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NORMAL FAMILIES OF MEROMORPHIC MAPPINGS OF SEVERAL COMPLEX VARIABLES INTO $\mathbf{P}^N(\mathbf{C})$

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Dedicated to Professor Hirotaka Fujimoto at his retirement

Abstract. The first aim in this article is to give some sufficient conditions for a family of meromorphic mappings of a domain D in $\mathbf{C}^n$ into $\mathbf{P}^N(\mathbf{C})$ omitting hypersurfaces to be meromorphically normal. Our result is a generalization of the results of Fujimoto and Tu. The second aim is to investigate extending holomorphic mappings into the compact complex space from the viewpoint of the theory of meromorphically normal families of meromorphic mappings.

§1. Introduction

Classically, a family $\mathcal{F}$ of holomorphic functions on a domain $D \subset \mathbf{C}$ is said to be (holomorphically) normal if every sequence in $\mathcal{F}$ contains a subsequence which converges uniformly on all the compact subsets of D.

In 1957 Lehto and Virtanen [LeVi] introduced the concept of normal meromorphic functions in connection with the study of boundary behaviour of meromorphic functions of one complex variable. Since then normal holomorphic maps has been studied intensively, resulting in an extensive development in the single complex variable context and in generalizations to several complex variables settings (see [Za], [JK1], [JK2], [AK] and the references cited in [Za] and [JK2]).

The first ideas and results on normal families of meromorphic mappings of several complex variables were introduced by Rutishauser [Rut] and Stoll [S].

The notion of a meromorphically normal family into the N-dimensional complex projective space is introduced by H. Fujimoto [Fu2]. He also gave

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some sufficient conditions for a family of meromorphic mappings of a domain D in $\mathbf{C}^n$ into $\mathbf{P}^N(\mathbf{C})$ to be meromorphically normal. As a weaker version of the concept of meromorphic normality, H. Fujimoto [Fu2] introduced the concept of quasi-normality. Recently, Z. Tu [Tu2] considered meromorphically normal families of meromorphic mappings of a domain D in $\mathbf{C}^n$ into $\mathbf{P}^N(\mathbf{C})$ omitting hyperplanes. See 3.1 for the actual definition of these concepts.

The first aim in this article is to give some sufficient conditions for a family of meromorphic mappings of a domain D in $\mathbf{C}^n$ into $\mathbf{P}^N(\mathbf{C})$ omitting hypersurfaces to be meromorphically normal or quasi-normal. These results are generalizations of the above Fujimoto's and Tu's results.

The second aim of this article is to investigate extending holomorphic mappings into compact complex spaces from the viewpoint of the theory of meromorphically normal families of meromorphic mappings. In order to state our main result, we need some preliminary.

First, for hypersurfaces H_i $(1 \le i \le q)$ of $\mathbf{P}^N(\mathbf{C})$ with $q \ge N+1$, let Q_i $(1 \le i \le q)$ be their defining polynomials, i.e., the homogeneous polynomials without multiple factors such that

$$H_i = \big\{z = (z_0 : z_1 : \cdots : z_N) : Q_i(z) = 0\big\}.$$

Here and below, throughout the article, we only consider homogeneous polynomials $Q(z) = \sum a_\nu z^\nu$ normalized so that $\sum |a_\nu|^2 = 1$. Now we define

$$\begin{aligned} D(H_1, \ldots, H_q) &= D(Q_1, \ldots, Q_q) \\ &= \prod_{1 \le i_1 < i_2 < \cdots < i_{N+1} \le q} \inf_{|z|=1} (|Q_{i_1}(z)|^2 + \cdots + |Q_{i_{N+1}}(z)|^2), \end{aligned}$$

where $\|z\| = \big(\sum |z_j|^2\big)^{1/2}$.

Next, let $\Lambda^d(S)$ denote the real d-dimensional Hausdorff measure of $S \subset \mathbf{C}^n$. For a formal $\mathbf{Z}$-linear combination $X = \sum_{i \in I} n_i X_i$ of analytic subsets $X_i \subset \mathbf{C}^n$ and for a subset $E \subset \mathbf{C}^n$, we call $\sum_{i\in I} \Lambda^d(X_i \cap E)$ (resp. $\sum_{i \in I} n_i \Lambda^d(X_i \cap E)$), the d-dimensional Lebesgue area of $X \cap E$ *regardless of multiplicities* (resp. *with counting multiplicities*).

Now we can state our main results.

THEOREM A. *Let $\mathcal{F}$ be a family of meromorphic mappings of a domain D in $\mathbf{C}^n$ into $\mathbf{P}^N(\mathbf{C})$. Suppose that for each $f \in \mathcal{F}$, there exist $q \geq 2N+1$ hypersurfaces $H_1(f), H_2(f), \ldots, H_q(f)$ in $\mathbf{P}^N(\mathbf{C})$ with*

$$\inf\{D(H_1(f), \ldots, H_q(f)); f \in \mathcal{F}\} > 0 \quad \textit{and}$$
$$f(D) \not\subset H_i(f) \quad (1 \leq i \leq N+1),$$

where q is independent of f, but the hypersurfaces $H_i(f)$ may depend on f, such that the following two conditions are satisfied:

i) For any fixed compact subset K of D, the $2(n-1)$-dimensional Lebesgue areas of $f^{-1}(H_i(f)) \cap K$ $(1 \leq i \leq N+1)$ with counting multiplicities for all f in $\mathcal{F}$ are bounded above.

ii) There exists a closed subset S of D with $\Lambda^{2n-1}(S) = 0$ such that for any fixed compact subset K of $D - S$, the $2(n-1)$-dimensional Lebesgue areas of $f^{-1}(H_i(f)) \cap K$ $(N+2 \leq i \leq q)$ with counting multiplicities for all f in $\mathcal{F}$ are bounded above.

Then $\mathcal{F}$ is a meromorphically normal family on D.

THEOREM B. *Let $\mathcal{F}$ be a family of meromorphic mappings of a domain D in $\mathbf{C}^n$ into $\mathbf{P}^N(\mathbf{C})$. Suppose that for each $f \in \mathcal{F}$, there exist $q \geq 2N+1$ hypersurfaces $H_1(f), H_2(f), \ldots, H_q(f)$ in $\mathbf{P}^N(\mathbf{C})$ with*

$$\inf\{D(H_1(f), \ldots, H_q(f)); f \in \mathcal{F}\} > 0,$$

where q is independent of f, but the hypersurfaces $H_i(f)$ may depend on f, such that for any fixed compact subset K of D, the $2(n-1)$-dimensional Lebesgue areas of $f^{-1}(H_i(f)) \cap K$ $(1 \leq i \leq q)$ with counting multiplicities for all $f \in \mathcal{F}$ are bounded above. Then $\mathcal{F}$ is a quasi-normal family on D.

THEOREM C. *Let Ω be a domain in $\mathbf{C}^n$ and $S \subset \Omega$ an analytic subset of codimension 1, whose singularities are normal crossings. Let M be a compact complex space. Let $f \in \mathrm{Hol}(\Omega - S, M)$. Suppose that there exist q hypersurfaces $H_1, \ldots, H_q$ in M and fixed positive integers $m_1, \ldots, m_q$ $(m_i \leq \infty$, $i = 1, \ldots, q)$ such that the family $\{(H_i, m_i)\}_{i=1}^q$ has the D-property (cf. Definition 4.3) and f intersects H_i with multiplicity at least m_i for each $1 \leq i \leq q$. Then f extends to a holomorphic mapping $f^* : \Omega \to M$.*

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§2. Notations

2.1. Let A be a non-empty open subset of a domain D in $\mathbf{C}^n$ such that $S = D - A$ is an analytic set in D. Let $f : A \to \mathbf{P}^N(\mathbf{C})$ be a holomorphic mapping. Let U be a non-empty connected open subset of D. A holomorphic mapping $\tilde{f} \not\equiv 0$ from U into $\mathbf{C}^{N+1}$ is said to be a representation of f on U if $f(z) = \rho(\tilde{f}(z))$ for all $z \in U \cap A - \tilde{f}^{-1}(0)$, where $\rho : \mathbf{C}^{N+1} - \{0\} \to \mathbf{P}^N(\mathbf{C})$ is the standard projective mapping. A holomorphic mapping $f : A \to \mathbf{P}^N(\mathbf{C})$ is said to be a meromorphic mapping from D into $\mathbf{P}^N(\mathbf{C})$ if and only if for any $z \in D$, there exists a representation of f on some neighborhood of z in D.

