb{Z} + \frac{1}{2}$ by definition 5.3. In this case the $\mathcal{B}$-module V is simple if and only if V is a simple $\mathcal{B}/J$-module. So the proposition follows from theorem 3.4. $\qquad\square$

5.3. The super $W(2,2)$ algebra

By definition, the super $W(2,2)$ algebra is the Lie superalgebra $SW(2,2) := \mathbb{C}\{L_m, I_m, G_r, Q_r, C_1, C_2 \mid m \in \mathbb{Z}, r \in \mathbb{Z} + \frac{1}{2}\}$ with the following

relations:

$$[L_m, L_n] = (n - m)L_{m+n} + \frac{1}{12}\delta_{m+n,0}(n^3 - n)C_1,$$

$$[L_m, I_n] = (n - m)I_{m+n} + \frac{1}{12}\delta_{m+n,0}(n^3 - n)C_2,$$

$$[G_r, G_s] = -2L_{r+s} + \frac{1}{3}\delta_{r+s,0}\left(r^2 - \frac{1}{4}\right)C_1,$$

$$[G_r, Q_s] = 2I_{r+s} + \frac{1}{3}\delta_{r+s,0}\left(r^2 - \frac{1}{4}\right)C_2,$$

$$[L_m, G_r] = -\left(\frac{m}{2} - r\right)G_{m+r}, \quad [L_m, Q_r] = -\left(\frac{m}{2} - r\right)Q_{m+r},$$

$$[I_m, G_r] = \left(\frac{m}{2} - r\right)Q_{m+r}, \tag{5.2}$$

for any $m, n \in \mathbb{Z}, r, s \in \mathbb{Z} + \frac{1}{2}$.

Note that $SW(2,2) = SW(2,2)_{\bar{0}} + SW(2,2)_{\bar{1}}$, where $SW(2,2)_{\bar{0}} := \mathbb{C}\{L_m, I_m, C_1, C_2 \mid m \in \mathbb{Z}\}$ and $SW(2,2)_{\bar{1}} = \mathbb{C}\{G_p, Q_p \mid p \in \mathbb{Z} + \frac{1}{2}\}$.

Clearly the subalgebra generated by $\{L_m, G_r, C \mid m \in \mathbb{Z}, r \in \mathbb{Z} + \frac{1}{2}\}$ is isomorphic to the $N = 1$ Neveu–Schwarz algebra $\mathfrak{S}$ (see [19]). From [26] we see that $S_{a,b}$ or $\Pi S_{a,b}$ is the Harich–Chandra module of intermediate series over $\mathfrak{S}$ for some $a, b \in \mathbb{C}$, where $S_{a,b}$ is defined as follows:

$$S_{a,b} := \sum_{i \in \mathbb{Z}} \mathbb{C}x_i + \sum_{k \in \mathbb{Z} + \frac{1}{2}} \mathbb{C}y_k \text{ with}$$

$$L_n x_i = (a + bn + i)x_{i+n}, \quad L_n y_k = \left(a + \left(b + \frac{1}{2}\right)n + k\right)y_{k+n},$$

$$G_r x_i = (a + i + 2rb)y_{r+i}, \quad G_r y_k = -x_{r+k},$$

for all $n, i \in \mathbb{Z}, r, k \in \mathbb{Z} + \frac{1}{2}$.

Moreover $S_{a,b}$ is simple if and only if $a \notin \mathbb{Z}$ or $a \in \mathbb{Z}$ and $b \neq 0, \frac{1}{2}$. We also use $S'_{a,b}$ to denote the simple sub-quotient of $S_{a,b}$.

Clearly the $\mathfrak{S}$-modules $S_{a,b}$ and $\Pi S_{a,b}$ become $SW(2,2)$-modules with trivial actions of $I_m, Q_{m+\frac{1}{2}}$ for any $m \in \mathbb{Z}$.

PROPOSITION 5.5. *Any simple cuspidal $SW(2,2)$-module V is isomorphic to $S'_{a,b}$ or $\Pi S'_{a,b}$ for some $a, b \in \mathbb{C}$.*

Proof. Set $\mathfrak{p} = \text{span}\{L_m, I_m, Q_r, C_1, C_2 \mid m \in \mathbb{Z}, r \in \mathbb{Z} + \frac{1}{2}\}$. By proposition 5.4 we can choose a simple $\mathfrak{p}$-module V' with $I_m V' = Q_r V' = C_2 V' = C_2 V' = 0$ for all $m \in \mathbb{Z}, r \in \mathbb{Z} + \frac{1}{2}$. In this case we have $V = \text{Ind}_{\mathfrak{p}}^{SW(2,2)} V'$. Moreover we have $I_m V = Q_r V = 0$ for all $m \in \mathbb{Z}, r \in \mathbb{Z} + \frac{1}{2}$ by definition 5.2. In this case the $SW(2,2)$-module V is simple if and only if V is a simple $\mathfrak{S}$-module. So the proposition follows from the main theorem in [26] (also see theorem 4.5 in [9]). $\square$

REMARK 5.6. We can easily prove that any simple Harish–Chandra module over the above Lie superalgebras is a cuspidal module, or a highest (or lowest) weight module as lemma 4.2. So all simple Harish–Chandra modules over the above Lie superalgebras are also classified.

REMARK 5.7. All indecomposable modules of the intermediate series and some other representations were studied in [**29**] and [**27**].

Acknowledgements

References

1   E. Arbarello, C. De Concini, V. Kac and C. Procesi. Moduli spaces of curves and representation theory. *Commun. Math. Phys.* **117** (1988), 1–36.

