SETS OF POLYNOMIALS ORTHOGONAL SIMULTANEOUSLY ON FOUR ELLIPSES

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1. Introduction. It has been shown by Walsh (**3**) and Szegö (**2**) that if a set of polynomials is orthogonal on both of two distinct curves, then one curve is a level curve of the other. Szegö (**2**) has determined all sets of polynomials which are orthogonal simultaneously on an entire family of level curves. There are five essentially different sets, two of which are orthogonal on concentric circles, and three of which are orthogonal on confocal ellipses. Merriman (**1**) has shown that the orthogonality of a set of polynomials on both of two concentric circles is sufficient to guarantee their orthogonality on the entire family of circles. In the present paper, we shall use a method akin to that of Merriman to show that the simultaneous orthogonality of a set of polynomials on four distinct confocal ellipses will ensure their orthogonality on the entire family of ellipses. Using $w = (z + z^{-1})/2$ and denoting the norm function (transformed into the z-plane) by $m(z)$, these latter three sets of polynomials $\{P_n(w),\ n = 0, 1, \ldots\}$ are:

(Type 1)
$$P_n(w) = 1 + \sum_{j=1}^{n} (z^j + z^{-j}),$$
$$m(z) = (z - 1)(R^2 - z)z^{-1}(R^2 + 1)^{-1},$$

(Type 2)
$$P_n(w) = z^n + z^{-n}, \qquad m(z) = 1,$$

(Type 3)
$$P_n(w) = \sum_{j=0}^{n} (1 + (-1)^{n-j})(z^j + z^{-j})/2,$$
$$m(z) = (z^2 - 1)(R^4 - z^2)z^{-2}(R^4 + 1)^{-1}.$$

2. Some required formulas. The set of polynomials

$$P_n(w) = \sum_{j=0}^{n} B_j^{(n)} w^j, \qquad n = 0, 1, \ldots, B_n^{(n)} \neq 0,$$

is said to be orthogonal on an ellipse E, with foci at 1 and -1, if

$$\int_E \overline{P_k(w)} P_h(w) n(w)\, |dw| = 0 \qquad (h \neq k),$$

where the norm function $n(w)$ is real and positive for w on E. The transformation $w = (z + z^{-1})/2$ takes the exterior of the "ellipse" $-1 \leqq w \leqq 1$ onto the exterior of the unit circle $|z| = 1$, and under this transformation, E corresponds to a circle C whose equation is $|z| = R$, where $R > 1$. When we

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transform the integral on E into an integral on C, the orthogonality criterion becomes

$$\int_C \bar{q}_k q_h m(z)\,|dz| = 0 \qquad (k \neq h),$$

where

$$q_k = q_k(z) = P_k((z + z^{-1})/2) \qquad (k = 0, 1, \ldots),$$
$$m(z) = n((z + z^{-1})/2)\,|1 - z^{-2}|/2.$$

Each q_k is a linear combination of the quantities $1, z + z^{-1}, \ldots, z^k + z^{-k}$. Hence, we can write

$$q_k = \sum_{j=0}^{k} a_j^{(k)} \zeta_j \qquad (k = 0, 1, \ldots),$$

where $\zeta_0 = 1$ and $\zeta_j = z^j + z^{-j}$ for $j = 1, 2, \ldots$. Since the leading coefficient of each polynomial $P_k(w)$ is non-vanishing, we can adjust each polynomial by a multiplicative constant so that the leading coefficient $a_k^{(k)}$ of each q_k is unity.

Since $n(w)$ is real and positive on E, $m(z)$ is real and positive on C. Hence, we can follow Merriman's lead (**1**) and expand it on C as

$$m(z) = \sum_{j=0}^{\infty} (A_j z^j + \bar{A}_j \bar{z}^j) = \sum_{j=0}^{\infty} (A_j z^j + \bar{A}_j R^{2j} z^{-j}),$$

where $m(z)$ has been adjusted by a multiplicative constant so that $A_0 = \frac{1}{2}$.

