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A BOUND FOR THE CHROMATIC NUMBER OF (P_5, GEM)-FREE GRAPHS

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Abstract

As usual, P_n ($n \geq 1$) denotes the path on n vertices. The gem is the graph consisting of a P_4 together with an additional vertex adjacent to each vertex of the P_4. A graph is called (P_5, gem)-free if it has no induced subgraph isomorphic to a P_5 or to a gem. For a graph G, $\chi(G)$ denotes its chromatic number and $\omega(G)$ denotes the maximum size of a clique in G. We show that $\chi(G) \leq \lfloor \frac{3}{2}\omega(G) \rfloor$ for every (P_5, gem)-free graph G.

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1. Introduction

In this paper, all graphs are finite, simple and undirected.

As usual, given a positive integer n, we denote the path on n vertices by P_n. For an integer $n \geq 3$, C_n is the cycle on n vertices. The gem is the graph consisting of a P_4 together with an additional vertex adjacent to each vertex of the P_4.

Given graphs G and H, we say that G is *H-free* if no induced subgraph of G is isomorphic to H. Given a graph G and a family $\mathcal{H}$ of graphs, we say that G is *$\mathcal{H}$-free* if G is H-free for all $H \in \mathcal{H}$.

A *clique* in a graph G is a set of pairwise adjacent vertices of G; a *stable set* is a set of pairwise nonadjacent vertices of G. The *clique number* of G, denoted by $\omega(G)$, is the maximum size of a clique in G. A *q-colouring* of G is a function $c : V(G) \longrightarrow \{1, \ldots, q\}$, such that for each edge uv of G, $c(u) \neq c(v)$. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number q for which there exists a q-colouring of G. A graph G is *perfect* if all its induced subgraphs H satisfy $\chi(H) = \omega(H)$.

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A class of graphs is called *hereditary* if it is closed under isomorphism and taking induced subgraphs. A hereditary class $\mathcal{G}$ of graphs is said to be *χ-bounded* if there exists a function f such that every graph $G \in \mathcal{G}$ satisfies $\chi(G) \le f(\omega(G))$; the function f is called a *χ-bounding function*. Gyárfás [6] introduced χ-bounded graph classes as a generalisation of perfect graphs.

Gyárfás [6] showed that for all positive integers n, the class of P_n-free graphs is χ-bounded. It is well known that P_4-free graphs are perfect [9], and thus are χ-bounded with identity χ-bounding function. However, for $n \ge 5$, the best χ-bounding function known for the class of P_n-free graphs is exponential: it was shown in [5] that every P_n-free graph G satisfies $\chi(G) \le (n-2)^{\omega(G)-1}$. If a second graph is forbidden in addition to forbidding a path, much better bounds are possible. Choudum, Karthick and Shalu [2] proved that every (P_6, gem)-free graph G satisfies $\chi(G) \le 8\omega(G)$ and that every (P_5, C_4)-free graph G satisfies $\chi(G) \le \lfloor \frac{5}{4}\omega(G) \rfloor$. Gaspers and Huang [4] showed that every (P_6, C_4)-free graph G satisfies $\chi(G) \le \lfloor \frac{3}{2}\omega(G) \rfloor$. This was recently improved by Karthick and Maffray to $\chi(G) \le \lfloor \frac{5}{4}\omega(G) \rfloor$ [7], which is an optimal χ-bounding function for the class. Chudnovsky and Sivaraman [3] proved that every (P_5, C_5)-free graph G satisfies $\chi(G) \le 2^{\omega(G)-1}$.

Choudum, Karthick and Shalu [2] proved that for any (P_5, gem)-free graph G, $\chi(G) \le 4\omega(G)$. In this note, we give a better bound by showing that $\chi(G) \le \lfloor \frac{3}{2}\omega(G) \rfloor$.

2. Definitions

Let $G = (V, E)$ be a graph. We use $|G|$ to denote $|V|$. For $U \subseteq V$, let $G[U]$ denote the subgraph of G *induced* by U. For $v \in V$, let $N(v)$ denote the open neighbourhood of v. The *degree* of v, denoted by $d(v)$, is $|N(v)|$. The *complement* of G is denoted by $\overline{G}$. Let G and H be two vertex-disjoint graphs and let x be a vertex of G. By *substituting H for x* we mean deleting x and joining every vertex of H to each of the vertices that was adjacent to x in G.