2.2. Let D be a domain in $\mathbf{C}^n$ and f a not identically zero holomorphic function on D. For a point $a = (a_1, a_2, \ldots, a_n) \in D$ we expand f as a compactly convergent series

$$f(u_1 + a_1, \ldots, u_n + a_n) = \sum_{m=0}^{\infty} P_m(u_1, \ldots, u_n)$$

on a neighborhood of a, where P_m is either identically zero or a homogeneous polynomial of degree m. The number

$$\nu_f(a) := \min\{m; P_m(u) \not\equiv 0\}$$

is said to be the zero multiplicity of f at a. By definition, a divisor on D is an integer-valued function ν on D such that for every $a \in D$ there are holomorphic functions $g(z)$ ($\not\equiv 0$) and $h(z)$ ($\not\equiv 0$) on a neighborhood U of a with $\nu(z) = \nu_g(z) - \nu_h(z)$ on U. We define the support $\operatorname{supp}\nu$ of the divisor ν on D by

$$\operatorname{supp}\nu := \overline{\{z \in D : \nu(z) \neq 0\}}.$$

We denote $\mathcal{D}^+(D) = \{\nu : \text{a non-negative divisor on } D\}$.

2.3. Let f be a meromorphic mapping of a domain D in $\mathbf{C}^n$ into $\mathbf{P}^N(\mathbf{C})$. Then for any $a \in D$, f always has a representation

$$\tilde{f}(z) = (f_0(z) : f_1(z) : \cdots : f_N(z))$$

on some neighborhood U of a with fixed homogeneous coordinates $(w_0 : w_1 : \cdots : w_N)$ on $\mathbf{P}^N(\mathbf{C})$ and holomorphic functions $f_i(z)$ $(0 \le i \le N)$ on U, where we can choose them so as to satisfy the condition

$$\operatorname{codim}\{f_0(z) = f_1(z) = \cdots = f_N(z) = 0\} \ge 2.$$

A representation of f satisfying this condition is refered to as an admissible representation of f on U in the following sections.

For a meromorphic mapping f into $\mathbf{P}^N(\mathbf{C})$ we denote by $I(f)$ the set of all points of indetermination of f, which is given by the condition

$$I(f) \cap U = \{z \in U : f_0(z) = f_1(z) = \cdots = f_N(z) = 0\}$$

if f has an admissible representation $f = (f_0 : f_1 : \cdots : f_N)$ on an open subset U of D. So, we have $\operatorname{codim} I(f) \geq 2$.

2.4. Take a hypersurface H in $\mathbf{P}^N(\mathbf{C})$ defined by

$$H := \{(z_0 : z_1 : \cdots : z_N) \in \mathbf{P}^N(\mathbf{C}) : Q^d(z_0, z_1, \dots, z_N) = 0\},$$

where Q^d is a homogeneous polynomial of degree d on $\mathbf{C}^{N+1}$.

Let D be a domain in $\mathbf{C}^n$. For any $a \in D$, taking an admissible representation $f = (f_0 : f_1 : \cdots : f_N)$ on a neighborhood U of a, we consider a holomorphic function

$$F := Q^d(f_0, f_1, \dots, f_N).$$

Then, the divisor $\nu(f, H)(z) := \nu_F(z)$ $(z \in U)$ is determined independently of a choice of admissible representations and hence is well-defined on the totality of D.

2.5. In 2.4, we defined the divisor $\nu(f, H)(z) := \nu_F(z)$ $(z \in U)$. Obviously, $\operatorname{supp} \nu(f, H)$ is either empty or a pure $(n-1)$-dimensional analytic set in D if $f(D) \not\subset H$ (i.e., $F(z) \not\equiv 0$ on U). We define $\nu(f, H) = \infty$ on D and $\operatorname{supp} \nu(f, H) = D$ if $f(D) \subset H$. Sometimes we identify $f^{-1}(H)$ with the divisor $\nu(f, H)$ on D. Rewrite $\nu(f, H)$ as the formal sum $\nu(f, H) = \sum_{i \in I} n_i X_i$, where X_i are the irreducible components of $\operatorname{supp} \nu(f, H)$ and n_i are the constant $\nu(f, H)(z)$ on $X_i \cap \operatorname{Reg}(\operatorname{supp} \nu(f, H))$, where $\operatorname{Reg}(\ )$ denotes the set of all the regular points.

We say that a meromorphic mapping f intersects H with multiplicity at least m on D if $f(D) \not\subset H$, $f(D) \cap H \neq \emptyset$, and $\nu(f, H)(z) \geq m$ for all $z \in \operatorname{supp} \nu(f, H)$ and that f intersects H with multiplicity ∞ on D if $f(D) \subset H$ or $f(D) \cap H = \emptyset$.

2.6. For each $x \in \mathbf{C}^n$ and $R > 0$, we set $B(x, R) = \{z \in \mathbf{C}^n : \|z - x\| < R\}$ and $B(0, R) = B(R)$.

§3. Criterions for meromorphically normal families

First of all, we recall some definitions.

Definition 3.1. Let D be a domain in $\mathbf{C}^n$.

i) (see [AK]) Let $\mathcal{F}$ be a family of holomorphic mappings of D into a complex manifold M. $\mathcal{F}$ is said to be *a holomorphically normal family on* D if any sequence in $\mathcal{F}$ contains a subsequence which converges uniformly on compact subsets of D to a holomorphic mapping of D into M.

ii) (see [Fu2]) A sequence $\{f^{(p)}(z)\}$ of meromorphic mappings from D into $\mathbf{P}^N(\mathbf{C})$ is said *to meromorphically converge on* D *to a meromorphic mapping* $f(z)$ if and only if, for any $z \in D$, each $f^{(p)}(z)$ has an admissible representation

$$\tilde{f}^{(p)} = (f_0^{(p)} : f_1^{(p)} : \cdots : f_N^{(p)})$$

on some fixed neighborhood U of z such that $\{f_i^{(p)}(z)\}_{p=1}^{\infty}$ converges uniformly on compact subsets of U to a holomorphic function f_i $(0 \leq i \leq N)$ on U with the property that $\tilde{f} = (f_0 : f_1 : \cdots : f_N)$ is a representation of f on U, where $f_{i_0} \not\equiv 0$ on U for some i_0.

iii) (see [Fu2]) Let $\mathcal{F}$ be a family of meromorphic mappings of D into $\mathbf{P}^N(\mathbf{C})$. $\mathcal{F}$ is said to be *a meromorphically normal family on* D if any sequence in $\mathcal{F}$ has a meromorphically convergent subsequence on D.

iv) (see [Fu2]) A sequence $\{f^{(p)}\}$ of meromorphic mappings from D into $\mathbf{P}^N(\mathbf{C})$ is said to be *quasi-regular on* D if and only if any $z \in D$ has a neighborhood U with the property that $\{f^{(p)}\}$ converges compactly on U outside a nowhere dense analytic subset S of U, i.e., for any domain $G \Subset U - S$, there is some p_0 such that $I(f^{(p)}) \cap G = \emptyset$ $(p \geq p_0)$ and $\{f^{(p)}|_G, p \geq p_0\}$ converges uniformly on G to a holomorphic mapping of G into $\mathbf{P}^N(\mathbf{C})$.