2   A. Bagchi. The BMS/GCA correspondence. *Phys. Rev. Lett.* **105** (2010), 171601.

3   G. Barnich, L. Donnay, J. Matulich and R. Troncoso. Asymptotic symmetries and dynamics of three-dimensional flat supergravity. *JHEP.* **08** (2014), 071.

4   Y. Billig. Jet modules. *Canad. J. Math.* **59** (2007), 721–729.

5   Y. Billig and V. Futorny. Classification of irreducible representations of Lie algebra of vector fields on a torus. *J. Reine Angew. Math.* **720** (2016), 199–216.

6   Y. Billig, V. Futorny, K. Iohara and I. Kashuba. Classification of simple strong Harish-Chandra $W(m, n)$-modules. e-print arXiv:2006.05618.

7   H. Bondi, M. G. J. Van der Burg and A. W. K. Metzner. Gravitational waves in general relativity, 7. Waves from axisymmetric isolated systems. *Proc. Roy. Soc. Lond. A.* **269** (1962), 21.

8   Y. Cai, D. Liu and R. Lü. Classification of simple Harish-Chandra modules over the $N = 1$ Ramond algebra. *J. Algebra.* **567** (2021), 114–127.

9   Y. Cai and R. Lü. Classification of simple Harish-Chandra modules over the Neveu-Schwarz algebra and its contact subalgebra. *J. Pure Appl. Algebra.* **226** (2022), 106866.

10   Y. Cai, R. Lü and Y. Wang. Classification of simple Harish-Chandra modules for map (super)algebras related to the Virasoro algebra. *J. Algebra.* **570** (2021), 397–415.

11   M. Dilxat, S. Gao and D. Liu. Whittaker modules for the $N = 1$ super-BMS$_3$ algebra. *J. Algebra Appl.* (2023), doi:10.1142/S0219498824500889.

12   D. B. Fuks. *Cohomology of Infinite-Dimensional Lie Algebras [in Russian]* (Moscow: Nauka, 1984). English transl. Plenum, New York (1986).

13   Y. Ge. Super Camassa-Holm-type systems associated to the Kuper-Ramond-Schwarz superalgebra. *J. Math. Phys.* **61** (2020), 103501.

14   X. Guo, R. Lü and K. Zhao. Simple Harish-Chandra modules, intermediate series modules, and Verma modules over the loop-Virasoro algebra. *Forum Math.* **23** (2011), 1029–1052.

15   Y. He, D. Liu and Y. Wang. Simple Harish-Chandra modules over the super affine-Virasoro algebras. e-print arxiv:2112.07448.

16   V. Kac. Some problems of infinite-dimensional Lie algebras and their representations. *Lecture Notes Math.* **933** (1982), 117–126.

17   I. Kaplansky and L. J. Santharoubane. Harish-Chandra modules over the Virasoro algebra. Infinite-dimensional groups with applications (Berkeley, Calif. 1984), pp. 217–231. Math. Sci. Res. Inst. Publ., 4, Springer, New York, (1985).

18   D. Liu. Classification of Harish-Chandra modules over some Lie algebras related to the Virasoro algebra. *J. Algebra.* **447** (2016), 548–559.

19   D. Liu, Y. Pei and L. Xia. Simple restricted modules for Neveu-Schwarz algebra. *J. Algebra.* **546** (2020), 341–356.

20   D. Liu, Y. Pei and L. Xia. A Category of restricted modules for the Ovisenko-Roger algebra. *Algebr. Represent. Theo.* **25** (2022), 777–791.

21   D. Liu, Y. Pei and L. Xia. Classification of simple weight modules for the $N = 2$ superconformal algebra. e-print arXiv:1904.08578v1.

22   R. Lü and K. Zhao. Classification of irreducible weight modules over the twisted Heisenberg-Virasoro algebra. *Commun. Contemp. Math.* **12** (2010), 183–205.

23   O. Mathieu. Classification of Harish-Chandra modules over the Virasoro Lie algebra. *Invent. Math.* **107** (1992), 225–234.

24   V. Ovsienko and C. Roger. Extension of Virasoro group and Virasoro algebra by modules of tensor densities on S^1. *Funct. Anal. Appl.* **30** (1996), 290–291.

25   S. E. Rao. Partial classification of modules for Lie algebra of diffeomorphisms of d-dimensional torus. *J. Math. Phys.* **45** (2004), 3322–3333.

26   Y. Su. Classification of Harish-Chandra modules over the super-Virasoro algebras. *Commun. Algebra.* **23** (1995), 3653–3675.

27   H. Wang, H. Fa and J. Li. Representations of super W(2,2) algebra $\mathfrak{L}$. e-print arxiv:1705.09452.

28   Y. Wang, Z. Chen and C. Bai. Classification of Balinsky-Novikov superalgebras with dimension 2|2. *J. Phys. A: Math. Theor.* **45** (2012), 225201.

29   Y. Wang, Q. Geng and Z. Chen. The superalgebra of $W(2,2)$ and its modules of the intermediate series. *Commun. Algebra.* **45** (2016), 749–763.

30   Y. Xue and R. Lü. Simple weight modules with finite-dimensional weight spaces over the Witt superalgebras. *J. Algebra.* **574** (2021), 92–116.

31   W. Zhang and C. Dong. W-algebra W(2,2) and the vertex operator algebra $L(1/2,0) \otimes L(1/2,0)$. *Commun. Math. Phys.* **285** (2009), 991–1004.