A set M of vertices with $2 \le |M| \le |V(G)| - 1$ is a *homogeneous set* in G if for each vertex $x \in V(G) \setminus M$, x is adjacent to all vertices of M or to no vertices of M. A graph that contains no homogeneous set is called *prime*. A homogeneous set M of G is said to be *maximal* if no other homogeneous set properly contains M. The graph G^* obtained from G by contracting every maximal homogeneous set of G to a single vertex is called the *characteristic graph* of G. Note that if G is prime, then $G^* = G$ by the definition.

We say that a graph G' is obtained from a graph G by *blowing up vertices of G into cliques* if G' consists of the disjoint union of cliques K_u, for every $u \in V(G)$, and all edges between cliques K_u and K_v exactly when $uv \in E(G)$. This is the same as substituting clique K_u for vertex u (for all u).

Let A and B be two disjoint sets of vertices of G. We say that A is *complete* to B if every vertex of A is adjacent to every vertex of B and we say that A is *anticomplete* to B if no vertex of A is adjacent to any vertex of B.

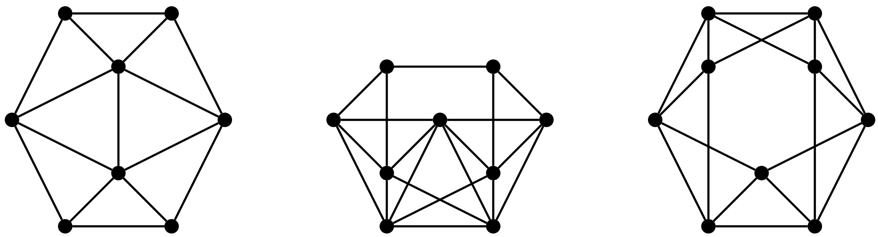

FIGURE 1. A specific graph is a graph shown here or one of its prime induced subgraphs.

A graph is called *co-connected* if its complement is connected. A graph is called *chordal* if it has no induced cycle on four or more vertices, and *co-chordal* if its complement is chordal. A vertex v is *simplical* if the set of vertices adjacent to v induces a clique. A vertex v is *co-simplicial* if the set of vertices not adjacent to v induces a stable set. A graph is said to be *matched co-bipartite* if its vertex set can be partitioned into two cliques C_1 and C_2 with $|C_1| = |C_2|$ or $|C_1| = |C_2| - 1$ such that the edges joining C_1 and C_2 are a matching and at most one vertex in each of C_1 and C_2 is not covered by the matching. Brandstädt and Kratsch [1] called a graph *specific* if it is one of the three graphs in Figure 1 or one of their prime induced subgraphs.

Consider the vertices of C_5 to be ordered v_1, v_2, v_3, v_4, v_5 where v_i is adjacent to v_{i+1} (mod 5). For a graph G and a vertex v of G, let the extension operation $\text{ext}(G, v)$ denote replacing v with a C_5 consisting of new vertices v_1, v_2, v_3, v_4, v_5 such that v_2, v_4 and v_5 have the same neighbourhood in G as v and the only neighbours of v_1 and v_3 are their neighbours in the cycle. For a set of vertices $U \subseteq V$ of G, let $\text{ext}(G, U)$ denote the result of repeatedly applying the extension operation to all vertices of U. For $k \geqslant 0$, let $\mathscr{C}_k$ be the class of prime graphs $G' = \text{ext}(G, Q)$ resulting from extending a co-chordal gem-free graph G by a clique Q of exactly k co-simplicial vertices of G.

3. Previous results

We will use the following known results to prove our result.

THEOREM 3.1 (Brandstädt and Kratsch [1]). *A connected and co-connected graph G is (P_5, gem)-free if and only if the following conditions hold.*

(1) *The homogeneous sets of G are P_4-free.*

(2) *For the characteristic graph G^* of G, one of the following conditions holds:*

  (a) *G^* is a matched co-bipartite graph;*

  (b) *$\overline{G^*}$ is a specific graph;*

  (c) *there is a $k \geqslant 0$ such that G^* is in $\mathscr{C}_k$.*

LEMMA 3.2 (Gaspers and Huang [4]). *Let G be a graph such that each homogeneous set of G is a clique. If the characteristic graph G^* of G satisfies $\chi(G^*) \leq 3$, then $\chi(G) \leq \lfloor \frac{3}{2}\omega(G) \rfloor$.*

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Lemma 3.3 (Lovász [8]). *The graph obtained by substituting perfect graphs for some vertices of a perfect graph is also perfect.*

4. Results

In this section, we prove our main result. First, we prove the following lemma.