Obviously a meromorphically convergent sequence on D is always quasi-regular sequence on D. But a quasi-regular sequence on D need not imply meromorphic convergence on D.

v) (see [Fu2]) Let $\mathcal{F}$ be a family of meromorphic mappings of D into $\mathbf{P}^N(\mathbf{C})$. $\mathcal{F}$ is said to be *a quasi-normal family on* D if any sequence in $\mathcal{F}$ has a subsequence so as to be quasi-regular on D.

vi) (see [S]) Let $\{\nu_i\}_{i\in I}$ be a directed set of non-negative divisors on D. It is said to converge to a non-negative divisor ν on D if and only if any $a \in D$ has a neighborhood U such that, for suitable holomorphic functions

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h_i ($\not\equiv 0$) and h ($\not\equiv 0$) on U, $\nu_i = \nu_{h_i}$, $\nu = \nu_h$ and $\{h_i\}_{i \in I}$ converges compactly to h on U.

LEMMA 3.2. ([S, Theorem 4.10]) *If a sequence* $\{\nu_i\}$ *converges to* ν *in* $\mathcal{D}^+(B(R))$, *then* $\{\operatorname{supp} \nu_i\}$ *converges to* $\operatorname{supp} \nu$ *in the sense that* $\operatorname{supp} \nu$ *coincides with the set of all* z *such that every neighborhood* U *of* z *intersects* $\operatorname{supp} \nu_i$ *for all but finitely many* i *and, simultaneously, with the set of all* z *such that every* U *intersects* $\operatorname{supp} \nu_i$ *for infinitely many* i.

LEMMA 3.3. ([S, Theorem 2.24]) *A sequence* $\{\nu_i\}$ *of non-negative divisors on a domain* D *in* $\mathbf{C}^n$ *is normal in the sense of the convergence of divisors on* D *if and only if the* $2(n-1)$*-dimensional Lebesgue areas of* $\nu_i \cap E$ $(i \geq 1)$ *with counting multiplicities are bounded above for any fixed compact set* E *of* D.

LEMMA 3.4. *Let* $\{f^{(p)}\}$ *be a sequence of meromorphic mappings of a domain* D *in* $\mathbf{C}^n$ *into* $\mathbf{P}^N(\mathbf{C})$ *and let* S *be a closed subset of* D *with* $\Lambda^{2n-1}(S) = 0$. *Suppose that* $\{f^{(p)}\}$ *meromorphically converges on* $D - S$ *to a meromorphic mapping* f *of* $D - S$ *into* $\mathbf{P}^N(\mathbf{C})$. *If there exists a hypersurface* H *in* $\mathbf{P}^N(\mathbf{C})$ *such that* $f(D-S) \not\subset H$ *and* $\{\nu(f^{(p)}, H)\}$ *is a convergent sequence of divisors on* D, *then* $\{f^{(p)}\}$ *is meromorphically convergent on* D.

Proof. Without loss of generality we may assume that $D = B(R)$, $0 \notin S$ and $\{f^{(p)}\}$ meromorphically converges on $B(R) - S$ to a meromorphic mapping $f : B(R) - S \to \mathbf{P}^N(\mathbf{C})$ with $f(0) \notin H$.

Let x_0 be any point of S. By [S, Theorem 2.7], for any r $(0 < r < \tilde{R} = R - \|x_0\|)$, we can choose holomorphic functions $h^{(p)} \not\equiv 0$ and $h \not\equiv 0$ on $B(x_0, r)$ such that $\nu(f^{(p)}, H) = \nu_{h^{(p)}}$, $\nu = \nu_h$ for the limit ν of $\{\nu(f^{(p)}, H)\}$ and $\{h^{(p)}\}$ converges uniformly on compact subsets of $B(x_0, r)$ to h. Then, each $f^{(p)}$ has an admissible representation on $B(x_0, r)$

$$f^{(p)} = (f_0^{(p)} : f_1^{(p)} : \cdots : f_N^{(p)})$$

with suitable holomorphic functions $f_i^{(p)}$ $(0 \leq i \leq N)$ on $B(x_0, r)$.

Let x be a point in $B(x_0, r) - (S \cup \{h = 0\})$. Choose a simply connected relatively compact neighborhood W_x of x in $B(x_0, r) - (S \cup \{h = 0\})$ such that there exists a sequence $\{u_x^{(p)}\}$ of nonvanishing holomorphic functions on W_x such that $\{u_x^{(p)} f_i^{(p)}\} \to f_i^x$ $(0 \leq i \leq N)$ on W_x and $f = (f_0^x : f_1^x : \cdots :$

f_N^x) on W_x. It may be assumed that $h^{(p)}$ $(p \geq 1)$ has no zero on W_x. Let Q be the defining polynomial of H. We have $Q(f^{(p)}) = Q(f_0^{(p)}, \ldots, f_N^{(p)}) = v^{(p)}h^{(p)}$, where $v^{(p)}$ is a nonvanishing holomorphic function on $B(x_0, r)$. This implies that $Q(u_x^{(p)} f_0^{(p)}, \ldots, u_x^{(p)} f_N^{(p)}) \neq 0$ on W_x. Since Q is a homogeneous polynomial, $Q(u_x^{(p)} f_0^{(p)}, \ldots, u_x^{(p)} f_N^{(p)}) \to Q(f_0^x, \ldots, f_N^x)$ on W_x. Since $f(B(R) - S) \not\subset H$, it implies that $Q(f_0^x, \ldots, f_N^x) \not\equiv 0$ on W_x, and hence $Q(f_0^x, \ldots, f_N^x) \neq 0$ on W_x. Assume that Q has degree d. Since

$$Q(u_x^{(p)} f_0^{(p)}, \ldots, u_x^{(p)} f_N^{(p)}) \text{ tends to } Q(f_0^x, \ldots, f_N^x) \text{ on } W_x \text{ and}$$
$$Q(u_x^{(p)} f_0^{(p)}, \ldots, u_x^{(p)} f_N^{(p)}) = (u_x^{(p)})^d \cdot v^{(p)} \cdot h^{(p)},$$

it follows that $(u_x^{(p)})^d \cdot v^{(p)} \cdot h^{(p)}$ tends to $Q(f_0^x, \ldots, f_N^x)$ on W_x. Since $v^{(p)} \neq 0$ on $B(x_0, r)$, $v^{(p)} = (k^{(p)})^d$, where $k^{(p)}$ is a nonvanishing holomorphic mapping on $B(x_0, r)$. We have

$$(u_x^{(p)})^d \cdot (k^{(p)})^d = (u_x^{(p)} \cdot k^{(p)})^d \to \frac{Q(f_0^x, \ldots, f_N^x)}{h} \text{ on } W_x.$$

Define
$$F^d := \frac{Q(f_0^x, \ldots, f_N^x)}{h} \text{ on } W_x.$$

Obviously $F^d \neq 0$ on W_x. So $(u_x^{(p)} \cdot k^{(p)})^d \to F^d$ on W_x, hence $(u_x^{(p)} \cdot k^{(p)}/F)^d$ tends to 1 on W_x. Therefore, there exist (or empty) infinite subsets $\{N_j^x\}_{j=0}^{d-1}$ of $\mathbf{N}$ such that

$$\mathbf{N} \text{ is a disjoint union of sets } N_j^x \text{ and}$$
$$\left\{\frac{u_x^{(p)} \cdot k^{(p)}}{F}\right\}_{p \in N_j^x} \to \theta_j = e^{i \cdot 2\pi j/d} \text{ for each } 0 \leq j \leq d-1.$$

This implies that $\{f_i^{(p)}/k^{(p)}\}_{p \in N_j^x} \to F_i^x/\theta_j$ on W_x, where $F_i^x = f_i^x/F$ on W_x.

