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On the Cohomology of Moduli of Vector Bundles and the Tamagawa Number of SL_n

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Abstract. We compute some Hodge and Betti numbers of the moduli space of stable rank r, degree d vector bundles on a smooth projective curve. We do not assume r and d are coprime. In the process we equip the cohomology of an arbitrary algebraic stack with a functorial mixed Hodge structure. This Hodge structure is computed in the case of the moduli stack of rank r, degree d vector bundles on a curve. Our methods also yield a formula for the Poincaré polynomial of the moduli stack that is valid over any ground field. In the last section we use the previous sections to give a proof that the Tamagawa number of SL_n is one.

1 Introduction

We will work over a ground field k. Let $\mathfrak{Y}$ be an algebraic stack defined over k. When we speak of its cohomology, we will mean its ℓ-adic cohomology in the smooth topology, except when $k = \mathbb{C}$, in which case we will mean the cohomology of the constant sheaf with values in $\mathbb{Q}$ with the usual topology. These constructions are reviewed in Section 2. We use the generic notation $H^*(\mathfrak{Y})$ for these cohomology theories, and it will be clear from the context what is meant. As we are working over a possibly non algebraically closed field, we remind the reader that the ℓ-adic cohomology is always defined by first passing to an algebraic closure, that is

$$H^i_{sm}(\mathfrak{Y}, \mathbb{Q}_\ell) \stackrel{\text{def}}{=} H^i_{sm}(\mathfrak{Y} \otimes_k \bar{k}, \mathbb{Q}_\ell).$$

The ground field k is detected only in the Galois action on these cohomology groups.

Let X be a smooth, geometrically connected, projective curve defined over k, with genus $g \geq 2$. Fix integers $r > 0$ and d and let $\mathfrak{M}^s_{r,d}$ be the moduli space of rank r and degree d stable vector bundles on this curve. We denote by $\mathrm{Bun}_{r,d}$ the moduli stack of rank r and degree d vector bundles on X. The integers r and d will frequently be omitted from the notation.

In this article, we will calculate the Betti numbers, $\dim H^i(\mathfrak{M}^s)$ (and the Hodge numbers for $k = \mathbb{C}$), when $i < 2(r-1)(g-1)$. For r and d coprime this question has been extensively studied, see [AB82, HN75, BGL94]. On the other hand, when r and d are no longer coprime, the question has remained open and only partial results exist, which we now describe. In rank two, a desingularization $\widetilde{\mathfrak{M}^{ss}}$ of $\mathfrak{M}^{ss}$ has been constructed by C. Seshadri. Its cohomology is studied in [Bal90, Bal93, BKN97]. In [AS01] the Hodge and Betti numbers of $H^i(\mathfrak{M}^s)$ are computed for

$$i < 2(r-1)g - (r-1)(r^2 + 3r + 1) - 7.$$

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Our method is to continue the study of the ind scheme **Div** that was started in [BGL94]. (See Section 3 for the definition of **Div**.) In this frequently cited paper the Poincaré polynomial of this ind scheme and its Shatz strata are computed. We will review this computation in Section 3. In Section 4 we show that the natural map

$$\mathbf{Div} \to \mathrm{Bun} \tag{1}$$

is a quasi-isomorphism. This allows us to compute the Betti numbers of the stack. Over $\mathbb{C}$ this was first done in [AB82]. In this paper the Poincaré polynomial of the classifying space of the gauge group is written down. A simple argument shows that in fact Bun and this classifying space have the same cohomology. In the introduction to [BGL94], the remark was made that **Div** and this classifying space have the same Poincaré polynomial and hence this coincidence is explained by the above isomorphism.

To obtain the Betti numbers of $\mathfrak{M}^s$ we prove a comparison theorem between the cohomology of Bun^s and $\mathfrak{M}^s$, see Section 5. As (1) holds for stable loci, this theorem reduces the study of the cohomology of the coarse moduli space $\mathfrak{M}^s$ to that of the fine moduli space $\mathbf{Div}^s$, where superscript s refers to the stable locus. We are unable to completely describe the cohomology of this ind scheme, so instead we provide an upper bound on the codimension of the complement of $\mathbf{Div}^s$ in **Div**.

Although not completely necessary here, it is desirable to provide a suitable theory of mixed Hodge structures for algebraic stacks. Our first task will be to sketch such a construction. Note that such a construction was first suggested in [Tel98] but has not been published, so it is provided here.

The construction of a functorial mixed Hodge structure on the cohomology of a stack is entirely analogous to that given in [Del74]. Given an algebraic stack $\mathfrak{X}$ and a smooth presentation

$$P \to \mathfrak{X},$$

we can form the simplicial algebraic space whose n-th term is

$$\underbrace{P \times_{\mathfrak{X}} P \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} P}_{n \text{ times}},$$

or in the notation of [Del74]

$$\mathrm{cosk}(P/\mathfrak{X}).$$

Essentially, the method for equipping such an algebraic space with a functorial mixed Hodge structure is given in [Del74], provided it is of finite type. This condition is a hindrance as the stack Bun is not of finite type. To remove this condition we construct

$$Y_\bullet \to \mathrm{cosk}(P/\mathfrak{X}),$$

such that $Y_\bullet$ is a disjoint union of schemes of finite type and the map is of cohomological descent. The finite type assumption is not really essential in [Del74]; what is important is that the cohomology of the stack be finite dimensional.

In the last section we use these results to give an essentially algebraic proof that the Tamagawa number of SL_n is 1 in the function field case. This fact was originally proved by Weil [Wei82]. The calculation here is based on the the Lefschetz trace formula for stacks, [Beh, Beh93, Beh03]. The relationship between this number and the cohomology of moduli spaces of bundles was first observed in [HN75], where the the Weil conjectures and the fact that the Tamagawa of SL_n is 1 are used to calculate Betti numbers in the moduli space in the coprime case. Here, we are reversing this process. The reader will observe that using the moduli stack as opposed to the moduli space simplifies matters considerably.

The interpretation of the Tamagawa number in terms of the Lefschetz trace formula on a moduli stack of torsors is valid for a large class of groups. This idea has been taken up in [Beh06] to prove a relationship between the Tamagawa number and the number of components of the moduli stack of G-torsors.

2 Hodge Theory for Algebraic Stacks

It is not practical to redo the entire contents of [Del71, Del74] here, as the modifications are only minor. We will therefore refer to these works for the bulk of the construction.

We begin with a few remarks regarding stacks and their presentations. If $\mathfrak{X} \to \mathrm{Spec}(k)$ is an algebraic stack with smooth presentation $P \to \mathfrak{X}$, then we can form a groupoid in an algebraic space (see [LMB00, p. 11]) with objects P and $P \times_{\mathfrak{X}} P$ and where the maps are the obvious projections and diagonals. The stack $\mathfrak{X}$ can be recovered from this groupoid via the construction $[-]$ in [LMB00, p. 17]. When $k = \mathbb{C}$, then P and $P \times_{\mathfrak{X}} P$ have underlying topological spaces so we may pass to a groupoid in topological spaces. The construction $[-]$ applied to this groupoid yields a topological stack that does not depend on the choice of presentation. This is called the underlying topological stack $\mathfrak{X}$ and is denoted $\mathfrak{X}^{\mathrm{top}}$

We now recall the definition of the cohomology of an algebraic stack $\mathfrak{X} \to \mathrm{Spec}(k)$. The stack $\mathfrak{X}$ is a category fibered over **schemes**$/k$. This second category has a smooth topology, so we define an arrow to be a cover if its image in **schemes**$/k$ is. This allows us to consider the ℓ-adic cohomology in the smooth topology on $\mathfrak{X}$. For details, see [Beh03] or [LMB00]. When $k = \mathbb{C}$, we may pass to the underlying topological stack

$$\mathfrak{X}^{\mathrm{top}} \to \mathbf{top}.$$

Similarly, one may define a Grothendieck topology on this stack by use of the big site on **top**. Given a coefficient ring F we denote by $\mathrm{H}^*(\mathfrak{X}, F)$, the cohomology of the constant sheaf with values in F on this site. (We remind the reader of our conventions, stated at the beginning of the article, for when $F = \mathbb{Q}$.) A good introduction to the cohomology of stacks can be found in Kai Behrend's talk at MSRI [Beh02].

These definitions are not completely necessary here, as we will be replacing our stack by a simplicial space and the cohomology of this simplicial space will be the same as that of the stack.

General references for simplicial objects and cohomological descent are [SD72, Del74]. For a simplicial object, denote by sk_{n} the n-th truncation functor and by cosk_n its right adjoint. Fix a locally finite stack $\mathfrak{X}$ over k and a smooth presentation $\alpha\colon P \to \mathfrak{X}$.

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Proposition 2.1

(i) *The map α is of universal cohomological descent for the smooth topology.*

(ii) *The map*

$$\alpha^{\mathrm{top}}\colon P^{\mathrm{top}} \to \mathfrak{X}^{\mathrm{top}}$$

is of universal cohomological descent for the usual topology.

Proof The proof of (i) can be found in [Beh03]. We give a sketch only of (ii) and leave the details to the reader. Recall that $\mathfrak{X}^{\mathrm{top}} = [P_\bullet^{\mathrm{top}}]$, where $P_\bullet^{\mathrm{top}}$ is the topological space in groupoids defined by

$$(P \times_{\mathfrak{X}} P)^{\mathrm{top}} \underset{t}{\overset{s}{\rightrightarrows}} P^{\mathrm{top}}.$$

Both the arrows s and t admit sections locally on X^{top} as they map underlying smooth morphisms of algebraic spaces. Using this fact, one shows that for every topological space T and every $T \to \mathfrak{X}^{\mathrm{top}}$ the map

$$T \times_{\mathfrak{X}^{\mathrm{top}}} P^{\mathrm{top}} \to T$$

admits sections locally on T. Now the result follows as the question is local on the base $\mathfrak{X}^{\mathrm{top}}$. ■

Corollary 2.2 *The natural augmentation map* $\operatorname{cosk}(P/\mathfrak{X}) \to \mathfrak{X}$ *induces an isomorphism* $\mathrm{H}^i(\operatorname{cosk}(P/\mathfrak{X})) \xrightarrow{\sim} \mathrm{H}^i(\mathfrak{X})$.