Lemma 4.1. *Let G be a connected (P_5, gem)-free graph and H a homogeneous set of G that is not a clique. Then there exists a connected induced subgraph G' of G with $|G'| < |G|$ such that $\chi(G') = \chi(G)$ and $\omega(G') = \omega(G)$.*

Proof. Let N and M be disjoint subsets of $V(G) \setminus H$ such that H is complete to N and anticomplete to M. Note that N is nonempty since G is connected. Since G is gem-free, it follows that $G[H]$ is P_4-free. It has been shown that the class of P_4-free graphs is perfect [9]. Construct G' from G by contracting the vertices of H to a clique K of size $\omega(G[H])$. Clearly G' is a connected induced subgraph of G. Since H is not a clique, it follows that $|G'| < |G|$. We now show that $\chi(G) = \chi(G')$ and $\omega(G) = \omega(G')$. Since G' is an induced subgraph of G, $\omega(G') \leq \omega(G)$ and $\chi(G') \leq \chi(G)$. So we must prove the reverse inequalities.

We first examine $\omega(G)$ and $\omega(G')$. Suppose that a largest clique in G contains a vertex of H. Then a largest clique in G would include a largest clique in H and some vertices in N. This clique would also appear in G', so $\omega(G) \leq \omega(G')$. Now suppose that the largest clique in G contains no vertex of H. Then the largest clique is some subset of $N \cup M$. Since $N \cup M \subseteq V(G')$ it follows that $\omega(G) \leq \omega(G')$. Therefore, $\omega(G) = \omega(G')$.

Next we examine $\chi(G)$ and $\chi(G')$. Colour G' with $q := \chi(G')$ colours. Let $S_1, \ldots, S_q$ be the colour classes. Since K is a clique, we may assume that the ith vertex k_i of K is in S_i for $1 \leq i \leq |K|$. Since $G[H]$ is perfect, $\chi(G[H]) = \omega(G[H]) = |K|$. Let $D_1, \ldots, D_{|K|}$ be a $|K|$-colouring of H. Since H contains K, we may assume that $k_i \in D_i$. Now $S_1 \cup D_1, \ldots, S_{|K|} \cup D_{|K|}, S_{|K|+1}, \ldots, S_q$ is a q-colouring of G. This shows that $\chi(G) \leq \chi(G')$. So, $\chi(G') = \chi(G)$. □

We are now ready to prove the main result of this paper.

Theorem 4.2. *Let G be a (P_5, gem)-free graph. Then $\chi(G) \leq \lfloor \frac{3}{2}\omega(G) \rfloor$.*

Proof. Recall that G^* denotes the characteristic graph of G. We prove the theorem by induction on $|G|$. If G is not connected, then we are done by applying the inductive hypothesis to each component of G. So, we may assume G is connected. If G is not co-connected, then $V(G)$ can be partitioned into two nonempty subsets V_1 and V_2 such that V_1 is complete to V_2. Since G is gem-free, it follows that $G[V_i]$ is P_4-free and so G is also P_4-free. Hence, $\chi(G) = \omega(G)$ and so the theorem holds. So, we may assume G is co-connected. If G contains a homogeneous set that is not a clique, then we are done by Lemma 4.1 and by the inductive hypothesis. So, we can assume that each homogeneous set of G is a clique. This implies that G is obtained from G^* by blowing up vertices of G^* into cliques.

Since G is connected and co-connected, it follows from Theorem 3.1 that G^* must satisfy the following:

(1) G^* is a matched co-bipartite graph;
(2) $\overline{G^*}$ is a specific graph;
(3) there is a $k \geqslant 0$ such that G^* is in $\mathscr{C}_k$.

We now consider each outcome of Theorem 3.1 and prove the claimed bound for each case.

Case 1. Suppose that G^* is a matched co-bipartite graph.