Take $a \in B(x_0, r) - (S \cup \{h = 0\})$. Then $\{f_i^{(p)}/k^{(p)}\}_{p \in N_j^a} \to F_i^a/\theta_j$ on W_a for each $0 \leq j \leq d-1$.

Take $b \in B(x_0, r) - (S \cup \{h = 0\})$ such that $W_a \cap W_b \neq \emptyset$. We will prove that $\{f_i^{(p)}/k^{(p)}\}_{p \in N_j^a} \to (F_i^b/\theta_j) \cdot c$ for each $0 \leq j \leq d-1$. Indeed, without loss of generality we may assume that $f_0^a \not\equiv 0$ on W_a. Then $f_0^x \not\equiv 0$ on W_x for each $x \in B(x_0, r) - (S \cup \{h = 0\})$. Hence $F_0^x \not\equiv 0$ on W_x for each $x \in B(x_0, r) - (S \cup \{h = 0\})$.

Consider $|N_j^a| = \infty$, where $|\cdot|$ denotes the cardinality of a set.

Assume that there exist N_1^b, N_2^b such that $|\tilde{N} = N_j^a \cap N_1^b| = |\tilde{\tilde{N}} = N_j^a \cap N_2^b| = \infty$. Since $\{f_0^{(p)}/k^{(p)}\}_{p\in\tilde{N}\subset N_1^b} \to F_0^b/\theta_1$ on W_b and $\{f_0^{(p)}/k^{(p)}\}_{p\in\tilde{N}\subset N_j^a} \to F_0^a/\theta_j$ on W_a, we have $F_0^b/\theta_1 = F_0^a/\theta_j$ on $W_a \cap W_b$. Similarly, $F_0^b/\theta_2 = F_0^a/\theta_j$ on $W_a \cap W_b$. This is a contradiction. Thus every infinite subset N_j^a intersects and only intersects infinitely with the subset $N_{\alpha(j)}^b$. Moreover, $|N_j^a \Delta N_{\alpha(j)}^b| < \infty$.

From this it follows that there exists a bijection $\alpha : \{0,1,\ldots,d-1\} \to \{0,1,\ldots,d-1\}$ such that

$$N_j^a = \emptyset \text{ if and only if } N_{\alpha(j)}^b = \emptyset,$$
$$\text{if } |N_j^a| = \infty \text{ then } |N_j^a \Delta N_{\alpha(j)}^b| < \infty.$$

On the other hand, since $\{f_0^{(p)}/k^{(p)}\}_{p\in N_j^a\cap N_{\alpha(j)}^b} \to F_0^a/\theta_j$ on W_a and $\{f_0^{(p)}/k^{(p)}\}_{p\in N_j^a\cap N_{\alpha(j)}^b} \to F_0^b/\theta_{\alpha(j)}$ on W_b, we have $F_0^a/\theta_j = F_0^b/\theta_{\alpha(j)}$ on $W_a \cap W_b$. This means that $F_0^a = F_0^b \circ (\theta_j/\theta_{\alpha(j)})$ on $W_a \cap W_b$ for each $0 \le j \le d-1$, and hence, $\theta_j/\theta_{\alpha(j)} \equiv c_b$: constant for each $0 \le j \le d-1$. It implies that $\{f_i^{(p)}/k^{(p)}\}_{p\in N_j^a\cap N_{\alpha(j)}^b} \to F_i^b/\theta_{\alpha(j)} = (F_i^b/\theta_j)\cdot c_b$ on W_b, and hence, $\{f_i^{(p)}/k^{(p)}\}_{p\in N_j^a} \to (F_i^b/\theta_j)\cdot c_b$ on W_b. Using the finite cover argument, we have $\{f_i^{(p)}/k^{(p)}\}_{p\in N_j^a} \to (F_i^x/\theta_j)\cdot c_x$ on W_x for each $x \in B(x_0,r) - (S \cup \{h=0\})$ and for each $0 \le j \le d-1$. For $p \in N_j^a$ put $\tilde{f}_i^{(p)} = f_i^{(p)}\cdot(\theta_j/k^{(p)})$ $(0 \le i \le N)$. Then $f^{(p)} = (\tilde{f}_0^{(p)},\ldots,\tilde{f}_N^{(p)})$ for all $p \in N_j^a$ and $0 \le j \le d-1$ and $\{\tilde{f}_i^{(p)}\}_{p=1}^\infty \to F_i^x \cdot c_x$ on W_x for each $0 \le i \le N$. Note that if $W_x \cap W_y \ne \emptyset$ $(x, y \in B(x_0,r) - (S\cup\{h=0\}))$ then $F_i^x \cdot c_x = F_i^y \cdot c_y$ for each $0 \le i \le N$.

Define the function $F_i : B(x_0,r) - (S\cup\{h=0\}) \to \mathbf{C}$ given by $F_i|_{W_x} = F_i^x \cdot c_x$. Then $\{\tilde{f}_i^{(p)}\}_{p=1}^\infty \to F_i$ on $B(x_0,r) - (S\cup\{h=0\})$ for each $0 \le i \le N$. We now prove that the sequence $\{f^{(p)}\}_{p=1}^\infty$ meromorphically converges on $B(x_0,r)$ to some meromorphic mapping $\tilde{F} = (\tilde{F}_0,\ldots,\tilde{F}_N)$. Indeed, let z_0 be any point of $S_1 = S \cup \{h=0\}$. Since $\Lambda^{2n-1}(S_1) = 0$, there exists a complex line l_{z_0} passing through z_0 such that $\Lambda^1(S_1 \cap l_{z_0}) = 0$. Put $l_{z_0} = \{z_0 + z\cdot u : z \in \mathbf{C}\}$. Then there exists $R > 0$ such that

$$\mathcal{C}_0 = \{z_0 + R\cdot e^{i\theta}\cdot u : \theta \in [0, 2\pi]\}$$

satisfying $\mathcal{C}_0 \subset B(x_0, r)$ and $\mathcal{C}_0 \cap S_1 = \emptyset$. By the maximum principle, it implies that the sequence $\{\tilde{f}_i^{(p)}(z_0)\}$ converges. Put $\lim_{p\to\infty} \tilde{f}_i^{(p)}(z_0) = \tilde{F}_i(z_0)$. This means that the mapping F_i extendes over $B(x_0, r)$ to the mapping $\tilde{F}_i$.

We now prove that the sequence $\{\tilde{f}_i^{(p)}(z)\}_{p=1}^{\infty}$ converges uniformly on compact subsets of $B(x_0, r)$ to $\tilde{F}_i(z)$. Indeed, assume that $\{z_j\} \subset B(x_0, r)$ converges to $z_0 \in B(x_0, r)$. As above, there exists a circle $\mathcal{C}_0 = \{z_0 + R\cdot e^{i\theta}\cdot u : \theta \in [0, 2\pi]\} \subset B(x_0, r)$ such that $\mathcal{C}_0 \cap S_1 = \emptyset$. Since $\mathcal{C}_0$ is a compact subset of $B(x_0, r) - S_1$, there exists $\epsilon_0 > 0$ such that

$$V(\mathcal{C}_0, \epsilon_0) = \{z \in \mathbf{C}^n : \mathrm{dist}(z, \mathcal{C}_0) < \epsilon_0\} \Subset B(x_0, r) - S_1.$$