It is worth noting that the following spectral sequence relates the cohomology of the components of $\operatorname{cosk}(P/\mathfrak{X})$ to that of $\mathfrak{X}$.

Proposition 2.3 *Let $Z_\bullet$ be a simplicial space. Then there is a spectral sequence with* $E_1^{pq} = \mathrm{H}^q(Z_p)$ *abutting to* $\mathrm{H}^{p+q}(Z_\bullet)$.

Proof See [SD72]. ■

For the remainder of this section we will take $k = \mathbb{C}$. Let **lfschemes**$/\mathbb{C}$ be the full subcategory of **schemes**$/\mathbb{C}$ consisting of schemes that are separated and are disjoint unions of schemes of finite type over $\mathbb{C}$. Let $\mathbf{lfss}_k$ be the category of k-truncated simplicial objects in **lfschemes**$/\mathbb{C}$. Our next task is to construct a smooth simplicial scheme $Y_\bullet$ in $\mathbf{lfss}_\infty$ with a map $Y_\bullet \to \operatorname{cosk}(P/\mathfrak{X})$ that is a hypercover. First let us recall the standard method for construction of hypercovers.

In what follows, a simplicial space could mean simplicial scheme, simplicial algebraic space or a simplicial topological space.

Consider an m-truncated simplicial space $X_\bullet$ augmented towards a stack $\mathfrak{S}$, *i.e.,* $a\colon X_\bullet \to \mathfrak{S}$. Recall that a is called a *hypercover* if the canonical maps deduced from adjunction

$$X_{n+1} \to (\mathrm{cosk\,skX_\bullet})_{\mathrm{n+1}} \quad \text{for } -1 \leq \mathrm{n} \leq \mathrm{m}-1,$$

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are of universal cohomological descent. This definition makes sense for $m = \infty$. Recall the following ([SD72, 3.3.3]):

Theorem 2.4 *If $a\colon X_\bullet \to \mathfrak{S}$ is a hypercover as above, then the natural map*

$$\operatorname{cosk}(X_\bullet/\mathfrak{S}) \to \mathfrak{S}$$

is of universal cohomological descent.

We describe below the main method for constructing hypercovers. A k-truncated simplicial space $X_\bullet$ is said to be *split* if there exists for each j, $k \geq j \geq 0$, a subobject NX_j of X_j such that the morphisms

$$\coprod s\colon \coprod_{i \leq n} \coprod_{s \in \operatorname{Hom}(\Delta_n, \Delta_i)} N(X_i) \to X_n$$

are isomorphisms, for $n \leq k$. This definition makes sense for $k = \infty$.

Let $X_\bullet$ be a split k-truncated simplicial space with k a finite number. We denote by $\alpha(X_\bullet)$ the triple (X', N, β), where

(i) X' is the $(k-1)$-truncated simplicial space obtained by restricting $X_\bullet$;
(ii) $N = NX_k$;
(iii) β is the canonical map $\beta\colon NX_k \to (\operatorname{cosk}_{k-1} \operatorname{sk}_{k-1}(X_\bullet))_k$.

The triple $\alpha(X) = (X', N, \beta)$ satisfies the following condition

(S) X' is a $(k-1)$-truncated split simplicial space and β is a map $\beta\colon N \to (\operatorname{cosk}_{k-1} X')_k$.

Proposition 2.5

(i) *Let (X', N, β) be a triple satisfying* (S). *Up to isomorphism, there exists a unique split k-truncated $X_\bullet$ with $\alpha(X) \cong (X', N, \beta)$.*
(ii) *In the setup of the previous part suppose Z is a k-truncated simplicial space. To give a map $f\colon X \to Z$ is the same as giving the following data:*
   (a) *a map $f'\colon X' \to \mathrm{sk}_{k-1}(Z)$,*
   (b) *a map $f''\colon N \to Z_k$ such that the following diagram commutes:*

$$\begin{array}{ccc} N & \longrightarrow & (\operatorname{cosk} X')_k \\ \downarrow & & \downarrow \\ Z_k & \longrightarrow & (\operatorname{cosk} \mathrm{sk}_{k-1} Z)_k. \end{array}$$

Proof This is Proposition 5.1.3 of [SD72]. ∎

Now recall our setup from earlier in this section: we had a stack $\mathfrak{X}$ and a smooth presentation $P \to \mathfrak{X}$. We construct our hypercover $Y_\bullet$ of $\operatorname{cosk}(P/\mathfrak{X})$ inductively as follows:

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$k = 0$: Let $P \to \mathfrak{X}$ be a presentation. We may assume that P is a scheme by replacing the algebraic space P by a presentation. As X is locally of finite type, we can assume that P is in **lfschemes**/$\mathbb{C}$, by replacing P by an open cover of P. We then take $Y^0_\bullet$ to be a resolution of singularities of P. We view $Y^0_\bullet$ as a 0-truncated simplicial space. Note that a smooth morphism locally admits sections and a resolution of singularities is proper and surjective so $Y^0_\bullet \to \mathfrak{X}$ is a hypercover.

$k = 1$: Let $Z_1 = (\operatorname{cosk}(Y^0_\bullet/\mathfrak{X}))_1$. We replace Z_1 by an open affine cover and then take a resolution of singularities of this cover to obtain a smooth scheme N_1 in **lfschemes**/$\mathbb{C}$, and a map $\beta\colon N_1 \to Z_1$. Apply Proposition 2.5 to the triple $(Y^0_\bullet, N_1, \beta)$ to obtain a smooth 1-truncated split simplicial scheme $Y^1_\bullet$.

$k > 1$: Inductively one produces for each k a split k-truncated simplicial scheme $Y^k_\bullet$ and an augmentation $Y^k_\bullet \to \mathfrak{X}$ such that

(1) The augmentation is a hypercover.
(2) Y^k_i is in **lfschemes**/$\mathbb{C}$.
(3) Y^k_i is smooth over $\mathbb{C}$.
(4) $\mathrm{sk_{k-1}(Y^k_\bullet) = Y^{k-1}_\bullet}$.

Condition (4) means that $Y^i_i = Y^{i+1}_i = \cdots$. We define Y^∞_i to be this stable value of Y^*_i. The Y^∞_i fit together to form a simplicial scheme that is in fact our required hypercover $Y_\bullet = Y^\infty_\bullet \to \mathfrak{X}$.

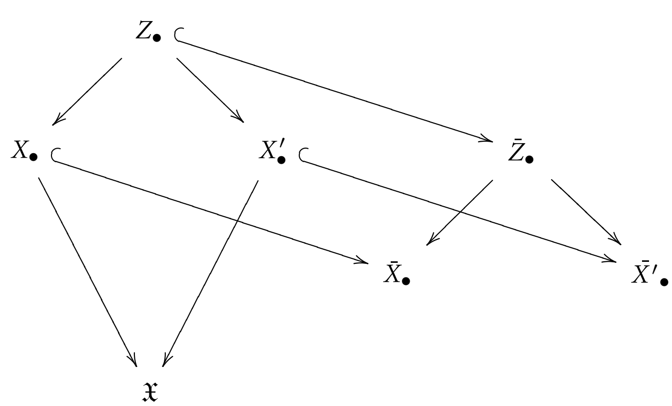
A *compactification* of a simplicial scheme $X_\bullet$ is a simplicial scheme $\bar{X}_\bullet$ and a morphism $j\colon X_\bullet \hookrightarrow \bar{X}_\bullet$ such that each of the maps j_n are compactifications.

A *divisor* $D_\bullet$ on a smooth simplicial scheme $X_\bullet$ is a closed simplicial subscheme $D_\bullet \hookrightarrow X_\bullet$ such that each of the morphisms $D_n \hookrightarrow X_n$ is a divisor. We say that $D_\bullet$ has *simple normal crossings* if each of the D_n do.

Theorem 2.6 *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be algebraic stacks locally of finite type.*

(i) *We can construct a hypercover $X_\bullet \to \mathfrak{X}$ with $X_\bullet$ smooth and a smooth compactification $\bar{X}_\bullet$ of $X_\bullet$ such that $\bar{X}_\bullet \setminus X_\bullet$ is a divisor with simple normal crossings and both of these simplicial schemes are in* **lfss**$_\infty$.

(ii) *If we have two such hypercover-compactification pairs $(X_\bullet, \bar{X}_\bullet)$ and $(X'_\bullet, \bar{X}'_\bullet)$ we can find a third pair $(Z_\bullet, \bar{Z}_\bullet)$ that satisfies the conditions of* (i) *and fits into a diagram*



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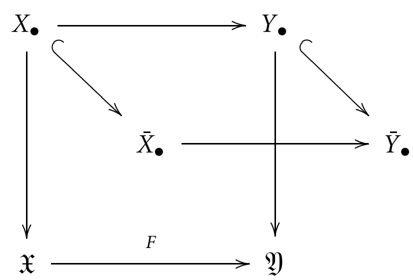
(iii) *Let* $F\colon \mathfrak{X} \to \mathfrak{Y}$ *be a morphism. Then there exists hypercover-compactification pairs* $(X_\bullet, \bar{X}_\bullet)$ *and* $(Y_\bullet, \bar{Y}_\bullet)$*as in* (i) *for* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *respectively, along with morphisms*

$$X_\bullet \to Y_\bullet \quad \bar{X}_\bullet \to \bar{Y}_\bullet$$

and a commutative diagram



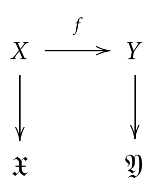
Proof The proofs are analogous to those in [Del71]. For the convenience of the reader we outline some of the proofs. (i) If X is a scheme that is a disjoint union of smooth, separated, finite type schemes over $\mathbb{C}$, we may find a compactification of it by [Nag62]. We may assume by [Hir64] that this compactification, $\bar{X}$, is smooth and $\bar{X} \setminus X$ is a simple normal crossings divisor. The result will know follow from the ideas in the discussion above.