In order to evaluate the integrals

$$\int_C \bar{q}_h q_s m(z)\,|dz|,$$

we shall need the values of the integrals

$$I_{hs} = I_{h,s} = (2\pi R)^{-1} \int_C \bar{\zeta}_h \zeta_s m(z)\,|dz|.$$

Since

$$\int_C z^k\,|dz|$$

vanishes for $k \neq 0$ and has the value $2\pi R$ for $k = 0$, the only terms in the product $\bar{\zeta}_h \zeta_s m(z)$ which contribute to the value of I_{hs} are the constant terms. When $h = s = 0$ we have that $\bar{\zeta}_0 \zeta_0 = 1$, so that

$$I_{00} = (2\pi R)^{-1} \int_C (A_0 + \bar{A}_0)\,|dz| = 1.$$

For $h = 0$ and $s \neq 0$ we have that $\bar{\zeta}_0 \zeta_s = z^s + z^{-s}$, so that

$$I_{0s} = (2\pi R)^{-1} \int_C (\bar{A} R^{2s} + A_s)\,|dz| = \bar{A}_s R^{2s} + A_s.$$

When $0 < h < s$ and z is on C, so that $\bar{z} = R^2 z^{-1}$, we have that

$$\bar{\zeta}_h \zeta_s = (\bar{z}^h + \bar{z}^{-h})(z^s + z^{-s}) = R^{2h} z^{s-h} + R^{-2h} z^{s+h} + R^{2h} z^{-s-h} + R^{-2h} z^{h-s},$$
$$I_{hs} = (2\pi R)^{-1} \int_C (R^{2s} \bar{A}_{s-h} + R^{2s} \bar{A}_{s+h} + R^{2h} A_{s+h} + R^{-2h} A_{s-h})\,|dz|$$
$$= (\bar{A}_{s-h} + \bar{A}_{s+h}) R^{2s} + A_{s-h} R^{-2h} + A_{s+h} R^{2h}.$$

For $h > s$, we evaluate I_{hs} by noting that

$$\bar{I}_{sh} = (2\pi R)^{-1} \int_C \zeta_s \bar{\zeta}_h\,|dz| = I_{hs}.$$

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For $h = s$, we find that

$$\begin{aligned} I_{ss} &= (2\pi R)^{-1}\int_C((R^{2s} + R^{-2s})(A_0 + \bar{A}_0) + R^{-2s}z^{2s} + R^{2s}z^{-2s})\,|dz| \\ &= R^{2s} + R^{-2s} + (A_{2s} + \bar{A}_{2s})R^{2s}. \end{aligned}$$

3. A second formula for the integrals I_{hs}. For convenience, we shall hereafter use $\int F$ to denote

$$(2\pi R)^{-1}\int_C Fm(z)\,|dz|.$$

From the orthogonality conditions

$$\int \bar{q}_h q_s = 0 \qquad (h \neq s)$$

we have, in particular, that

$$\int \bar{q}_0 q_1 = \int \bar{\zeta}_0(a_0^{(1)}\zeta_0 + \zeta_1) = a_0^{(1)}I_{00} + I_{01} = a_0^{(1)} + I_{01} = 0.$$

The coefficient $a_0^{(1)}$ is independent of R. Hence, we can set $I_{01} = k_1$, where k_1 is a complex constant which is independent of R. Similarly,

$$\int \bar{q}_0 q_2 = \int \bar{\zeta}_0(a_0^{(2)}\zeta_0 + a_1^{(2)}\zeta_1 + \zeta_2) = a_0^{(2)}I_{00} + a_1^{(2)}I_{01} + I_{02} = a_0^{(2)} + a_1^{(2)}k_1 + I_{02} = 0.$$

It now appears that I_{02} must also be a constant which is independent of R. Thus, we set $I_{02} = k_2$. Proceeding in this way, we can establish that $I_{0n} = k_n$ for all n, with $k_0 = 1$, and with all the k_n independent of R.