Consider the circles $\mathcal{C}_j = \{z_j + R\cdot e^{i\theta}\cdot u : \theta \in [0, 2\pi]\}$. It is easy to see that $\mathrm{dist}(\mathcal{C}_0, \mathcal{C}_j) = \|z_j - z_0\| \to 0$ as $j \to \infty$. Thus, without loss of generality, we may assume that $\mathcal{C}_j \subset V(\mathcal{C}_0, \epsilon_0) \Subset B(x_0, r) - S_1$. By the hypothesis, $\forall \epsilon > 0$, $\exists N = N(\epsilon)$ such that

$$\sup\{\|\tilde{f}_i^{(p)}(z) - F_i(z)\| : z \in V(\mathcal{C}_0, \epsilon_0), p \geq N\} < \epsilon.$$

By the maximum principle, we have $\limsup_{j\to\infty} \|\tilde{f}_i^{(j)}(z_j) - F_i(z_j)\| = 0$. This implies that the sequence $\{\tilde{f}_i^{(p)}\}_{p=1}^{\infty}$ converges uniformly on compact subsets of $B(x_0, r)$ to $\tilde{F}_i$. ∎

LEMMA 3.5. *Let $\{f^{(p)}\}$ be a sequence of meromorphic mappings of a domain D in $\mathbf{C}^n$ into $\mathbf{P}^N(\mathbf{C})$ and let S be a closed subset of D with $\Lambda^{2n-1}(S) = 0$. Suppose that $\{f^{(p)}\}$ meromorphically converges on $D - S$ to a meromorphic mapping f of $D - S$ into $\mathbf{P}^N(\mathbf{C})$. Suppose that, for each $f^{(p)}$, there exist $N+1$ hypersurfaces $H_1(f^{(p)}), \ldots, H_{N+1}(f^{(p)})$ in $\mathbf{P}^N(\mathbf{C})$, where the hypersurfaces $H_i(f^{(p)})$ may depend on $f^{(p)}$, such that the following two conditions are satisfied.*

i) $\inf\{D(H_1(f^{(p)}), \ldots, H_{N+1}(f^{(p)})) : p \geq 1\} > 0$.

ii) The $2(n-1)$-dimensional Lebesgue areas of ${f^{(p)}}^{-1}(H_k(f^{(p)})) \cap E$ $(1 \leq k \leq N+1; p \geq 1)$ with counting multiplicities are bounded above for any fixed compact subset E of D.

Then $\{f^{(p)}\}$ has a meromorphically convergent subsequence on D.

Proof. We also assume that $D = B(R)$, $0 \notin S$ and $f(0) \notin H$. By the hypothesis ii) in Lemma 3.5 and by Lemma 3.3, we can assume that $\nu(f^{(p)}, H_k(f^{(p)}))$ converges in $\mathcal{D}^+(B(R))$ $(1 \leq k \leq N+1)$.

As in the proof of Lemma 3.4, there exist holomorphic mappings $h_k^{(p)} \not\equiv 0$ and $h_k \not\equiv 0$ on $B(r)$ such that $\nu(f^{(p)}, H_k(f^{(p)})) = \nu_{h_k^{(p)}}$, $\nu_k = \nu_{h_k}$ for the limit ν_k of $\{\nu(f^{(p)}, H_k(f^{(p)}))\}$ for each $1 \leq k \leq N{+}1$ $(0 < r < R)$ and $\{h_k^{(p)}\}$ converges uniformly on compact subsets of $B(r)$ to h_k $(1 \leq k \leq N+1)$. Take $x \in B(r) - (S \cup \bigcup_{k=1}^{N+1}\{h_k = 0\})$. Choose a simply connected relatively compact neighborhood W_x of x in $B(r) - (S \cup \bigcup_{k=1}^{N+1}\{h_k = 0\})$. Then, each $f^{(p)}$ has an admissible representation $f^{(p)} = (f_0^{(p)} : f_1^{(p)} : \cdots : f_N^{(p)})$ with suitable holomorphic functions $f_i^{(p)}$ $(0 \leq i \leq N)$ on $B(r)$. At the same time, there exist representations of $f^{(p)}$ $(p \geq 1)$ on W_x: $f^{(p)} = (u^{(p)} f_0^{(p)} : u^{(p)} f_1^{(p)} : \cdots : u^{(p)} f_N^{(p)})$ such that $u^{(p)}$ are nonvanishing holomorphic functions on W_x and $u^{(p)} f_i^{(p)} \to f_i$ $(0 \leq i \leq N)$ on W_x and $(f_0 : f_1 : \cdots : f_N)$ is a representation of f on W_x.

For each $1 \leq k \leq N+1$, let $Q_k^{d_k}(f^{(p)})$ be the defining polynomial of $H_k(f^{(p)})$, where the superscript d_k indicates the degree of the polynomial: $\deg Q_k^{d_k}(f^{(p)}) = d_k$. Put $Q_k^{(p)} := Q_k^{d_k}(f^{(p)})$. Then there exists a subsequence $\{Q_k^{(p_j)}\}_{j=1}^{\infty}$ of $\{Q_k^{(p)}\}_{p=1}^{\infty}$ such that $\{Q_k^{(p_j)}\}_{j=1}^{\infty}$ converges uniformly on compact subsets of $\mathbf{C}^{N+1}$ to a homogeneous polynomial Q_k. Without loss of generality we can assume $\{Q_k^{(p)}\}_{p=1}^{\infty}$ converges uniformly on compact subsets of $\mathbf{C}^{N+1}$ to Q_k as $p \to \infty$. Therefore $Q_k^{(p)}(u^{(p)} f_0^{(p)}, \ldots, u^{(p)} f_N^{(p)}) \to Q_k(f_0, \ldots, f_N)$ on compact subsets of W_x as $p \to \infty$. Let

$$H_k := \{(z_0 : z_1 : \cdots : z_N) \in \mathbf{P}^N(\mathbf{C}) : Q_k(z_0, z_1, \ldots, z_N) = 0\} \qquad (1 \leq k \leq N+1).$$

Then $D(H_1, \ldots, H_{N+1}) \geq \liminf_{p\to\infty} D(H_1(f^{(p)}), \ldots, H_{N+1}(f^{(p)}))$. By the hypothesis, we have $\inf\big\{D(H_1(f^{(p)}), \ldots, H_{N+1}(f^{(p)})) : p \geq 1\big\} > 0$. Thus $D(H_1, \ldots, H_{N+1}) > 0$, i.e. hypersurfaces $H_1, H_2, \ldots, H_{N+1}$ have no common point. Hence, there exists $k_0 \in \{1, \ldots, N+1\}$ such that $f(D - S) \not\subset H_{k_0}$. Since $Q_{k_0}^{(p)}(u^{(p)} f_0^{(p)}, \ldots, u^{(p)} f_N^{(p)})$ converges uniformly on compact subsets of W_x to $Q_{k_0}(f_0, \ldots, f_N)$ and by the same argument as in the proof of Lemma 3.4, it implies that $\{f^{(p)}\}$ meromorphically converges on D. □

3.6. Proof of Theorem A

Without loss of generality, we may assume that $D = \Delta^n$. Let $\{f_i\} \subset \mathcal{F}$. By Lemmas 3.2, 3.3 and by passing if necessary to a subsequence, we may assume that the sequence $\{f_i\}$ satisfies

$$\lim_{i\to\infty} f_i^{-1}(H_k(f_i)) = S_k \quad (1 \leq k \leq N+1)$$

as a sequence of closed subsets of Δ^n, where S_k are either empty or pure $(n-1)$-dimensional analytic sets in Δ^n, and

$$\lim_{i\to\infty} f_i^{-1}(H_k(f_i)) - S = S_k \quad (N+2 \leq k \leq q),$$

as a sequence of closed subsets of $\Delta^n - S$, where S_k are either empty or a pure $(n-1)$-dimensional analytic sets in $\Delta^n - S$. Let $E := \bigcup_{k=1}^q S_k - S$. Then E is either empty or a pure $(n-1)$-dimensional analytic set in $\Delta^n - S$. For each $1 \leq k \leq q$, let $Q_k^{d_k}(f_i)$ be the defining polynomial of $H_k(f_i)$. Put $Q_k^{(i)} := Q_k^{d_k}(f_i)$. Without loss of generality we can assume $\{Q_k^{(i)}\}_{i=1}^\infty$ converges uniformly on compact subsets of $\mathbf{C}^{N+1}$ to Q_k as $i \to \infty$. Let

$$H_k := \{(z_0 : z_1 : \cdots : z_N) \in \mathbf{P}^N(\mathbf{C}) : Q_k(z_0, z_1, \ldots, z_N) = 0\} \quad (1 \leq k \leq q).$$