(ii) The proof of this result is similar to that of (iii) so we only give the proof of (iii).

(iii) Let $Y \to \mathfrak{Y}$ be a presentation of $\mathfrak{Y}$. We may assume that Y is a disjoint union of separated schemes of finite type over $\mathbb{C}$. The stack $\mathfrak{X} \times_{\mathfrak{Y}} Y$ is algebraic and

$$\mathfrak{X} \times_{\mathfrak{Y}} Y \to \mathfrak{X}$$

is a representable surjective and smooth morphism. So a presentation for this stack gives a presentation for $\mathfrak{X}$ by composition. We obtain a diagram



where the two vertical arrows are of universal cohomological descent and X and Y are in **lfschemes**/$\mathbb{C}$. We may further assume that X and Y are smooth. To do this, first resolve Y to Y' and then resolve $X \times_Y Y'$ and note that the projection $X \times_Y Y' \to X$ is of universal cohomological descent.

We claim that there are smooth compactifications of X and Y denoted $\bar{X}$ and $\bar{Y}$, respectively, such that f extends to a morphism $\bar{f}\colon \bar{X} \to \bar{Y}$ and

$$\bar{X} \setminus X, \quad \bar{Y} \setminus Y$$

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are simple normal crossings divisors. To do this choose any compactifications $\bar{Y}$ of Y and $\bar{X}'$ of X. Let $\overline{\Gamma_f} \subseteq \bar{X}' \times \bar{Y}$ be the closure of the graph of f. It is compact, and after applying [Hir64] to it we may assume that in addition the complement of the inclusion $X \subseteq \overline{\Gamma_f}$ has simple normal crossings. We take $\bar{X} = \overline{\Gamma_f}$, and this proves the claim.

We take $X_0 = X$, $Y_0 = Y$, $\bar{X}_0 = \bar{X}$ and $\bar{Y}_0 = \bar{Y}$. To construct the next level of the required simplicial schemes form a diagram

$$\begin{array}{ccc} N' & \xrightarrow{f_1} & N \\ \big\downarrow{\scriptstyle p'} & & \big\downarrow{\scriptstyle p} \\ \operatorname{cosk}(X/\mathfrak{X})_1 & \longrightarrow & \operatorname{cosk}(Y/\mathfrak{Y})_1, \end{array}$$

where N and N' are smooth schemes in **lfschemes**$/\mathbb{C}$ and the vertical arrows are of universal cohomological descent. We may compactify N and N' as above, so that f_1 extends to a morphism on the compactifications. Now apply Proposition 2.5 as in the discussion preceding this theorem. One continues by induction and the required diagram is constructed. ■

Consider the category whose objects are pairs $(X_\bullet, \bar{X}_\bullet,)$ where $X_\bullet$ and $\bar{X}_\bullet$ are smooth simplicial schemes in $\mathbf{lfss}_\infty$ and $\bar{X}_\bullet$ is a compactification of $X_\bullet$ with simple normal crossings on the boundary. We will now construct a functor from this category to $\mathbb{Q}$-mixed Hodge structures. The underlying vector space of this mixed Hodge structure will be $\mathrm{H}^*(X_\bullet, \mathbb{Q})$.

Once this functor is constructed, Theorem 2.6 will show that a stack $\mathfrak{X}$ has a canonical functorial mixed Hodge structure. Note that a morphism of mixed Hodge structures that is an isomorphism on underlying vector spaces is in fact an isomorphism of mixed Hodge structures, so (ii) shows that the construction is independent of the choice of hypercover-compactification. Functoriality follows from (iii).

There is one *very* minor complication here. As $\mathrm{H}^i(X_\bullet, \mathbb{Q})$ may not be of finite type, we may not directly apply [Del71, Del74]. However, we claim that once the definitions of these papers are relaxed as outlined below, the results of these papers still hold.

An *infinite $\mathbb{Q}$-Hodge structure of weight n* is a $\mathbb{Q}$-vector space V and a finite decreasing filtration F on $V \otimes_{\mathbb{Q}} \mathbb{C} = V_{\mathbb{C}}$ such that the filtrations F and $\bar{F}$ are n-opposed, that is

$$\mathrm{Gr}_F^p \mathrm{Gr}_{\bar{F}}^q (V_{\mathbb{C}}) = 0$$

for $p + q \neq n$. We do not require that V be finite dimensional.

An *infinite $\mathbb{Q}$-mixed Hodge structure* consists of the following data:

(i) a $\mathbb{Q}$-module V,
(ii) a finite increasing filtration W on V, called the weight filtration,
(iii) a finite decreasing filtration F on $V \otimes_{\mathbb{Q}} \mathbb{C} = V_{\mathbb{C}}$ called the Hodge filtration,

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This data is required to satisfy the following axiom: F induces a weight n infinite Hodge structure on $\mathrm{Gr}_n^W(V)$.

A morphism $f\colon V \to V'$ of infinite mixed Hodge structures is a map of Abelian groups that induces maps that are compatible with the filtrations.

A weight *n infinite Hodge complex* consists of

(α) A complex $K^\bullet$ of $\mathbb{Q}$-modules.
(β) A filtered complex $(K^\bullet_{\mathbb{C}}, F)$ in $D^+F(\mathbb{C})$ and an isomorphism

$$K^\bullet \otimes \mathbb{C} \xrightarrow{\sim} K^\bullet_C \quad \text{in } D^+(\mathbb{C}).$$

This data is required to satisfy the following axiom: For all k, the filtration on $\mathrm{H}^k(K^\bullet_{\mathbb{C}})$ induced by F, defines a weight $n+k$ infinite Hodge structure.

In the above $D^+F(\mathbb{C})$ is the filtered derived category as defined in [Del74]. In particular the filtration F is biregular, that is it a finite filtration on each component of the complex $K^\bullet_{\mathbb{C}}$.

An *infinite mixed Hodge complex* consists of

(α) A filtered complex (K, W) of $\mathbb{Q}$-vector spaces in $D^+F(\mathbb{Q})$.
(β) A bifiltered complex $(K^\bullet_{\mathbb{C}}, W, F)$ a complex of $\mathbb{C}$ vector spaces, W an increasing biregular filtration, F a decreasing biregular filtration and an isomorphism

$$\mathbb{C} \otimes_{\mathbb{Q}} K^\bullet \xrightarrow{\sim} K^\bullet_{\mathbb{C}} \quad \text{in } D^+F(\mathbb{C}).$$

This data is required to satisfy the following axiom: The data consisting of the complex $\mathrm{Gr}_n^W K^\bullet_{\mathbb{Q}}$ and the quasi isomorphism

$$\mathrm{Gr}_n^W K^\bullet \otimes \mathbb{C} \xrightarrow{\sim} \mathrm{Gr}_n^W K^\bullet_{\mathbb{C}}$$

is a weight n infinite Hodge complex.

We will now proceed to show that the cohomology of an infinite mixed Hodge complex inherits a canonical infinite mixed Hodge structure. We first need to recall some facts from [Del71].

Let $(K^\bullet, W, F)$ be a bifiltered complex. On the terms $E_r^{pq}(K^\bullet, W)$ of the spectral sequence associated to the filtered complex $(K^\bullet, W)$, we have three filtrations induced by F:

(i) *The first direct filtration*, F_d, is formed by viewing E_r^{pq} as a quotient of a subobject of K^{p+q}.
(ii) *The second direct filtration*, F_{d^*}, is formed by viewing E_r^{pq} as a subobject of a quotient object of K^{p+q}.
(iii) *The recursive filtration*, F_r, is formed by defining,

$$\text{on } E_0^{pq}, \quad F_r = F_d = F_{d*} \text{ (see below)},$$
$$\text{on } E_r^{pq}, \quad F_r = \text{ the filtration induced by the direct filtration on } E_{r-1}^{pq}.$$

Proposition 2.7

(i) *On E_0 and E_1 the three filtrations coincide.*

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(ii) *The differentials d_r are compatible with F_d and F_{d^*}.*
(iii) $F_d \subseteq F_r \subseteq F_{d^*}$.

Proof See [Del71, p. 17]. ■

Theorem 2.8 *Let $(K^\bullet, W, F)$ be a bifiltered complex. We let $E_r^{pq} = E_r^{pq}(K^\bullet, W)$ be the terms of the spectral sequence. Suppose that F is biregular and for $0 \le r \le r_0$ the differentials d_r are strictly compatible with F_r. Then on E_{r_0+1} we have $F_d = F_r = F_{d*}$.*

Proof See [Del71, p. 18]. ■

Given a complex $K^\bullet$ with an increasing filtration W, we define a new shifted filtration $\operatorname{Dec} W$ on $K^\bullet$ by $\operatorname{Dec} W_n K^i = W_{n-i} K^i$.