If we now return to our previously established formula for I_{0n}, we can now solve for the quantities A_n in terms of the k_n and R. We have that

$$I_{0n} = \bar{A}_n R^{2n} + A_n = k_n, \qquad \bar{A}_n + A_n R^{2n} = \bar{k}_n,$$

so that

$$A_n = (\bar{k}_n R^{2n} - k_n)/(R^{4n} - 1).$$

We can now express the integrals I_{hs} in terms of the constants k_n and R. We find that the real and imaginary parts of I_{hs} are, for $0 < h < s$, given by

$$I_{hs} + \bar{I}_{hs} = (k_{s-h} + \bar{k}_{s-h})\frac{R^{s+h} + R^{-s-h}}{R^{s-h} + R^{-s+h}} + (k_{s+h} + \bar{k}_{s+h})\frac{R^{s-h} + R^{-s+h}}{R^{s+h} + R^{-s-h}},$$

$$I_{hs} - \bar{I}_{hs} = (k_{s-h} - \bar{k}_{s-h})\frac{R^{s+h} - R^{-s-h}}{R^{s-h} - R^{-s+h}} + (k_{s+h} - \bar{k}_{s+h})\frac{R^{s-h} - R^{-s+h}}{R^{s+h} - R^{-s-h}}.$$

When $h = s \neq 0$, we have that

$$I_{ss} = R^{2s} + R^{-2s} + \frac{k_{2s} + \bar{k}_{2s}}{R^{2s} + R^{-2s}}.$$

4. Determination of the early functions. Thus far we have that $q_0 = 1$, $q_1 = -k_1 + \zeta_1$, and we have seen that the orthogonality conditions $\int \bar{q}_0 q_s = 0$ give rise to the equations

$$\sum_{j=0}^{s} a_j^{(s)} k_j = 0 \qquad (s = 1, 2, \ldots). \tag{1}$$

Our next task is to determine the coefficients of q_2 in terms of the k_j. We can write

(2) $\int \bar{q}_1 q_2 = \int (\overline{a_0^{(1)}}\bar{\zeta}_0 + \bar{\zeta}_1) q_2 = \overline{a_0^{(1)}} \int \bar{q}_0 q_2 + \int \bar{\zeta}_1 q_2 = \int \bar{\zeta}_1 q_2 =$

$$\int \bar{\zeta}_1 (a_0^{(2)} \zeta_0 + a_1^{(2)} \zeta_1 + \zeta_2) = a_0^{(2)} I_{10} + a_1^{(2)} I_{11} + I_{12} = 0.$$

We already know that $I_{10} = \bar{k}_1$, and we find that

$$I_{11} = \rho + \frac{k_2 + \bar{k}_2}{\rho},$$

$$I_{12} + \bar{I}_{12} = (k_1 + \bar{k}_1)(\rho - 1) + \frac{k_3 + \bar{k}_3}{\rho - 1},$$

$$I_{12} - \bar{I}_{12} = (k_1 - \bar{k}_1)(\rho + 1) + \frac{k_3 - \bar{k}_3}{\rho + 1},$$

where $\rho = R^2 + R^{-2}$. Using

$$A = a_1^{(2)}, \qquad B = a_0^{(2)} \bar{k}_1, \qquad C = a_1^{(2)}(k_2 + \bar{k}_2),$$

we can write (2) as

$$A\rho + B + C\rho^{-1} + I_{12} = 0.$$

Setting both the real and the imaginary parts of this last equation equal to zero, putting in the values of $I_{12} + \bar{I}_{12}$ and $I_{12} - \bar{I}_{12}$, and reducing the left side of each equation to a polynomial in ρ, we have the two equations

$$\begin{aligned} &\rho^3(A + \bar{A} + k_1 + \bar{k}_1) + \rho^2(B + \bar{B} - A - \bar{A} - 2k_1 - 2\bar{k}_1) \\ &\qquad + \rho(C + \bar{C} - B - \bar{B} + k_1 + \bar{k}_1 + k_3 + \bar{k}_3) - C - \bar{C} = 0, \end{aligned}$$