For any fixed point z_0 in $(\Delta^n - S) - E$, choose a relatively compact neighborhood U_{z_0} in $(\Delta^n - S) - E$. Then $\{f_i|_{U_{z_0}}\} \subset \mathrm{Hol}(U_{z_0}, \mathbf{P}^N(\mathbf{C}))$. We now prove that the family $\{f_i|_{U_{z_0}}\}$ is a holomorphically normal family. Indeed, suppose that the family $\{f_i|_{U_{z_0}}\}$ is not holomorphically normal. By [TTH, Theorem 2.5], there exist $p_0 \in U_{z_0}$, $\{p_j\} \subset U_{z_0}$ with $p_j \to p_0$, $\{\rho_j\} \subset (0, \infty)$ with $\rho_j \to 0^+$ such that the sequence of holomorphic maps

$$g_j(z) = f_j(p_j + \rho_j z) : \Delta^n_{r_j} \to \mathbf{P}^N(\mathbf{C}) \quad (r_j \uparrow \infty)$$

converges uniformly on compact subsets of $\mathbf{C}^n$ to a nonconstant holomorphic map $g : \mathbf{C}^n \to \mathbf{P}^N(\mathbf{C})$. Since $\{g_j\}$ converges uniformly on compact subsets of $\mathbf{C}^n$ to g, it follows that there exist admissible representations $\tilde{g}_j = (g_j^0, \ldots, g_j^N)$ and $\tilde{g} = (g^0, \ldots, g^N)$ of g_j and g, respectively, such that $\{\tilde{g}_j\}$ converges uniformly on compact subsets of $\mathbf{C}^n$ to $\tilde{g}$. This implies that $\{Q_k^{(j)} \circ \tilde{g}_j\}$ converges uniformly on compact subsets of $\mathbf{C}^n$ to $Q_k \circ \tilde{g}$. Thus, by the Hurwitz's theorem, one of the following two assertions holds:

i) $Q_k \circ \tilde{g} \neq 0$ on $\mathbf{C}^n$, i.e. $g(\mathbf{C}^n) \cap H_k = \emptyset$.

ii) $Q_k \circ \tilde{g} \equiv 0$ on $\mathbf{C}^n$, i.e. $g(\mathbf{C}^n) \subset H_k$.

Hence there exists a subset $I \subset \{1, 2, \ldots, q\}$ such that $g(\mathbf{C}^n) \subset \big(\bigcap_{i\in I} H_i\big) \setminus \big(\bigcup_{i\notin I} H_i\big)$.

By [No-Wi, Corollary 1.4 (ii)], $\big(\bigcap_{i\in I} H_i\big) \setminus \big(\bigcup_{i\notin I} H_i\big)$ is hyperbolic, and hence g is constant. This is a contradiction. Thus $\{f_i\}$ is a holomorphically normal family on U_{z_0}. Therefore, by the usual diagonal argument, we can find a subsequence (again denoted by $\{f_i\}$) which converges uniformly on compact subsets of $(\Delta^n - S) - E$ to a holomorphic mapping f

of $(\Delta^n - S) - E$ into $\mathbf{P}^N(\mathbf{C})$. By Lemma 3.5, $\{f_i\}$ has a meromorphically convergent subsequence (again denoted by $\{f_i\}$) on $\Delta^n - S$ and again by Lemma 3.5, $\{f_i\}$ has a meromorphically convergent subsequence on Δ^n. Then $\mathcal{F}$ is a meromorphically normal family on Δ^n. The proof of Theorem A is completed. □

Remark 3.7. By the same argument as the proof of Theorem A, we get an another criterion for meromorphic normality.

THEOREM A′. *Let $\mathcal{F}$ be a family of meromorphic mappings of a domain D in $\mathbf{C}^n$ into $\mathbf{P}^N(\mathbf{C})$. Suppose that there exist $q \geq 2N+1$ hypersurfaces $H_1, \ldots, H_q$ in $\mathbf{P}^N(\mathbf{C})$ with*

$$D(H_1, \ldots, H_q) > 0 \ \textit{and} \ f(D) \not\subset H_i \quad (1 \leq i \leq N+1),$$

such that the following three conditions are satisfied.

i) For any fixed compact subset K of D, the $2(n-1)$-dimensional Lebesgue areas of $f^{-1}(H_i) \cap K$ $(1 \leq i \leq N+1)$ with counting multiplicities for all f in $\mathcal{F}$ are bounded above.

ii) There exists a closed subset S of D with $\Lambda^{2n-1}(S) = 0$ such that for any fixed compact subset K of $D - S$, the $2(n-1)$-dimensional Lebesgue areas of

$$\{z \in \operatorname{supp} \nu(f, H_j); \nu(f, H_j)(z) < m_j\} \cap K \quad (N+2 \leq j \leq q)$$

regardless of multiplicities for all f in $\mathcal{F}$, are bounded above, where $\{m_j\}_{j=N+2}^{q}$ are fixed positive integers and may be ∞.

iii) Any holomorphic mapping $\varphi : \mathbf{C} \to \mathbf{P}^N(\mathbf{C})$ which intersects H_j with mutiplicitiy at least m_j $(N+2 \leq j \leq q)$, must be constant.

Then $\mathcal{F}$ is a meromorphically normal family on D.

We now give a corollary of Theorem A.

COROLLARY 3.8. *Let $\mathcal{F}$ be a family of holomorphic mappings of a domain D in $\mathbf{C}^n$ into $\mathbf{P}^N(\mathbf{C})$. Suppose that for each $f \in \mathcal{F}$, there exist $q \geq 2N+1$ hypersurfaces $H_1(f), H_2(f), \ldots, H_q(f)$ in $\mathbf{P}^N(\mathbf{C})$ with*

$$\inf\{D(H_1(f), \ldots, H_q(f)) : f \in \mathcal{F}\} > 0,$$

where q is independent of f, but the hypersurfaces $H_i(f)$ may depend on f, such that the following two conditions are satisfied

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i) $f(D) \cap H_i(f) = \emptyset$ $(1 \leq i \leq N+1)$ for any f in $\mathcal{F}$.

ii) There exists a closed subset S of D with $\Lambda^{2n-1}(S) = 0$ such that for any fixed compact subset K of $D - S$, the $2(n-1)$-dimensional Lebesgue areas of $f^{-1}(H_i(f)) \cap K$ $(N+2 \leq i \leq q)$ with counting multiplicities for all $f \in \mathcal{F}$ are bounded above.

Then $\mathcal{F}$ is a holomorphically normal family on D.

In order to prove this corollary, we need the following lemma.

LEMMA 3.9. *Let $\{f^{(p)}\}$ be a meromorphically convergent sequence of holomorphic mappings of polydisc Δ^n in $\mathbf{C}^n$ into $\mathbf{P}^N(\mathbf{C})$. If for each $f^{(p)}$, there exist $N+1$ hypersurfaces $H_1(f^{(p)}), \dots, H_{N+1}(f^{(p)})$ in $\mathbf{P}^N(\mathbf{C})$ such that*

$$\inf\big\{D(H_1(f^{(p)}), \dots, H_{N+1}(f^{(p)})) : p \geq 1\big\} > 0$$

and

$$f^{(p)}(\Delta_n) \cap H_k(f^{(p)}) = \emptyset \quad (1 \leq k \leq N+1; p \geq 1),$$

where the hypersurfaces $H_i(f^{(p)})$ may depend on $f^{(p)}$.