Theorem 2.9 *Assume $(K^\bullet, W, \alpha, K^\bullet_{\mathbb{C}}, F)$ is an infinite mixed Hodge complex. Then* $\operatorname{Dec}(W)$ *and F induce a mixed Hodge structure on* $\mathrm{H}^i(K^\bullet)$.

Proof Consider the decreasing filtration $\widetilde{W}$ on K defined by $\widetilde{W}^p = W_{-p}$. This filtration gives a spectral sequence with $E_1^{pq} = \mathrm{H}^{p+q}(\mathrm{Gr}^W_{-p}(K))$, abutting to $\mathrm{H}^{p+q}(K)$. By Proposition 2.7 the three filtrations on E_1^{pq} coincide and the differential is compatible with this filtration. As d_1 is defined over $\mathbb{Q}$, this differential is compatible with the conjugate filtration and therefore is strictly compatible with the filtration. So $d_1 : E_1^{pq} \to E_1^{p+1,q}$ is a morphism of Hodge structures of weight q.

Hence E_2^{pq} has a weight q Hodge structure. By Theorem 2.8 the three filtrations coincide on E_2 and d_2 is compatible with it. As before, we conclude that d_2 is strictly compatible with this filtration. However, $d_2 : E_2^{pq} \to E_2^{p+2,q-1}$ is a morphism of Hodge structure of different weights so it vanishes. Hence $E_2^{pq} = E_\infty^{pq}$ and so $\mathrm{Gr}^W_{-p}\mathrm{H}^{p+q}(K)$ has a weight q Hodge structure. One checks that $\mathrm{Gr}_q^{\mathrm{Dec}}\mathrm{H}^{p+q}(K) = \mathrm{Gr}^W_{-p}\mathrm{H}^{p+q}(K)$ and we are done. ■

One can now proceed to define infinite complexes of sheaves as in [Del74, pp. 28–38]. The results will carry over verbatim to this setting. In particular, the analogue of Proposition 8.1.20 [Del74] constructs a functorial mixed Hodge structure on the cohomology of a hypercover-compactification pair.

3 The Cohomology of the Ind Scheme of Matrix Divisors

For the remainder of this paper, X is a smooth geometrically connected projective curve defined over our ground field k.

The primary purpose of this section is to recall the results in [BGL94] regarding the cohomology of **Div** and provide a bound on the codimension of the complement $\mathbf{Div}^{ss} \setminus \mathbf{Div}^{s}$.

Let Λ be the partially ordered set of effective divisors on X. Fix $D \in \Lambda$ and consider the functor

$$\mathrm{Div}^{r,d}(D)^\flat : \mathbf{schemes}/k \to \mathbf{sets}$$

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whose S-points are equivalence classes of inclusions $\mathcal{F}\hookrightarrow\mathcal{O}_{X\times S}(D)^r$, where $\mathcal{F}$ is a family of rank r degree d bundles on $X\times S$. This functor is representable by a Quot scheme that we denote by $\mathrm{Div}^{r,d}(D) = \mathrm{Div}(D)$. These Quot schemes fit together to form an ind scheme denoted by $\mathbf{Div}^{r,d} = \mathbf{Div}$.

Let $\mathbf{m} = (m_1, m_2, \dots, m_r)$ be a partition of the integer $r.\deg D - n$, by non negative integers. Then the product of Hilbert schemes of points

$$H^{\mathbf{m}} = \mathrm{Hilb}(m_1, C) \times \mathrm{Hilb}(m_2, C) \times \cdots \times \mathrm{Hilb}(m_r, C)$$

sits canonically inside of $\mathrm{Div}(D)$. Recall that over an algebraically closed field, the Hilbert scheme of points of a smooth curve is just a symmetric power of the curve.

The torus $\mathbb{G}_m^r$ acts on $\mathrm{Div}(D)$ and the above products of Hilbert schemes are clearly fixed by this action. The converse is also true.

Theorem 3.1

(i) *The fixed points of this action are precisely the schemes $H^{\mathbf{m}}$ as $\mathbf{m}$ varies over all partitions of $r.\deg D - n$.*

(ii) *The cohomology of* **Div** *stabilizes and its Poincaré polynomial is given by*

$$P(\mathbf{Div}; t) = \frac{\prod_{i=1}^{r}(1 + t^{2j-1})^{2g}}{(1 - t^{2r})\prod_{i=1}^{r-1}(1 - t^{2j})^2}.$$

The fact that the cohomology stabilizes means that the inverse limit

$$\varprojlim_{\Lambda} \mathrm{H}^i(\mathrm{Div}(D), \mathbb{Q})$$

is in fact finite.

(iii) *When $k = \mathbb{C}$, the Hodge–Poincaré polynomial of* **Div** *is*

$$P_H(\mathbf{Div}; x, y) = \frac{(1 + x)^g(1 + y)^g}{(1 - x^r y^r)} \prod_{i=1}^{r-1} \frac{(1 + x^{i+1} y)^g(1 + xy^{i+1})^g}{(1 - x^i y^i)^2}.$$

Proof The first part is proved in [Bif89]. The second part follows from the first by some theorems of A. Białynicki-Birula and some deformation theory. For details see [BB73, BB74] and [BGL94, Proposition 4.2]. The last part follows by noting that the Białynicki-Birula decomposition is compatible with, among other things, Hodge theory. A nice exposition of these ideas can be found in [dB01]. The formula we have written down follows directly from Proposition 4.4 of that paper. ∎

For a vector bundle $\mathcal{E}$ on X with rank r and degree d, its Harder–Narasimhan

$$\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \cdots \subseteq \mathcal{E}_l = E.$$

filtration is unique. So the sequence of pairs of numbers $(r_1, d_1), (r_2, d_2), \dots, (r_l, d_l)$, where r_i is rank of E_i and d_i its degree, is unique. If these points are plotted in $\mathbb{R}^2$ and

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the line segments from (r_i, d_i) to (r_{i+1}, d_{i+1}) are joined, then one obtains a polygonal curve from the origin to (r, d) such that the slope of each successive line segment decreases. Such a curve will be called a *Shatz polygon* for (r, d). We denote the set of Shatz polygons for (r, d) by $\mathcal{P}^{r,d} = \mathcal{P}$. If one thinks of these polygons as graphs of functions $[0, r]\to\mathbb{R}$, then this collection has a natural partial order determined by the partial order on the set of functions with domain $[0, r]$ and codomain $\mathbb{R}$. For a vector bundle $\mathcal{E}$, we let $s(\mathcal{E})$ denote its Shatz polygon.

Now consider a family of vector bundles $\mathcal{E}$ on $X \times T$ of rank r and degree d, with T in **lfschemes**$/k$. Fix a Shatz polygon P for (r, d) and recall the following results:

(i) The locus $T^P = \{t \in T \mid s(\mathcal{E}_t) > P\}$ is closed.
(ii) The locus $\{t \in T \mid s(\mathcal{E}_t) = P\}$ is closed in the open set $T \setminus T^P$.

To prove these statements one considers the relative flag scheme $\mathrm{Flag}^P(\mathcal{E}/T)$ over T, whose fiber over $t \in T$ is a parameter space for flags of $\mathcal{E}_t$ with rank and degree data specified by P. It is proper over T so it has closed image in T. The above results follow by use of this fact. Complete details can be found in [Bru83].

Denote by $\mathrm{Div}^P(D)$ the open locus inside $\mathrm{Div}(D)$ parameterizing subbundles of $\mathcal{O}_X(D)^r$ whose Shatz polygon is not bigger than P, *i.e.*, the complement of the closed set in (i) defined by taking $T = \mathrm{Div}(D)$. We can consider the corresponding ind schemes $\mathbf{Div}^P$. We denote by $\mathbf{Div}^{ss}$ the semistable locus, corresponding to taking P equal to the straight line from $(0, 0)$ to (r, d).

Denote by $S^P(D)$ the locally closed locus inside $\mathrm{Div}(D)$ parameterizing bundles with Shatz polygon exactly P. These fit together to form an ind scheme $\mathbf{S}^P$. For $\deg D$ large enough $S_P(D)$ is smooth. If P has vertices $(r_0 = 0, d_0 = 0), (r_1, d_1), \dots, (r_l = r, d_l = d)$ and $\deg D$ large, then the codimension of this stratum is given by

$$d_P = \sum_{i<j} r_i r_j(\mu_i - \mu_j + g - 1)$$

where $\mu_i = d_i/r_i$.

Theorem 3.2 *Let P be a Shatz polygon with vertices*

$$(r_0 = 0, d_0 = 0), (r_1, d_1), \dots, (r_l = r, d_l = d)$$

Set $r_i' = r_i - r_{i-1}$ and $d_i' = d_i - d_{i-1}$. There is a closed immersion

$$\delta\colon \mathbf{Div}^{r_1',d_1',ss} \times \mathbf{Div}^{r_2',d_2',ss} \times \cdots \times \mathbf{Div}^{r_l',d_l',ss} \to \mathbf{S}^P$$

$$(\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_l) \mapsto \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \mathcal{E}_l$$

that induces an isomorphism in cohomology.

Proof This is [BGL94, Proposition 7.1]. ■

Let I be a subset of the collection of all matrix divisors. We say that I is *open* if $P \in I$ and $P' \le P$ implies $P' \in I$. If P is a minimal element of the complement of I then $J = I \cup \{P\}$ is also open. If I is open then the locus $S^I = \bigcup_{P\in I} S^P$ is an open subset of $\mathrm{Div}(D)$.

