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A NONABELIAN FROBENIUS–WIELANDT COMPLEMENT

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We recall the following definition (see [**1**]):

A finite group G is said to be a *Frobenius–Wielandt* group provided that there exists a proper subgroup H of G and a proper normal subgroup N of H such that $H \cap H^g \leqq N$ if $g \in G - H$. Then H/N is said to be the *complement* of (G, H, N) (see [**1**] for more details and notation).

An important particular case of this situation is the following:

The finite group H acts faithfully on a vector space V over a finite field and the elements of $H-N$ act fixed point freely on V.

In this case let G be the semidirect product of V by H. Now (G, H, N) is an F–W group. This particular situation was first considered by Lou and Passman in [**2**]. With their notation the group $H/\langle C_H(v) \mid 0 \neq v \in V \rangle$ is said to be the *generalized Frobenius complement*, GFC, in short.

Recently, in his thesis at the University of Chicago, C. M. Scoppola showed that GFCs for abelian-by-cyclic p-groups of odd order are abelian (see Theorem B of [**4**]). As he points out the result is false for $p=2$, the quaternion group being a counterexample.

The purpose of this note is to construct a *metabelian* p-group for p odd having a nonabelian generalized Frobenius complement (in particular, a nonabelian F–W complement). Furthermore our example is an extension of an abelian p-group by the elementary abelian p-group of order p^2. Thus Scoppola's result is best possible in some sense.

Example. Let p be an odd prime. There exists a p-group P with the following properties:

(i) P is an extension of an abelian group A by the elementary abelian group of order p^2, $C_p \times C_p$. In particular, P is metabelian.

(ii) $P/\Omega_1(A)$ is extraspecial of order p^3 and exponent p^2, where $\Omega_1(A) = \langle x \in A \mid x^p = 1 \rangle$.

(iii) There exists a faithful irreducible KP-module V, K being a finite field, such that the elements of $P - \Omega_1(A)$ act f.p.f. on V.

Thus the GFC of P with respect to V is nonabelian.

Proof. We start with the elementary abelian p-group

$$A_0 = \langle a_{i,j} \mid 1 \leqq i \leqq p-1, 1 \leqq j \leqq p, (i,j) \neq (p-1, p) \rangle \times \langle z \rangle \simeq C_p^{p(p-1)-1+1} = C_p^{(p-1)p}.$$

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Consider $\langle x\rangle \simeq C_{p^3}$ acting on A as follows:

$$a_{i,j}^x = a_{i,j+1}, \quad z^x = z,$$

where, by definition, $a_{i,p+1} = a_{i,1}$ if $i < p-1$ and $a_{p-1,p} = a_{p-1,1}^{-1} \dots a_{p-1,p-1}^{-1} z^{-1}$. Observe that the image of $a_{p-1,p}$ under x is $a_{p-1,1}$. Hence, for each i, the group $\langle x\rangle$ permutes the set $\{a_{i,j} \mid 1 \leqq j \leqq p\}$ cyclically. Observe equally that x^p centralizes A_0. Let $\langle x\rangle A_0$ be the natural semidirect product of A_0 by $\langle x\rangle$. Consider $B = \langle x\rangle A_0/\langle z^{-1}x^{p^2}\rangle$. Identify the elements of $\langle x\rangle A_0$ with their images in B.

Take $\langle y\rangle \simeq C_{p^2}$ acting on B as follows:

$$a_{i,j}^y = \begin{cases} a_{i,j}a_{i+1,j} & \text{if } \ i \neq p-1. \\ a_{p-1,j} & \text{if } \ i = p-1. \end{cases}$$

$$x^y = x^{1+p}a_{1,1}, \quad z^y = z.$$

We show that this action is well defined. We must prove that the image of a^y under x^y is equal to the image of a^x under y for all $a \in A_0$. As $\langle x^p, A_0\rangle$ is abelian then the image of a^y under x^y is equal to the image of a^y under x. Thus we will prove that $a^{xy} = a^{yx}$ for all $a \in A_0$. As this is clear for z we may suppose that $a = a_{i,j}$ for some i, j.