Then $\{f^{(p)}\}$ converges uniformly on compact subsets of Δ^n to a holomorphic mapping of Δ^n into $\mathbf{P}^N(\mathbf{C})$.

Proof. Let $z_0 \in \Delta_n$. Then every $f^{(p)}$ has an admissible representation

$$f^{(p)}(z) = (f_0^{(p)}(z) : f_1^{(p)}(z) : \cdots : f_N^{(p)}(z)) \quad (p \geq 1)$$

on a fixed neighborhood $U(z_0)$ of z_0 such that $f^{(p)}$ converges uniformly on compact subsets of $U(z_0)$ to $f(z) = (f_0(z) : f_1(z) : \cdots : f_N(z))$, $f(z) \not\equiv 0$ on $U(z_0)$ (by Lemma 3.5).

We now show that $f(z) \neq 0$ everywhere on $U(z_0)$ and hence $\{f^{(p)}\}$ converges uniformly on compact subsets of Δ^n to a holomorphic mapping of Δ^n into $\mathbf{P}^N(\mathbf{C})$. As in the proof of Lemma 3.5, there exists k_0 such that $f(\Delta^n) \not\subset H_{k_0}$. By denoting as in the proof of Lemma 3.5, we have $Q_{k_0}^{(p)}(f_0^{(p)}, \dots, f_N^{(p)})$ converges uniformly on compact subsets of U to $Q_{k_0}(f_0, \dots, f_N)$ as p tends to infinity. Since $f^{(p)}(\Delta^n) \cap H_k(f^{(p)}) = \emptyset$ $(1 \leq k \leq N+1; p \geq 1)$, and $Q_{k_0}^{(p)}(f_0^{(p)}(z), \dots, f_N^{(p)}(z)) \neq 0$ for all $z \in \Delta^n$, it implies that $Q_{k_0}^{(p)}(f_0^{(p)}(z), \dots, f_N^{(p)}(z)) \neq 0$ on U. On the other hand, since $Q_{k_0}(f_0, \dots, f_N) \not\equiv 0$ on U, $Q_{k_0}(f_0, \dots, f_N) \neq 0$ everywhere on U. This implies that, for any $z \in U$, there exists $l \in \{0, \dots, N\}$ such that $f_l(z) \neq 0$. Thus f is a holomorphic mapping on U and hence, $\{f^{(p)}\}$ converges uniformly on compact subsets of Δ^n to a holomorphic mapping $f : \Delta^n \to \mathbf{P}^N(\mathbf{C})$. $\square$

3.10. Proof of Corollary 3.11

By Theorem A, $\mathcal{F}$ is a meromorphically normal family on D and hence by Lemma 3.9, $\mathcal{F}$ is a holomorphically normal family on D. □

LEMMA 3.11. *Let f be a meromorphic mapping from a domain D in $\mathbf{C}^n$ into $\mathbf{P}^N(\mathbf{C})$. If there exist $q \geq 2N+1$ hypersurfaces $H_1, \ldots, H_q$ in $\mathbf{P}^N(\mathbf{C})$ such that*

$$D(H_1, \ldots, H_q) > 0 \text{ and } f(D) \cap H_j = \emptyset \quad (1 \leq j \leq q).$$

Then f is actually a holomorphic mapping from domain D into $\mathbf{P}^N(\mathbf{C})$.

Proof. By [E-S] or [Ru], $\mathbf{P}^N(\mathbf{C}) - \bigcup_{i=1}^{2n+1} H_i$ is hyperbolic. Since f is a meromorphic mapping from D into the hyperbolic space $\mathbf{P}^N(\mathbf{C}) - \bigcup_{i=1}^{2n+1} H_i$, by the theorem of Kodama (see [Ko, Theorem 6.3.19, p. 288]), f is actually a holomorphic mapping. □

3.12. Proof of Theorem B

Take any sequence $\{f_i\} \subset \mathcal{F}$. By the assumption and Lemmas 3.2 and 3.3, we can find a subsequence (again denoted by $\{f_i\}$) such that $\lim_{i\to\infty} f_i^{-1}(H_k(f_i)) = S_k$ $(1 \leq k \leq q)$ as a sequence of closed subsets of D, where S_k are either empty or a pure $(n-1)$-dimensional analytic sets in D. Let $E := \bigcup_{k=1}^{q} S_k$. Then E is either empty or a pure $(n-1)$-dimensional analytic set of D, and hence E is a nowhere dense analytic set of D. We now prove that $\{f_i\}_{i=1}^{\infty}$ has a compactly convergent subsequence on $D - E$.

For any fixed point z_0 in $D - E$, there exist an integer i_0 and a neighborhood U_{z_0} in $D - E$ such that $f_i^{-1}(H_k(f_i)) \cap U_{z_0} = \emptyset$ $(1 \leq k \leq q)$ for all $i \geq i_0$. By using Lemma 3.11, $\{f_i\}_{i=i_0}^{\infty}$ is a sequence of holomorphic mappings of U_{z_0} into $\mathbf{P}^N(\mathbf{C})$ and by using again the argument in the proof of Theorem A, $\{f_i\}_{i=i_0}^{\infty}$ has a subsequence which converges uniformly on compact subsets of U_{z_0} to a holomorphic mapping of U_{z_0} into $\mathbf{P}^N(\mathbf{C})$. Therefore, by the usual diagonal argument, we can find a subsequence $\{f_{i_j}(z)\}$ so as to converge uniformly on compact subsets of $D - E$ to a holomorphic mapping of $D - E$ into $\mathbf{P}^N(\mathbf{C})$ and hence $\{f_{i_j}(z)\}$ is a quasi-regular on D. The proof of Theorem B is completed. □

§4. Extending holomorphic mappings into compact complex spaces

Modifying the notions in [JK1], we give the following definition.

DEFINITION 4.1. Let X be a complex space and M a compact complex space. We say that the family $\mathcal{F} \subset \mathrm{Hol}(X, M)$ is *uniformly normal* if $\mathcal{F} \circ \mathrm{Hol}(\Delta, X) := \{f \circ \varphi : f \in \mathcal{F} \text{ and } \varphi \in \mathrm{Hol}(\Delta, X)\} \Subset \mathrm{Hol}(\Delta, M)$, and that a mapping $f \in \mathrm{Hol}(X, M)$ is *a normal mapping* if the family $\{f\}$ is uniformly normal. Here $\mathrm{Hol}(Y, Z)$ is the space of holomorphic mappings from a complex space Y to a complex space Z with the compact - open topology.

LEMMA 4.2. ([Noc]) *Suppose that $q \geq 2N + 1$ hyperplanes $H_1, \ldots, H_q$ are given in general position in $\mathbf{P}^N(\mathbf{C})$, along with q positive integers $m_1, \ldots, m_q$ $(m_i \leq \infty, i = 1, \ldots, q)$. If*

$$\sum_{i=1}^{q} \frac{1}{m_i} < \frac{q - (N+1)}{N}$$

then there does not exist a nonconstant holomorphic mapping $f : \mathbf{C} \to \mathbf{P}^N(\mathbf{C})$ such that f intersects H_i with multiplicity at least m_i $(i = 1, \ldots, q)$.