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Theorem 3.3 *Suppose P is a minimal element of the complement of J with J open. Set $I = J \cup \{P\}$. The Gysin sequences*

$$\cdots \to \mathrm{H}^{i-2d_P}(\mathbf{S}^P, \mathbb{Q}) \to \mathrm{H}^i(\mathbf{S}^I, \mathbb{Q}) \to \mathrm{H}^i(\mathbf{S}^J, \mathbb{Q}) \to \cdots$$

split into short exact sequences. Hence the following relation among Poincaré polynomials holds:

$$P(\mathbf{Div}; t) = \sum_{P \in \mathcal{P}} P(\mathbf{S}^P; t) t^{2d_P}.$$

Proof See [BGL94, Proposition 10.1]. ■

The above three theorems yield recursive formulas for the Hodge and Betti numbers of the ind varieties of matrix divisors associated to Shatz polygons.

In the remainder of this section we provide a dimension bound for the complement $\mathbf{Div}^{ss} \setminus \mathbf{Div}^s$.

We consider pairs of sequences of integers

$$(\underline{r}, \underline{d}) = ((r_1, r_2, \dots, r_l), (d_1, d_2, \dots, d_l))$$

satisfying the following conditions

$$(\sharp) \qquad 0 < r_1 < r_2 < \cdots < r_l = r, \qquad d_i = \frac{dr_i}{r}.$$

We denote by $\mathrm{Flag}^{(\underline{r},\underline{d})}(D)$ the scheme representing the functor

$$T \mapsto \{\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \cdots \subseteq \mathcal{E}_l \subseteq \mathcal{O}_X(D)^r \mid rk\mathcal{E}_i = r_i \deg \mathcal{E}_i = d_i\}$$

See [BGL94] for the existence of such a scheme. There is a proper morphism

$$\pi^{(\underline{r},\underline{d})} \colon \mathrm{Flag}^{(\underline{r},\underline{d})}(D) \to \mathrm{Div}(D).$$

There is an open subset $JH^{(\underline{r},\underline{d})}(D) \subseteq \mathrm{Flag}^{(\underline{r},\underline{d})}(D)$ parameterizing semistable flags with $\mathcal{E}_i/\mathcal{E}_{i-1}$ a stable bundle for all i. By the existence of Jordan–Holder filtrations we have

$$\mathrm{Div}^s(D) = \mathrm{Div}(D) \setminus \bigcup_{(\underline{r},\underline{d})} \pi^{(\underline{r},\underline{d})}(JH^{(\underline{r},\underline{d})}).$$

To find a dimension bound on the complement $\mathrm{Div}(D) \setminus \mathrm{Div}^s(D)$, we need only bound the dimensions of each of the open sets $JH^{(\underline{r},\underline{d})}(D)$.

Theorem 3.4

$$\dim JH^{(\underline{r},\underline{d})}(D) \leq r^2 \deg D - rd - (g-1)(r-1),$$

where g is the genus of the curve.

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Proof Consider a point $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_l \subseteq \mathcal{O}_X(D)^r$ of $JH^{(\underline{r},\underline{d})}(D)$. Following [BGL94] we denote by $\widetilde{\mathcal{E}}_i$ the sheaf $\mathcal{O}_X(D)^r/\mathcal{E}_i$. From [BGL94], the tangent space to $JH^{(\underline{r},\underline{d})}(D)$ at the above point is identified with the vector subspace of $\operatorname{Hom}(\mathcal{E}_1, \widetilde{\mathcal{E}}_1) \oplus \operatorname{Hom}(\mathcal{E}_2, \widetilde{\mathcal{E}}_2) \oplus \cdots \oplus \operatorname{Hom}(\mathcal{E}_l, \widetilde{\mathcal{E}}_l)$ consisting of l-tuples $(x_1, x_2, \ldots, x_l)$ satisfying the following condition:

- The images of x_i and x_{i+1} agree in $\operatorname{Hom}(\mathcal{E}_i, \widetilde{\mathcal{E}}_{i+1})$.

(See [BGL94].)

We have exact sequences

$$0 \to \mathcal{E}_1 \to R_{i+1} \to L_i \to 0 \quad \text{and} \quad 0 \to L_i \to \widetilde{\mathcal{E}}_i \to \widetilde{\mathcal{E}}_{i+1} \to 0$$

where L_i is a stable bundle of rank $r_{i+1} - r_i$. These sequences give rise to long exact sequences

$$0 \to \operatorname{Hom}(\mathcal{E}_i, L_i) \to \operatorname{Hom}(\mathcal{E}_i, \widetilde{\mathcal{E}}_i) \to \operatorname{Hom}(E_i, \widetilde{\mathcal{E}}_{i+1}) \to \cdots$$

and

$$0 \to \operatorname{Hom}(L_i, \widetilde{\mathcal{E}}_{i+1}) \to \operatorname{Hom}(\mathcal{E}_{i+1}, \widetilde{\mathcal{E}}_{i+1}) \to \operatorname{Ext}^1(\mathrm{L_i}, \widetilde{\mathcal{E}}_{\mathrm{i+1}}) \to \cdots .$$

As L_i is stable and E_i is semistable $\operatorname{Hom}(E_i, L_i) = 0$ and for $\deg D$ large enough $\operatorname{Ext}^1(\mathrm{L_i}, \widetilde{\mathcal{E}}_{\mathrm{i+1}})$ vanishes. It follows that

$$\begin{aligned}\dim JH^{(\underline{r},\underline{d})}(D) \leq \dim \operatorname{Hom}(\mathcal{E}_1, \widetilde{\mathcal{E}}_1) + \dim \operatorname{Hom}(L_1, \widetilde{\mathcal{E}}_2) \\ + \dim \operatorname{Hom}(L_2, \widetilde{\mathcal{E}}_3) + \cdots + \dim \operatorname{Hom}(L_{l-1} \widetilde{\mathcal{E}}_l).\end{aligned}$$

In bounding the right-hand side above, we will freely make use of [Ful98, §5, §15]. We have

$$Td(C) = 1 + \frac{1}{2} c_1(-K),$$
$$ch(\mathcal{E}_1^{\vee}) = r_1 - c_1(E_1),$$
$$ch(\widetilde{\mathcal{E}}_1) = r - r_1 + rc_1(D) - c_1(E_1),$$
$$ch(\widetilde{\mathcal{E}}_1 \otimes \mathcal{E}_1^{\vee}) = r_1(r - r_1) + r_1 rc_1(D) - rc_1(D).$$

Hence,

$$\chi(\widetilde{\mathcal{E}}_1 \otimes E_1^{\vee}) = r_1(r - r_1)(1 - g) + r_1 r \deg D - r_1 d.$$

Similarly,

$$\begin{aligned}ch(L_i^{\vee} \otimes \widetilde{\mathcal{E}}_{i+1}) = (r - r_{i+1})(r_{i+1} - r_i) \\ + (r - r_{i+1}) c_i(\mathcal{E}_i)(r_i - r) c_1(\mathcal{E}_{i+1}) + (r_{i+1} - r_i) rc_1(D).\end{aligned}$$

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Hence

$$\begin{aligned}\chi(L_i^\vee \otimes \widetilde{\mathcal{E}}_{i+1}) &= (r - r_{i+1})(r_{i+1} - r_i)(1-g)\\ &\qquad + (r - r_{i+1})d_i + (r_i - r)d_{i+1} + (r_{i+1} - r_i)r \deg D\\ &= (r_{i+1} - r_i)r \deg D + (r - r_{i+1})(r_{i+1} - r_i)(1-g) + (r_i - r_{i+1})d.\end{aligned}$$

So

$$\begin{aligned}&\dim JH^{(\underline{r},\underline{d})}(D)\\ &\quad\le r_1(r-r_1)(1-g) + r_1 r \deg D - r_1 d + (r_2 - r_1) r \deg D\\ &\qquad + ((r-r_2)(r_2-r_1)(1-g) + (r_1 - r_2)d + (r_3 - r_2) r \deg D\\ &\qquad + (r-r_3)(r_3-r_2)(1-g) + (r_2-r_3)d + \cdots + (r_l - r_{l-1}) r \deg D\\ &\qquad + (r-r_l)(r_l - r_{l-1})(1-g) + (r_{l-1} - r_l)d\\ &\quad = r^2 \deg D - rd + (1-g)(r_1(r-r_1) + (r-r_2)(r_2-r_1) + (r-r_l)(r_l-r_{l-1})\\ &\quad\le r^2 \deg D - rd + (1-g)(r-1).\end{aligned}$$

To see why the last inequality holds, first observe that for integers s and t with $1 \le s < t$ we have $\frac{s(t-s)}{t-1} \ge 1$. Since $1-g<0$, the inequality in question is equivalent to showing that $r_1(r_2-r_1) + r_2(r_3-r_2) + \cdots + r_{l-1}(r-r_{l-1}) \ge r-1$ which follows from the above inequality. ■

Corollary 3.5 *The inclusion* $\mathrm{Div}^s(D) \hookrightarrow \mathrm{Div}(D)$ *induces an isomorphism in cohomology*

$$\mathrm{H}^i(\mathrm{Div}(D), \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^i(\mathrm{Div}^s(D), \mathbb{Q})$$

for $i < 2(g-1)(r-1)$.

Proof We calculate the dimension of $\mathrm{Div}(D)$ to be $r^2 \deg D - rd$. The result follows from the Gysin sequence and the dimension bound above, see for example [Mil80, p. 268]. ■

4 The Cohomology of the Stack

Let $\mathcal{E}$ be a family of vector bundles on $X \times S$ with S/k a smooth scheme. We say that the family $\mathcal{E}$ is *complete* if the Kodaira–Spencer map $T_sS \rightarrow \mathrm{Ext}^1(\mathcal{E}_s, \mathcal{E}_s)$ is surjective.