(3)

$$\begin{aligned} &\rho^3(A - \bar{A} + k_1 - \bar{k}_1) + \rho^2(A - \bar{A} + B - \bar{B} + 2k_1 - 2\bar{k}_1) \\ &\qquad + \rho(B - \bar{B} + C - \bar{C} + k_1 - \bar{k}_1 + k_3 - \bar{k}_3) + C - \bar{C} = 0. \end{aligned}$$

Now, suppose that the original polynomials are orthogonal on four distinct confocal ellipses. Then equations (3) both hold for four distinct values of R which all exceed unity. Hence they also hold for four distinct values of ρ, since $\rho_1 - \rho_2 = (R_1^2 - R_2^2)(1 - R_1^{-2} R_2^{-2})$ cannot vanish for distinct R_1 and R_2 which both exceed unity. Since the cubics (3) vanish for four distinct values of ρ, they vanish identically and their coefficients are all zero. It follows at once that $A + k_1 = B - \bar{k}_1 = C = k_3 = 0$. Using this information, along with equation (1) with $s = 2$, we find that

$$a_0^{(2)} = k_1^2 - k_2, \quad a_1^{(2)} = -k_1,$$

$$k_1(k_2 + \bar{k}_2) = 0, \quad k_1(k_1^2 - k_2 - 1) = 0, \quad k_3 = 0.$$

We can now separate the functions q_s into two types.

(Type 1) $\qquad k_1 \neq 0, \qquad k_2 = k_1^2 - 1, \qquad k_2 + \bar{k}_2 = k_3 = 0,$

$$a_0^{(2)} = 1, \qquad a_1^{(2)} = -k_1.$$

(Type 2) $\qquad k_1 = k_3 = 0, \qquad a_0^{(2)} = -k_2, \qquad a_1^{(2)} = 0.$

Next, we use the orthogonality conditions involving q_3. We have that

$$\int \bar{q}_1 q_3 = \int \bar{\zeta}_1 q_3 = a_0^{(3)} I_{10} + a_1^{(3)} I_{11} + a_2^{(3)} I_{12} + I_{13} = 0. \tag{4}$$

We find that

$$I_{13} + \bar{I}_{13} = (k_2 + \bar{k}_2)\left(\rho - \frac{2}{\rho}\right) + (k_4 + \bar{k}_4)\frac{\rho}{\rho^2 - 2},$$

$$I_{13} - \bar{I}_{13} = (k_2 - \bar{k}_2)\rho + \frac{k_4 - \bar{k}_4}{\rho},$$

and we now have that $I_{12} = k_1\rho - \bar{k}_1$, since $k_3 = 0$ for both types of functions. Using the real and imaginary parts of (4) gives us the two equations

$$\begin{aligned} (A + \bar{A})\rho + B + \bar{B} + \frac{C + \bar{C}}{\rho} + (k_2 + \bar{k}_2)\left(\rho - \frac{2}{\rho}\right) \\ + (k_4 + \bar{k}_4)\frac{\rho}{\rho^2 - 2} = 0, \\ (A - \bar{A})\rho + B - \bar{B} + \frac{C - \bar{C}}{\rho} + (k_2 - \bar{k}_2)\rho + \frac{k_4 - \bar{k}_4}{\rho} = 0, \end{aligned} \tag{5}$$

where

$$A = a_1^{(3)} + k_1 a_2^{(3)}, \qquad B = (a_0^{(3)} - a_2^{(3)})\bar{k}_1, \qquad C = a_1^{(3)}(k_2 + \bar{k}_2).$$

For functions of the first type, we have that $C = k_2 + \bar{k}_2 = 0$, and equations (5) can be reduced to the polynomial equations

$$\rho^3(A + \bar{A}) + \rho^2(B + \bar{B}) + \rho(k_4 + \bar{k}_4 - 2A - 2\bar{A}) - 2B - 2\bar{B} = 0,$$
$$\rho^2(A - \bar{A} + 2k_2) + \rho(B - \bar{B}) + k_4 - \bar{k}_4 = 0.$$

Satisfaction of these two equations, both of degree less than four in ρ, for four distinct values of ρ means that their coefficients vanish. It follows that $A + k_2 = B = k_4 = 0$. This information, together with equation (1) with $s = 3$, allows us to solve for $a_0^{(3)}$ and $a_1^{(3)}$ in terms of $a_2^{(3)}$ and to find that $k_2 = 0$ and $k_1^2 = 1$.