Let H be an analytic hypersurface of a compact complex space M. For every $a \in H$ denote by $\mathcal{F}_a$ the set of all pair (U, φ), where U is an open neighbourhood of a in M and φ is a holomorphic function on U such that $U \cap H = \{z \in U : \varphi(z) = 0\}$. Recall that the holomorphic mapping f on a domain $D \subset \mathbf{C}^n$ intersects the hypersurface H with multiplicity at least m $(m < \infty)$ if $f(D) \not\subset H$, $f(D) \cap H \neq \emptyset$ and, for every $z_0 \in D \cap f^{-1}(H)$ and for every $(U, \varphi) \in \mathcal{F}_{f(z_0)}$, z_0 is the zero with multiplicity $\geq m$ of the holomorphic function $\varphi(f(z))$. We say that the holomorphic mapping f intersects the hypersurface H with multiplicity ∞ on D if $f(D) \subset H$ or $f(D) \cap H = \emptyset$.

DEFINITION 4.3. Let $H_1, \ldots, H_q$ be q hypersurfaces in a compact complex space M and $m_1, \ldots, m_q$ be fixed positive integers $(m_i \leq \infty,\ i = 1, \ldots, q)$. We say that the family $\{(H_i, m_i)\}_{i=1}^q$ has *the D-property* if every $f \in \mathrm{Hol}(\mathbf{C}, M)$ such that f intersects H_i with multiplicity at least m_i $(i = 1, \ldots, q)$ is constant.

It is easy to see that the family $\{(H_i, m_i)\}_{i=1}^q$ has *the D-property* iff any $f \in \mathrm{Hol}(\mathbf{C}, M)$ such that $f|_{\mathbf{C}^*}$ intersects H_i with multiplicity at least m_i $(i = 1, \ldots, q)$, then f is constant.

EXAMPLES. • Suppose that $q \geq 2N+1$ hyperplanes $H_1, \dots, H_q$ are given in general position in $\mathbf{P}^N(\mathbf{C})$, along with q positive integers $m_1, \dots, m_q$ ($m_i \leq \infty, i = 1, \dots, q$) such that

$$\sum_{i=1}^{q} \frac{1}{m_i} < \frac{q-(N+1)}{N}.$$

Then, by Nochka's theorem (cf. Lemma 4.2), the family $\{(H_i, m_i)\}_{i=1}^q$ has *the D-property*.

• Let $\mathbf{H_0} \to \mathbf{P}^N(\mathbf{C})$ be the hyperplane bundle and $H_j \in |\mathbf{H_0^d}|$ ($1 \leq j \leq q$) hyperbolic non-singular analytic hypersurfaces such that $H = \sum_{j=1}^q H_j$ has only normal crossings and $c_1([H]) + c_1(K(\mathbf{P}^N(\mathbf{C}))) > 0$. Let $m_1, \dots, m_q$ ($m_i \leq \infty, i = 1, \dots, q$) be positive integers such that

$$\sum_{j=1}^{q} \left(1 - \frac{1}{m_j}\right) > \frac{N+1}{d}.$$

Then, by the Ramification Theorem [NO, Example 5.5.56, p. 217], the family $\{(H_i, m_i)\}_{i=1}^q$ has *the D-property*.

4.4. Proof of Theorem C

Since the problem is local, we may assume that $\Omega = \Delta^n$ and $\Omega - S = (\Delta^*)^n$. We first show that the family $\mathcal{F} := \{f \circ \varphi : \varphi \in \mathrm{Hol}(\Delta, (\Delta^*)^n)\}$ is a holomorphically normal family.

Indeed, suppose that $\mathcal{F}$ is not normal. By using Zalcman's theorem [AK], [TTH], there exist sequences $\{p_j\} \subset \Delta$ with $\{p_j\} \to p_0 \in \Delta$, $\{f_j\} \subset \mathcal{F}$, $\{\rho_j\} \subset \mathbf{R}$ with $\rho_j > 0$ and $\{\rho_j\} \to 0$ such that

$$g_j(\xi) = f_j(p_j + \rho_j \xi), \quad \xi \in \mathbf{C}$$

converges uniformly on compact subsets of $\mathbf{C}$ to a nonconstant holomorphic map $g : \mathbf{C} \to M$.

Since f intersects H_i with multiplicity at least m_i ($i = 1, \dots, q$), it is easy to see that g_j also intersects H_i with multiplicity at least m_i ($i = 1, \dots, q; j = 1, 2, \dots$).

By using Hurwitz theorem, it follows that g intersects H_i with multiplicity at least m_i ($i = 1, \dots, q$). Since $\{(H_i, m_i)\}_{i=1}^q$ has the D-property, g is constant. This is impossible.

Thus $\mathcal{F}$ is holomorphically normal, i.e, f is a normal mapping.

By [JK1, Theorem 2.3], f extends to a holomorphic mapping $f^* : \Delta^n \to M$. □

Remark 4.5. (i) By the same argument as the proof of Theorem C and by [JK1, Theorem 2.3 and Corollary 2.5], we have the following.

THEOREM C$'$. *Let M be a compact complex space. Let $\mathcal{F} \subset \mathrm{Hol}((\Delta^*)^n, M)$. Suppose that there exist q hypersurfaces $H_1, \ldots, H_q$ in M and fixed positive integers $m_1, \ldots, m_q$ $(m_i \leq \infty, i = 1, \ldots, q)$ such that the family $\{(H_i, m_i)\}_{i=1}^q$ has the D-property (cf. Definition 4.3) and f intersects H_i with multiplicity at least m_i $(1 \leq i \leq q)$ for each $f \in \mathcal{F}$.*

Then every $f \in \mathcal{F}$ extends to a holomorphic mapping $f^* : \Delta^n \to M$ and the family $\mathcal{F}^* := \{f^* : f \in \mathcal{F}\} \subset \mathrm{Hol}(\Delta^n, M)$ is uniformly normal.

(ii) Let S be an analytic subset in Ω. Denote by $\mathrm{Sing}(S)$ the set of singular points of S. By Theorem C, f extends to a holomorphic mapping of $\Omega - \mathrm{Sing}(S)$ into M. Thus f can be regarded as a meromorphic mapping into M. Since $\dim \mathrm{Sing}(S) \leq n-2$, f extends to a meromorphic mapping in Ω.

We now discuss the problem posed in [Ja, Remark 3]. First of all, we recall the following.

DEFINITION 4.6. Let Ω be a hyperbolic domain in $\mathbf{C}^n$. Let M be a complete Hermitian complex space with a length function E_M. Let $f \in \mathrm{Hol}(\Omega, M)$. We say that f is *a normal holomorphic mapping in the sense of Lehto - Virtanen* provided that there exists some positive constant c such that, for all $z \in \Omega$ and all $\xi \in T_z\Omega$ it holds that

$$E_M(f(z), df(z)(\xi)) \leq cF_K^{\Omega}(z, \xi),$$

where $df(z)$ is the tangent mapping from $T_z\Omega$ to $T_{f(z)}M$ induced by f and F_K^{Ω} denotes the infinitesimal Kobayashi metric on Ω.

By [AK], if M is compact then f is normal if and only if f is normal in the sense of Lehto - Virtanen.

By Lemma 4.2 and Theorem C$'$, we have the partial answer for the above problem of Jarvi.

COROLLARY 4.7. *Let $f : (\Delta^*)^n \to \mathbf{P}^1(\mathbf{C})$ be a holomorphic mapping such that the following are satisfied.*

(i) There are positive integers m_0, m_1, m_2 $(m_i \leq \infty)$ such that $\frac{1}{m_0} + \frac{1}{m_1} + \frac{1}{m_2} < 1$.

(ii) There exist distinct points $a_0, a_1, a_2 \in \mathbf{P}^1(\mathbf{C})$ such that f has multiplicity at least m_i at a_i $(0 \leq i \leq 2)$.

Then f extends to a normal holomorphic mapping $f^* : \Delta^n \to \mathbf{P}^1(\mathbf{C})$.

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