Lemma 4.1 *Let S and T be schemes smooth over k and let $\mathcal{E}_S$ (resp., $\mathcal{E}_T$) be a complete family of bundles on $X \times S$ (resp., $X \times T$). Assume also that the induced maps $S \rightarrow \mathrm{Bun}$ and $T \rightarrow \mathrm{Bun}$ are smooth. Then the induced family on $S \times_{\mathrm{Bun}} T$ is complete.*

Proof This is mostly a matter of unwinding definitions. Recall that if A is a k-algebra then an A-point on $S \times_{\mathrm{Bun}} T$ consists of a triple (s, t, α) where s (resp., t) is an A-point of S (resp., T) and α is an isomorphism $\alpha\colon (s \times 1)^*\mathcal{E}_S \xrightarrow{\sim} (t \times 1)^* E_T$.

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So consider a closed point (s_0, t_0, α_0) of the fibered product. We have a diagram of Kodaira–Spencer maps

$$\begin{array}{ccc} T_{s_0}S & \longrightarrow & \mathrm{Ext}^1(\mathcal{E}_{s_0}, \mathcal{E}_{s_0}) \\ & & \downarrow \wr \\ T_{t_0}T & \longrightarrow & \mathrm{Ext}^1(\mathcal{E}_{t_0}, \mathcal{E}_{t_0}), \end{array}$$

where the vertical arrow is an isomorphism and the horizontal maps are surjective. Fix an extension class and choose $k[\epsilon]$-points of S and T lying above it. Call these point s and t, respectively. The bundles $(s \times 1)^*\mathcal{E}_S$ and $(t \times 1)^*E_T$ are isomorphic, as they correspond to the same extension class. It is possible to choose an isomorphism between these bundles that restricts to α upon specialization to the closed point of $k[\epsilon]$. Such an isomorphism gives a $k[\epsilon]$-point of the fibered product that maps onto the extension class we chose earlier. ■

We recall how a presentation of Bun was constructed in [LMB00]. Let $p(x) = rx + d + r(1 - g)$. For every integer m we define an open subscheme

$$Q^m \hookrightarrow \mathrm{Quot}(\mathcal{O}_X^{p(m)}, p(x+m))$$

by requiring that

(i) the quotients parameterized by Q^m be vector bundles;
(ii) for every T-point of Q^m defined by the family $\tau\colon \mathcal{O}_{X\times T}^{p(m)} \to \mathcal{F} \to 0$, we have

$$R^1\pi_{T,*}\mathcal{F} = 0 \quad \text{and} \quad \pi_{T,*}\colon \mathcal{O}_{X\times T}^{p(m)} \xrightarrow{\sim} \pi_{T,*}\mathcal{F} \text{ is an isomorphism.}$$

It follows from (ii) that if the quotient

$$0 \to \mathcal{G} \to \mathcal{O}_X^{p(m)} \to \mathcal{F} \to 0$$

represents a point of Q^m, then we have $\mathrm{H}^1(\mathcal{F} \otimes \mathcal{G}^\vee) = 0$, *i.e.*, Q^m is smooth.

We have maps $Q^n \to \mathrm{Bun}$ and $\mathcal{F} \mapsto \mathcal{F}(-n)$. Then

$$Q = \coprod_m Q^m \to \mathrm{Bun}$$

is a smooth presentation.

Proposition 4.2 *The family on Q is complete and hence, by the lemma,* $\mathrm{cosk}(Q/\mathrm{Bun})$ *is a simplicial algebraic space each of whose components defines a smooth family.*

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Proof Let

$$0 \to \mathcal{G}_n \to \mathcal{O}_{X\times Q^n}^{p(n)} \to \mathcal{F}_n \to 0,$$

be the universal family on $Q_n \times X$. The Kodaira–Spencer map is identified with the connecting homomorphism

$$\mathrm{Hom}(\mathcal{G}_n, \mathcal{F}_n) \to \mathrm{Ext}^1(\mathcal{F}_\mathrm{n}, \mathcal{F}_\mathrm{n}) = \mathrm{Ext}^1(\mathcal{F}_\mathrm{n}(-\mathrm{n}), \mathcal{F}_\mathrm{n}(-\mathrm{n})).$$

The next term in the sequence vanishes and the result follows. ■

We note the following theorem from [Pot97, p. 206].

Theorem 4.3 *Let E be a complete family of vector bundles X parameterized by S. Assume S is smooth. Let* $P = ((d_1, r_1), \dots, (d_l, r_l))$ *be a Shatz polygon for* (r, d). *Then the subvariety* S_P *of S parameterizing bundles with polygon P is locally closed and has codimension*

$$\mathrm{cod}(P) \stackrel{\mathrm{def}}{=} \sum_{i<j} r_i r_j (\mu_i - \mu_j + g - 1),$$

where $\mu_i = d_i / r_i$.

Let Bun^P be the open substack of Bun parameterizing bundles whose Shatz polygon is not bigger than P.

Theorem 4.4 *We have*

$$\varprojlim_P \mathrm{H}^i(\mathrm{Bun}^P, \mathbb{Q}) = \mathrm{H}^i(\mathrm{Bun}, \mathbb{Q}),$$

and in fact the limit on the left stabilizes.

Proof First some notation, if $\mathcal{E}$ is a family of rank r degree d bundles on $T \times X$, denote by T^P the open locus consisting of points $t \in T$ such that $s(\mathcal{E}_t)$ is not bigger than P. From definitions we have

$$(S \times_{\mathrm{Bun}} T)^P = (S^P \times_{\mathrm{Bun}^P} T^P).$$

For each fixed integer i there are only finitely many Shatz polygons Q having $\mathrm{cod}(Q) < i$. Let P_0 be a Shatz polygon greater than all of the Shatz polygons in this finite set. Let $P \geq P_0$. It suffices to show that the natural map $\mathrm{Bun}^P \to \mathrm{Bun}$ induces an isomorphism on degree i cohomology. By Propositions 2.3 and 4.2, it suffices to show that if $\mathcal{E}$ is a complete family of vector bundles on $X \times T$, then the natural inclusion $T^P \hookrightarrow T$ induces an isomorphism in cohomology of degree j for all $j \leq i$. But this follows by the Gysin sequence and choice of P. ■

The virtue of the above theorem is that the family of bundles parameterized by Bun^P is bounded. To see this last statement, note that only finitely many Shatz polygons appear in Bun^P and that the collection of bundles with a particular Shatz polygon is bounded. We will now proceed to exploit this.

The proof of Theorem 4.6 will rely on the following lemma, (see [BGL94, Lemma 8.2]).

Lemma 4.5 *Let $\mathcal{E}$ and $\mathcal{F}$ be rank r bundles on X such that $\operatorname{Ext}^1(\mathcal{E},\mathcal{F}) = 0$. Then for any effective divisor D, the codimension c_D of the closed locus in $\operatorname{Hom}(\mathcal{E},\mathcal{F}(D))$ consisting of non-injective homomorphisms satisfies $c_D \geq \deg D$.*

Theorem 4.6 *The natural map* $\mathbf{Div}\to \mathrm{Bun}$ *is a quasi-isomorphism,* i.e., *it induces an isomorphism on cohomology groups.*

Proof By Theorem 4.4, it suffices to show that the natural map $\mathbf{Div}^P\to \mathrm{Bun}^P$, is a quasi-isomorphism. As this last stack is of finite type, it suffices to show, by Proposition 2.3, that for all schemes T of finite type and all maps $T\to \mathrm{Bun}^P$, the map p_T below is a quasi-isomorphism:

$$\begin{array}{ccc} T \times_{\mathrm{Bun}^P} \mathbf{Div}^P & \xrightarrow{p_T} & T \\ \downarrow & & \downarrow \\ \mathbf{Div}^P & \longrightarrow & \mathrm{Bun}^P . \end{array}$$

Let $\mathcal{F}$ be the family of bundles on $X \times T$ defining the map $T\to \mathrm{Bun}^P$. For D large enough we have $\mathrm{H}^1(\mathcal{F}_t^\vee(D)) = 0$ for each $t \in T$. So, by the standard results on base change, for D large enough, we have that an S-point of $T \times_{\mathrm{Bun}^P} \mathrm{Div}^P(D)$ consists of a map $\phi\colon S\to T$ and an injection

$$(\phi \times 1)^*\mathcal{F}\hookrightarrow\mathcal{O}_{X\times S}(D)^r.$$

Hence $T \times_{\mathrm{Bun}^P} \mathrm{Div}^P(D)$ is an open subset of the vector bundle

$$\pi_{T,*}(\mathcal{H}om(\mathcal{F}, \mathcal{O}_{X\times T}(D)^r.$$

The result follows by Lemma 4.5 and a Gysin sequence. ■

Corollary 4.7 *When $k = \mathbb{C}$ or a finite field, the cohomology of* Bun *is pure and of the correct weight.*

5 The Cohomology of the Stack Versus That of the Moduli Space

Proposition 5.1 *Let G be a geometrically connected group scheme over k and let $f\colon P\to Y$ be a G-torsor. Then the local systems $Rf_*\mathbb{Q}$ or for the etále site $Rf_*\mathbb{Q}_l$, are constant.*

Proof This result is from [Beh93, §1.4]. For the convenience of the reader we give an outline of the ideas.