(Type 1) $k_1^2 = 1, \quad k_2 = k_3 = k_4 = 0, \quad a_0^{(3)} = a_2^{(3)}, \quad a_1^{(3)} = -k_1 a_2^{(3)}$.

For functions of the second type, we have that $B = 0$ and equations (5) can be put in the form

$$\begin{aligned} \rho^4(A + \bar{A} + k_2 + \bar{k}_2) + \rho^2(C + \bar{C} - 2A - 2\bar{A} - 4k_2 - 4\bar{k}_2 + k_4 + \bar{k}_4) \\ - 2C - 2\bar{C} + 4k_2 + 4\bar{k}_2 = 0, \end{aligned}$$
$$\rho^2(A - \bar{A} + k_2 - \bar{k}_2) + C - \bar{C} + k_4 - \bar{k}_4 = 0.$$

Again, since one is a quadratic in ρ and the other is a quadratic in ρ^2, the coefficients must vanish. It follows that $A + k_2 = k_4 + \bar{k}_4 = 0$, $C = k_2 + \bar{k}_2 - k_4$. From this we find that

$$k_4 = (k_2 + \bar{k}_2)(1 + k_2), \qquad (k_2 + \bar{k}_2)(k_2 + \bar{k}_2 + 2) = 0.$$

Using this last condition on k_2, we can now split the functions q_s of the second

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type into two types, one of which we shall continue to call Type 2. Note, for later use, that $k_4 + \bar{k}_4 = 0$ and $k_4 = (k_2 + \bar{k}_2)(k_2 + 1)$ for both types.

$$\text{(Type 2)} \qquad k_1 = k_3 = k_4 = 0, \qquad k_2 + \bar{k}_2 = 0,$$
$$a_0^{(3)} = -k_2 a_2^{(3)}, \qquad a_1^{(3)} = -k_2.$$

$$\text{(Type 3)} \qquad k_1 = k_3 = 0, \qquad k_2 + \bar{k}_2 = -2, \qquad k_4 = -2k_2 - 2,$$
$$a_0^{(3)} = -k_2 a_2^{(3)}, \qquad a_1^{(3)} = -k_2.$$

We now use the orthogonality condition

$$\int \bar{q}_2 q_3 = \int \bar{\zeta}_2 q_3 = a_0^{(3)} I_{20} + a_1^{(3)} I_{21} + a_2^{(3)} I_{22} + I_{23} = 0. \tag{6}$$

We find that

$$I_{22} = \rho^2 - 2,$$
$$I_{23} + \bar{I}_{23} = 2k_1(\rho^2 - \rho - 1) + \frac{k_5 + \bar{k}_5}{\rho^2 - \rho - 1},$$
$$I_{23} - \bar{I}_{23} = \frac{k_5 - \bar{k}_5}{\rho^2 + \rho - 1},$$

where we have used the fact that k_1 is real and $k_4 + \bar{k}_4 = 0$ for all three types of functions. Equation (6) becomes

$$A\rho^2 + B\rho + C + I_{23} = 0, \tag{7}$$

where

$$A = a_2^{(3)}, \qquad B = a_1^{(3)} k_1, \qquad C = a_0^{(3)} \bar{k}_2 - a_1^{(3)} k_1 - 2a_2^{(3)}.$$