Proof. Let G^* be a matched co-bipartite graph. Co-bipartite graphs are perfect. It follows from Lemma 3.3 that G is also perfect. Thus, $\chi(G) = \omega(G)$. □

Case 2. $\overline{G^*}$ is a specific graph.

Proof. From Lemma 3.2 it is enough to show that G^* is 3-colourable. It can be readily checked that each of the graphs in Figure 1 can be partitioned into 3 cliques. So, their complements are 3-colourable, as are all of their prime induced subgraphs. Thus, $\chi(G) \leq \lfloor \frac{3}{2}\omega(G) \rfloor$. □

Case 3. There is a $k \geqslant 0$ such that G^* is in $\mathscr{C}_k$.

Proof. If $k = 0$, then $G^* \in \mathscr{C}_0$ and so G^* is a prime co-chordal gem-free graph. Co-chordal graphs are perfect. It follows from Lemma 3.3 that G is perfect. Now suppose that $k \geqslant 1$. Then G^* is obtained from some prime co-chordal gem-free graph by applying the extension operation at least once. Let G' be the graph before applying the last extension operation and $G^* = \mathrm{ext}(G', v)$ for some $v \in V(G')$. Note that G^* has the structure illustrated in Figure 2. Then $\{v_1, v_2, v_3, v_4, v_5\}$ induces a C_5 in G^* and v_2, v_4 and v_5 are adjacent to the neighbours of v, and the only neighbours of v_1 and v_3 are their neighbours in the cycle. The degree of v_1 and of v_3 in G^* is 2. Recall that G can be obtained from G^* by blowing up vertices into cliques, and let V_i be the clique that was substituted for v_i for $i = 1, 2, 3, 4, 5$ when G was obtained from G^*. Since $V_4 \cup V_5$ is a clique in G, it follows that $|V_4| + |V_5| \leq \omega(G)$. Thus at least one of V_4 and V_5 has size at most $\frac{1}{2}\omega(G)$, say V_5. (If it is V_4, then apply the following argument with V_1 replaced by V_3.) Also, $V_1 \cup V_2$ has size at most $\omega(G)$. Thus, any vertex $u \in V_1$ has degree at most $\frac{3}{2}\omega(G) - 1$ since it has at most $\frac{1}{2}\omega(G)$ neighbours in V_5 and $\omega(G) - 1$ neighbours in $V_1 \cup V_2$. By the induction hypothesis, $\chi(G - v) \leq \frac{3}{2}\omega(G - v) \leq \frac{3}{2}\omega(G)$. Colour all vertices of G except v with $\lfloor \frac{3}{2}\omega(G) \rfloor$ colours. Since $d(v) \leq \lfloor \frac{3}{2}\omega(G) \rfloor - 1$ there is some colour among the $\lfloor \frac{3}{2}\omega(G) \rfloor$ colours which was not used to colour any neighbour of v. Colour v with this colour. This gives a colouring of G with $\lfloor \frac{3}{2}\omega(G) \rfloor$ colours, and thus shows that $\chi(G) \leq \lfloor \frac{3}{2}\omega(G) \rfloor$. □

Therefore, any (P_5, gem)-free graph G satisfies $\chi(G) \leq \lfloor \frac{3}{2}\omega(G) \rfloor$. □

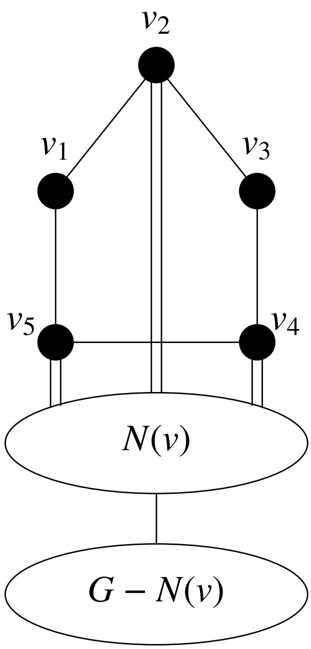

FIGURE 2. The structure of $G^* \in \mathscr{C}_k$ ($k \geqslant 1$) for some extended vertex v.

Note that this bound is tight for general (P_5, gem)-free graphs since the bound is attained by C_5 and the Petersen graph.

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