First, the action of G on itself by left multiplication induces an action of G on $\mathrm{H}^i(G)$. This action is trivial as it comes from an action of G on the discrete spaces $\mathrm{H}^i(G,\mathbb{Z})$, if $k = \mathbb{C}$, or for general k, $\mathrm{H}^i_{\text{ét}}(G \otimes_k \bar{k}, \mathbb{Z}/l^n\mathbb{Z})$. The fibration $g\colon P \times_G \mathrm{H}^i(G)\to Y$ is hence trivial over Y and one shows that $Rf_*\mathbb{Q} = Rg_*\mathbb{Q}$. ■

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Proposition 5.2 *Consider the natural map* $\Phi\colon \mathrm{GL}_n \to \mathrm{PGL}_n \times \mathbb{G}_m$ *that is the projection on the first factor and the determinant on the second factor. Then* Φ *is a quasi-isomorphism.*

Proof Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{GL}_n & \xrightarrow{\Phi} & \mathrm{PGL}_n \times \mathbb{G}_m \\ & \searrow^{f} & \downarrow g \\ & & \mathrm{PGL}_n, \end{array}$$

where f and g are the projections. It suffices to show that map induced by Φ on the Leray spectral sequences for f and g is an isomorphism at the E_2 level. We have

$$\Phi^*\colon E_2^{pq}(g) = \mathrm{H}^p(\mathrm{PGL}_n, \mathrm{H}^q(\mathbb{G}_m, \mathbb{Q})) \to E_2^{pq}(f) = \mathrm{H}^p(\mathrm{PGL}_n, Rf_*^q\mathbb{Q}).$$

By the above Proposition the local system on the right is constant, and it suffices to observe that the n-th power map $\mathbb{G}_m \to \mathbb{G}_m$ induces an isomorphism in cohomology. ■

Let G be an algebraic group over k acting on a scheme X over $\mathbb{C}$. Let the action be given by $\sigma\colon X \times G \to G$. The map $X \to [X/G]$ is a presentation and we wish to describe the simplicial space $\mathrm{cosk}(X/[X/G])$. The n-th term of the simplicial scheme $\mathrm{cosk}(X/[X/G])$ is of the form

$$X \times \underbrace{G \times G \times \cdots \times G}_{n \text{ times}}.$$

The i-th face map is given by

$$\delta_i(x, g_1, g_2, \dots, g_n) = \begin{cases} (x, g_1, \dots, \hat{g}_i, \dots, g_n) & \text{for } i > 0, \\ (xg_1, g_1^{-1}g_2, \dots, g_1^{-1}g_n) & \text{for } i = 0. \end{cases}$$

Let Bun^s be the open substack of Bun parameterizing stable bundles. The following is well known:

Proposition 5.3 *There is a scheme Q and a commutative diagram of stacks*

$$\begin{array}{ccc} [Q/\mathrm{GL}] & \longrightarrow & [Q/\mathrm{PGL}] \\ \downarrow & & \downarrow \\ \mathrm{Bun}^s & \longrightarrow & \mathfrak{M}^s \end{array}$$

in which the two vertical arrows are isomorphisms.

Proof The Q in the above theorem is the open locus inside the quot scheme that is both a presentation for Bun^s and the GIT quotient of it by PGL is the moduli space. For details, see [Gom, Proposition 3.3]. ■

Theorem 5.4 *There is an isomorphism* $H^*(\mathrm{Bun}^s, \mathbb{Q}) \xrightarrow{\sim} H^*(\mathfrak{M}^s, \mathbb{Q}) \otimes H^*(B\mathbb{G}_m, \mathbb{Q})$.

Proof We have a map $\operatorname{cosk}(Q/[Q/\mathrm{GL}]) \to \operatorname{cosk}(Q/[Q/\mathrm{PGL}])$. We define a map $\operatorname{cosk}(Q/[Q/\mathrm{GL}]) \to \operatorname{cosk}(\mathrm{point}/B\mathbb{G}_m)$ by projecting onto $\operatorname{cosk}(\mathrm{point}/B\mathrm{GL})$ and then taking the determinant. Hence we have a map

$$\operatorname{cosk}(Q/[Q/\mathrm{GL}]) \to \operatorname{cosk}(Q/[Q/\mathrm{PGL}]) \times \operatorname{cosk}(\mathrm{point}/B\mathbb{G}_m).$$

We see that this map induces an isomorphism in cohomology by using the standard spectral sequence, Proposition 2.3, and Proposition 5.2. ■

Corollary 5.5 *The natural map* $\mathrm{Div}^s \to \mathfrak{M}^s$ *is a quasi-isomorphism. The Betti and Hodge numbers of* $\mathrm{H}^i(\mathfrak{M}^s)$ *can be calculated for* $i < 2(r-1)(g-1)$. *If* k *is a finite field or* $\mathbb{C}$ *these cohomology groups are pure and of the correct weight.*

Proof First, the natural map $\mathrm{Bun}^s \to \mathfrak{M}^s$ is a quasi-isomorphism, as the proof of Theorem 4.6 carries over to this case verbatim. The result now follows from the above theorem and Corollary 3.5. ■

6 The Tamagawa Number of SL_n

In the remainder of this paper k will be a finite field of cardinality q. Let K be the function field of X and let $\mathbb{A}$ be its adele ring. Let $\mathfrak{K}$ be the canonical maximal compact in $\mathrm{SL}_n(\mathbb{A})$. We have a standard bijection between the double coset space $\mathfrak{K}\backslash \mathrm{SL}_n(\mathbb{A})/\mathrm{SL}_n(K)$ and the set of SL_n-torsors on X. To see this first observe that every SL_n-torsor is rationally trivial, after all a SL_n-torsor over a field is a vector space with a trivialization of its top exterior power, and these structures are all abstractly isomorphic. Next an SL_n-torsor can be described by descent data, and we may assume that one component of our étale cover is a Zariski open set. Such an étale cover can always be refined to a flat cover of the form $U \cup \bigcup \operatorname{Spec}\widehat{\mathcal{O}_{X,x_i}}$, where the union above is over a finite number of points x_i and U is a Zariski open in X and $\widehat{\mathcal{O}_{X,x_i}}$ is the completion of the local ring at x_i. It follows from faithfully flat descent that the points of $\mathrm{SL}_n(\mathbb{A})$ are in bijection with the collection of triples $(P, \phi, (\rho_x)_{x \text{ closed in } X})$, where P is an SL_n-torsor, ϕ is a generic trivialization, and a trivialization ρ_x is a family of trivializations over each $\operatorname{Spec}\widehat{\mathcal{O}_{X,x_i}}$. From this the above bijection follows.

Before proceeding further we will briefly recall the construction of the Tamagawa measure on $G(\mathbb{A})$, where G is a semisimple algebraic group over the function field K. The details of this construction can be found in [Wei82, Oes84]. Given a differential form ω on G of highest degree and a closed point $x \in X$, there is a way to produce a Haar measure on the locally compact group $G(\widehat{k(x)})$: here $\widehat{k(x)}$ is the quotient field of $\widehat{\mathcal{O}_{X,x}}$. Multiplying ω by $f \in K$ multiplies the Haar measure by f, thinking of K as

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a subfield of $\widehat{k(x)}$. From the product formula it follows that the limit of the product measures on $G(\mathbb{A})$ does not depend on the choice of top form ω. The Tamagawa measure is this measure multiplied by a factor of $q^{(1-g)\dim G}$, where g is the genus of X. The group $G(K)$ is a discrete subgroup of $G(\mathbb{A})$, and the Tamagawa number is defined to be the volume of $G(\mathbb{A})/G(K)$ under the above measure.

We now recall the Siegel formula for the Tamagawa number of G (quasi-split), denoted $\tau(G)$. We have

$$\begin{aligned}
\tau(G) &= \operatorname{vol}(G(\mathbb{A}))/G(K)) \\
&= \sum_x \operatorname{vol}(\mathfrak{K}xG(K)/G(K)) \\
&\qquad\qquad \text{(as } x \text{ runs over a collection of double coset representatives)} \\
&= \sum_x \operatorname{vol}(\mathfrak{K}) \frac{1}{|x\mathfrak{K}x^{-1} \cap G(K)|} \\
&= \operatorname{vol}(\mathfrak{K}) \sum_{x \in \operatorname{Bun}_G(k)} \frac{1}{|\operatorname{Aut}(x)|} \quad \text{(where the sum is over isomorphism classes)} \\
&= \operatorname{vol}(\mathfrak{K}) q^{\dim \operatorname{Bun}_G} \sum_{i=0}^{\infty} (-1)^i \operatorname{Tr} \Phi|_{\mathrm{H}^i(\operatorname{Bun}_G), \mathbb{Q}_l}, \qquad \text{(by [Beh]).}
\end{aligned}$$

In the last line, Φ is the arithmetic Frobenius and the last equality is by the Lefschetz trace formula for stacks. We will show that right-hand side above is in fact 1.

When $G = \mathrm{SL}_n$ in the above, we have

$$\operatorname{vol}(\mathfrak{K}) = q^{-(n^2-1)(g-1)} \prod_{x \in X} (1 - \frac{1}{q^{2 \deg x}}) \cdots (1 - \frac{1}{q^{n \deg x}}),$$

by [Wei82, p. 31]. The product above is over all closed points of X. In summary:

Proposition 6.1 *We have the following formula for the Tamagawa number of* SL_n*;*

$$\tau(\mathrm{SL}_n) = \Big(\prod_{x \in X} \Big(1 - \frac{1}{q^{2 \deg x}}\Big) \cdots \Big(1 - \frac{1}{q^{n \deg x}}\Big)\Big) \Big(\sum_{i=0}^{\infty} (-1)^i \operatorname{Tr} \Phi|_{\mathrm{H}^i(\operatorname{Bun}_{\mathrm{SL}_n}, \mathbb{Q}_l)} \Big).$$

For D an effective divisor on X we define $\operatorname{Div}_{\mathrm{SL}_n}(D)$ by the Cartesian square

$$\begin{array}{ccc}
\operatorname{Div}_{\mathrm{SL}_n}(D) & \longrightarrow & \operatorname{Div}^{n,0}(D) \\
\downarrow & & \downarrow \\
\operatorname{Bun}_{\mathrm{SL}_n} & \longrightarrow & \operatorname{Bun}_{n,0},
\end{array}$$

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where the lower horizontal map is induced by the standard faithful representation of SL_n. A point of $\mathrm{Div}_{\mathrm{SL}_n}(D)$ consists of a triple $(\mathcal{E}, i, r)$, where $\mathcal{E}$ is a rank n degree 0 bundle on X, i is inclusion of $\mathcal{E}$ into $\mathcal{O}_X(D)^n$ and r is a reduction of the structure group of $\mathcal{E}$ to SL_n. Now $\mathrm{GL}_n / \mathrm{SL}_n = \mathbb{G}_m$ is an affine algebraic group and as X is projective every morphism from X to $\mathbb{G}_m$ is constant. It follows that $\mathrm{Div}_{\mathrm{SL}_n}(D) \to S(D)$ is a $\mathbb{G}_m$-torsor, where $S(D) \subseteq \mathrm{Div}(D)$ is the locus of bundles with trivial determinant. Furthermore, by arguments similar to those as in Section 4, one shows that the natural map $\mathbf{Div}_{\mathrm{SL}_n} \to \mathrm{Bun}_{\mathrm{SL}_n}$, is a quasi-isomorphism. Here $\mathbf{Div}_{\mathrm{SL}_n}$ is the obvious ind scheme.