Using the values of $a_0^{(3)}$ and $a_1^{(3)}$ already found for Type 1, we find that $B = C = -A$. The real and imaginary parts of (7) furnish the two equations

$$(A + \bar{A})(\rho^2 - \rho - 1) + 2k_1(\rho^2 - \rho - 1) + \frac{k_5 + \bar{k}_5}{\rho^2 - \rho - 1} = 0,$$

$$(A - \bar{A})(\rho^2 - \rho - 1) + \frac{k_5 - \bar{k}_5}{\rho^2 + \rho - 1} = 0,$$

which can be reduced to

$$(A + \bar{A} + 2k_1)(\rho^2 - \rho - 1)^2 + k_5 + \bar{k}_5 = 0,$$
$$(A - \bar{A})(\rho^4 - 3\rho^2 + 1) + k_5 - \bar{k}_5 = 0.$$

Now, $\rho^2 - \rho - 1$ cannot assume the same value for two distinct values of ρ since $(\rho_1^2 - \rho_1 - 1) - (\rho_2^2 - \rho_2 - 1) = (\rho_1 - \rho_2)(\rho_1 + \rho_2 - 1)$ and ρ_1 and ρ_2 both exceed two. Thus, as before, the coefficients of the two quadratics (8) must vanish. This yields $A + k_1 = k_5 = 0$, from which we readily find the $a_j^{(3)}$ in terms of k_1. This, in fact, establishes the general pattern for functions of the first type.

$$\text{(Type 1)} \qquad k_1^2 = 1, \qquad k_2 = k_3 = k_4 = k_5 = 0,$$
$$a_0^{(3)} = a_2^{(3)} = -k_1, \qquad a_1^{(3)} = 1.$$

For both Types 2 and 3, we have that $k_1 = B = 0$, and the two equations arising from (7) are

$$(A + \bar{A})(\rho^4 - \rho^3 - \rho^2) + (C + \bar{C})(\rho^2 - \rho - 1) + k_5 + \bar{k}_5 = 0,$$
$$(A - \bar{A})(\rho^4 + \rho^3 - \rho^2) + (C - \bar{C})(\rho^2 + \rho - 1) + k_5 - \bar{k}_5 = 0.$$

For three distinct values of ρ for which it holds, the first of these equations gives rise to a system of three linear homogeneous equations in the three unknowns $A + \bar{A}$, $C + \bar{C}$, $k_5 + \bar{k}_5$. The determinant of this system is

$$\begin{vmatrix} \rho_1^4 - \rho_1^3 - \rho_1^2 & \rho_1^2 - \rho_1 - 1 & 1 \\ \rho_2^4 - \rho_2^3 - \rho_2^2 & \rho_2^2 - \rho_2 - 1 & 1 \\ \rho_3^4 - \rho_3^3 - \rho_3^2 & \rho_3^2 - \rho_3 - 1 & 1 \end{vmatrix} =$$
$$(\rho_2 - \rho_1)(\rho_3 - \rho_1)(\rho_2 - \rho_3)[\rho_1^2(\rho_2 - 1) + \rho_2^2(\rho_3 - 1) + \rho_3^2(\rho_1 - 1)$$
$$+ \rho_1\rho_2(\rho_2 - 2) + \rho_2\rho_3(\rho_3 - 2) + \rho_3\rho_1(\rho_1 - 2) + 2\rho_1\rho_2\rho_3 + \rho_1 + \rho_2 + \rho_3 + 1].$$

Since the distinct values ρ_1, ρ_2, and ρ_3 all exceed two, this determinant cannot vanish. Hence, the only solution the corresponding system of equations can have is the trivial one, namely, $A + \bar{A} = C + \bar{C} = k_5 + \bar{k}_5 = 0$. The system of equations which arises from the imaginary equation has the determinant

$$\begin{vmatrix} \rho_1^4 + \rho_1^3 - \rho_1^2 & \rho_1^2 + \rho_1 - 1 & 1 \\ \rho_2^4 + \rho_2^3 - \rho_2^2 & \rho_2^2 + \rho_2 - 1 & 1 \\ \rho_3^4 + \rho_3^3 - \rho_3^2 & \rho_3^2 + \rho_3 - 1 & 1 \end{vmatrix} =$$
$$(\rho_2 - \rho_1)(\rho_3 - \rho_1)(\rho_2 - \rho_3)[\rho_1^2\rho_2 + \rho_2^2\rho_3 + \rho_3^2\rho_1 + \rho_1\rho_2^2 + \rho_2\rho_3^2 + \rho_3\rho_1^2$$
$$+ 2\rho_1\rho_2\rho_3 + 2\rho_1\rho_2 + 2\rho_2\rho_3 + 2\rho_3\rho_1 + \rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_1 + \rho_2 + \rho_3 - 1].$$