If $i \neq p-1$ then $a_{i,j}^{xy} = a_{i,j+1}^y = a_{i,j+1}a_{i+1,j+1}$ and $a_{i,j}^{yx} = (a_{i,j}a_{i+1,j})^x = a_{i,j+1}a_{i+1,j+1}$.

If $i = p-1$ then $a_{p-1,j}^{xy} = a_{p-1,j+1} = a_{p-1,j}^{yx}$.

It is easy to see that y centralizes x^{p^2}. Thus the action of $\langle y\rangle$ on B is well defined. Consider the semidirect product $\langle y\rangle B$ of B by $\langle y\rangle$. We will prove the following relation:

$$u^{y^{p-1}+\dots+y+1} = 1 \quad \text{if} \quad u \in \langle x^p, A_0\rangle. \tag{*}$$

Once (*) is proven it is clear that y^p centralizes $\langle x^p, A_0\rangle$. Furthermore $x^y = xu$ for $u \in \langle x^p, A_0\rangle$ and then $x^{y^p} = xuu^y \dots u^{y^{p-1}} = x$ and y^p is central in $\langle y\rangle B$. Consider $P = \langle y\rangle B/\langle y^p z^{-1}\rangle$. We identify the elements of $\langle y\rangle B$ with their images in P. Let A be the subgroup of P generated by x^p and A_0. Then A is abelian and $A_0 = \Omega_1(A)$. Observe that $Z(P)$ is generated by z.

Assuming that (*) is true we show that if $t \in P - A_0$ then t has a nontrivial power in $Z(P)$. Clearly P/A_0 is extraspecial of order p^3 and exponent p^2. As p is odd then if $t \in P - A\langle y\rangle$ we have that $t^p \in A - A_0$ and t^{p^2} is a nontrivial element of $Z(P)$. If $t \in A - A_0$ then t^p is nontrivial and central. Now (*) assures that $(yu)^p = y^p = z$ if $u \in A$ and thus our claim is verified.

To prove (*) consider y as a linear map of A_0. Then y acts on each $A_j = \langle a_{i,j} \mid 1 \leqq i \leqq p-1\rangle$. As $|A_j| = p^{p-1}$ it is clear that the minimum polynomial of y on A_j divides $(X-1)^{p-1}$. Thus (*) is proven for $u \in A_0$. As A is abelian and it is generated by x^p and A_0 it only remains to check that (*) is true for x^p. But $(x^p)^y = (x^y)^p = (x^{1+p}a_{1,1})^p = x^p z a_{1,1} \dots a_{1,p}$. Put $b_i = \prod_{j=1}^p a_{i,j}$, $1 \leqq i \leqq p-1$. Observe that $b_{p-1} = z^{-1}$ by the definition of $a_{p-1,p}$. Now

$$(x^p)^y = x^p b_1 z, \quad b_1^y = b_1 b_2, \dots, b_{p-2}^y = b_{p-2}z^{-1}.$$

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Put $\bar{b}_1 = b_1 z$ and $\bar{b}_i = b_i$ for $i > 1$. Then we have that $(x^p)^y = x^p \bar{b}_1$ and the remaining relations are valid replacing b_i by $\bar{b}_i$. By induction on i it is easy to see that

$$(x^p)^{y^i} = x^p \bar{b}_1^{\binom{i}{1}} \cdots \bar{b}_i^{\binom{i}{i}}, \quad \text{for } i > 0.$$

Now the exponent of $\bar{b}_j$ in the first member of (*) is $\sum_{i=j}^{p-1} \binom{i}{j}$. This number equals $\binom{p}{j+1}$ and hence it is divisible by p if $j < p-1$. As $\bar{b}_{p-1} = z^{-1}$ and $x^{p^2} = z$ then (*) is valid for x^p also.

Hence our claim is verified and every element of $P - A_0$ has a nontrivial power in $Z(P)$. Let α be a faithful irreducible representation of $Z(P)$ over a finite field K, $\text{char}(K) \neq p$, and put $\rho = \alpha^P$. It is clear that every element of $P - A_0$ acts f.p.f. under ρ since z does. Furthermore, as A_0 is elementary abelian, we have that A_0 is the subgroup of P generated by the elements having nontrivial fixed points under ρ. Thus P/A_0 is a nonabelian GFC for P and the proof is finished.

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