Writing $\mathrm{Tr}(\Phi|_X)$ for the alternating sum of the traces of the arithmetic Frobenius on the cohomology of X, we have

$$\begin{aligned}(2)\qquad \mathrm{Tr}(\Phi|_{\mathrm{Div}_{\mathrm{SL}_n}(D)}) &= \mathrm{Tr}(\Phi|_{\mathbb{G}_m})\,\mathrm{Tr}(\Phi|_{S(D)}) \\ &= \frac{q-1}{q}\,\mathrm{Tr}(\Phi|_{S(D)}).\end{aligned}$$

A standard deformation theory argument shows that $S(D)$ is smooth and the tangent space at a point

$$0 \to \mathcal{E} \to \mathcal{O}_X(D)^n \to Q \to 0$$

is the the subspace of $\mathrm{Hom}(\mathcal{E}, Q)$ consisting of maps whose image under the connecting homomorphism is inside $\mathrm{H}^1(\mathrm{ad}_{\mathrm{SL}_n}\mathcal{E}) \hookrightarrow \mathrm{H}^1(\mathrm{ad}\mathcal{E})$.

A point in the fixed locus of the torus action on $\mathrm{Div}^{n,0}(D)$ is of the form

$$\bigoplus_{i=1}^{n} \mathcal{O}_X(D - F_i) \hookrightarrow \mathcal{O}_X(D)^n.$$

Hence the connected components of the fixed point locus are parameterized by partitions of $n \deg D = nd$. If $\mathbf{m} = (m_1, m_2, \ldots, m_n)$, $m_i \geq 0$, is such a partition, then the corresponding fixed point locus is $\mathrm{Hilb}(m_1, X) \times \cdots \times \mathrm{Hilb}(m_n, X)$.

Let $S_{\mathbf{m}}(D)$ be its intersection with $S(D)$. If

$$\bigoplus_{i=1}^{n} \mathcal{O}_X(D - F_i) \hookrightarrow \mathcal{O}_X(D)^n$$

is a point of $S_{\mathbf{m}}(D)$, its tangent space to $\mathrm{Div}^{n,0}(D)$ is

$$\bigoplus_{i,j} \mathrm{Hom}(\mathcal{O}_X(D - F_i), \mathcal{O}_{F_j})$$

and the tangent space to $S(D)$ is a proper subspace that we do not write down. The bundle positive weights inside the normal bundle to the fixed locus inside $\mathrm{Div}^{n,0}(D)$ is

$$(3)\qquad \bigoplus_{i>j} \mathrm{Hom}(\mathcal{O}_X(D - F_i), \mathcal{O}_{F_j}).$$

One checks that the bundle of positive weights of the normal bundle of $S_{\mathbf{m}}(D)$ in $S(D)$ is the same thing.

The Lefschetz trace formula for the arithmetic Frobenius on a smooth variety X over k reads

$$\frac{1}{|X(k)|}q^{\dim X} = \operatorname{Tr}(\Phi|_X),$$

where $|X(k)|$ is the number of k-rational points on X. As $\dim S(D) = n^2 \deg D - g$ we have

$$\operatorname{Tr}(\Phi|_{S(D)}) = \sum_{\mathbf{m}} \frac{|S^+_{\mathbf{m}}(D)(k)|}{q^{n^2 \deg D - g}}, \tag{4}$$

where the sum is over all partitions of $n \deg D = nd$ and $S^+_{\mathbf{m}}(D)$ is the strata corresponding to $S_{\mathbf{m}}(D)$.

Before proceeding further we record a few elementary remarks regarding Hilbert schemes and zeta functions. Let $\zeta(s)$ be the zeta function of X, so $\zeta(s) = Z(q^{-s})$, where $Z(t)$ is a zeta function in the sense of Weil. Let N_i be the number of closed points of degree i on X. To give a k point of $\operatorname{Hilb}(m, X)$ is the same as giving a partition of m of the form

$$m = (x_{11} + x_{12} + \cdots + x_{1,N_1}) + 2(x_{21} + x_{22} + \cdots + x_{2,N_2}) + \cdots$$

with $x_{ij} \geq 0$. Let $c(\mathbf{N}, m)$ be the number of such partitions and let $c(\mathbf{N}, \mathbf{m})$ be the number of k points on $\operatorname{Hilb}(m_1, X) \times \cdots \times \operatorname{Hilb}(m_n, X)$, when $\mathbf{m} = (m_1, m_2, \ldots, m_n)$.

Lemma 6.2 *We have*

$$\zeta(2)\zeta(3)\cdots\zeta(n) = \prod_{x \in X}\Big(1 - \frac{1}{q^{2 \deg x}}\Big)^{-1} \cdots \Big(1 - \frac{1}{q^{(n+1)\deg x}}\Big)^{-1},$$

and in fact the product of the right converges (absolutely). For a positive integer α let $A_\alpha = \{(m_2, \ldots, m_n) \mid \sum i m_i = \alpha\}$. The coefficient of $q^{-\alpha}$ in the above product is

$$\sum_{\mathbf{m} \in A_\alpha} c(\mathbf{m}, \mathbf{N}) \stackrel{\text{def}}{=} B_\alpha.$$

Proof The remark about special values of zeta functions is by definition and the convergence statement is well known, for example it follows from Weil's analogue of the Riemann hypothesis. To see the second part¡ expand each term in the product as a geometric series and then expand using the combinatorics described above. ■

Corollary 6.3 *For any subset of the positive integers I, we have*

$$\zeta(2)^{-1} \cdots \zeta(n)^{-1} \geq \sum_{i \in I} B_\alpha q^{-i}.$$

We write $d = \deg D$. In order to calculate the Tamagawa number we just need to calculate the right-hand side of (4), which we now do. We are only interested in the limit as $d \to \infty$. The calculation is broken into two cases and the second case will disappear as d become large.

Case 1, $m_1 > 2g-2$: We have a $\mathbb{P}^{m_1-g}$ bundle $S_{\mathbf{m}} \to \mathrm{Hilb}(m_2, X) \times \cdots \times \mathrm{Hilb}(m_n, X)$. The weight positive weight space has dimension $m_1(n-1) + \sum_{i\geq 2}(n-i)m_i$. Remembering that $\mathbf{m}$ is a partition of nd, we have

$$\frac{|S_{\mathbf{m}}(k)|}{q^{n^2d-g}} = \frac{q}{q-1}c(\mathbf{m}^\circ, \mathbf{N})(q^{-\sum_{i\geq 2} im_i} - \text{error}),$$

where $\mathbf{m}^\circ = (m_2, m_3, \dots, m_n)$. It is straightforward to check that the sum of the errors goes to zero as d becomes large, using the corollary above. (See below also).

Case 2, $m_1 \leq 2g-2$: We wish to show that the sum of the terms in this case goes to zero as d increases, so it is assumed that $d > n(2g-2)$. Let $0 \leq k \leq 2g-2$ and let $2 \leq l \leq n$. Let ϵ_{kl} be the sum of the terms contributing to (4), in this case with $m_1 = k$ and $m_k > 2g-2$. It suffices to show that ϵ_{kl} goes to 0 as d increases. For this we may assume that $l = 2$. Consider the projection

$$S_{\mathbf{m}} \to \mathrm{Hilb}(m_1 = k, X) \times \mathrm{Hilb}(m_3, X) \times \cdots \times \mathrm{Hilb}(m_n, X).$$

Counting fibers and points as before we find that

$$\frac{S_{\mathbf{m}}}{q^{n^2d-g}} = \frac{c(k, \mathbf{N})c(\mathbf{m}^\sharp, \mathbf{N})q}{q-1}(q^{-2nd-\sum_{i\geq 3}(i-1)m_i} - q^{-1+k-2nd-\sum_{i\geq 3}(i-2)m_i+g}),$$

where $\mathbf{m}^\sharp = (m_1, m_2, \dots, m_n)$. Summing over the possibilities and applying the corollary, we find $\epsilon_{k2} \leq (\text{Constant})q^{-2nd}$.

Theorem 6.4 *The Tamagawa number of* SL_n *is* 1.

Proof By the above calculation and Lemma 6.2 we have

$$\lim_{d\to\infty} \mathrm{Tr}(\Phi|_{S(D)}) = \frac{q}{q-1}\zeta(2)\cdots\zeta(n).$$

The result follows from (2), Proposition 6.1 and the remarks immediately following it. ∎

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