This determinant also cannot vanish; whence, we have that $A - \bar{A} = C - \bar{C} = k_5 - k_5 = 0$. Thus, we have that $A = C = k_5 = 0$ for both Types 2 and 3. This, along with our previous information, lets us conclude that for both types, $a_0^{(3)} = a_2^{(3)} = 0$.

(Type 2) $\qquad k_1 = k_3 = k_4 = k_5 = 0, \qquad k_2 + \bar{k}_2 = 0,$
$$a_0^{(3)} = a_2^{(3)} = 0, \qquad a_1^{(3)} = -k_2.$$

(Type 3) $\quad k_1 = k_3 = k_5 = 0, \qquad k_2 + \bar{k}_2 = -2, \qquad k_4 = -2k_2 - 2,$
$$a_0^{(3)} - a_2^{(3)} = 0, \qquad a_1^{(3)} = -k_2.$$

Since the general pattern for functions of the first type has already been established, we need to consider the orthogonality conditions involving q_4 only for functions of the second two types. We have that

(9) $$\int \bar{q}_1 q_4 = a_0^{(4)} I_{10} + a_1^{(4)} I_{11} + a_2^{(4)} I_{12} + a_3^{(4)} I_{13} + I_{14} = 0.$$

Upon substitution of the relations among the k_j which are known to hold for

Types 2 and 3, we have that $I_{10} = I_{13} = I_{14} = 0$, $I_{13} = k_2 I_{11}$. Thus, (9) becomes

$$(a_1^{(4)} + a_3^{(4)} k_2) I_{11} = 0.$$

Now, $I_{11} = \rho^{-1}(\rho^2 + k_2 + \bar{k}_2)$ can vanish for at most one positive value of ρ. Hence, we must have that $a_1^{(4)} = -a_3^{(4)} k_2$. Using (1) with $s = 4$, we also have that $a_0^{(4)} = -k_2 a_2^{(4)} - k_4$.

Next, we have that

$$(10) \qquad \int \bar{q}_2 q_4 = a_0^{(4)} I_{20} + a_1^{(4)} I_{21} + a_2^{(4)} I_{22} + a_3^{(4)} I_{23} + I_{24} = 0$$

and we find that

$$I_{24} + \bar{I}_{24} = (k_2 + \bar{k}_2)(\rho^2 - 3) + \frac{k_6 + \bar{k}_6}{\rho^2 - 3},$$

$$I_{24} - \bar{I}_{24} = (k_2 - \bar{k}_2)(\rho^2 - 1) + \frac{k_6 - \bar{k}_6}{\rho^2 - 1}.$$

From (10) we have the two equations

$$\rho^4(A + \bar{A} + k_2 + \bar{k}_2) + \rho^2(-3A - 3\bar{A} + B + \bar{B} - 6k_2 - 6\bar{k}_2) \\ -3B - 3\bar{B} + 9k_2 + 9\bar{k}_2 + k_6 + \bar{k}_6 = 0,$$

$$\rho^4(A - \bar{A} + k_2 - \bar{k}_2) + \rho^2(-A + \bar{A} + B - \bar{B} - 2k_2 + 2\bar{k}_2) \\ -B + \bar{B} + k_2 - \bar{k}_2 + k_6 - \bar{k}_6 = 0,$$

where

$$A = a_2^{(4)}, \qquad B = a_0^{(4)} \bar{k}_2 - 2a_2^{(4)}.$$

The coefficients of both these quadratics in ρ^2 must vanish, furnishing the following result:

$$A = -k_2, \qquad B = 2k_2 + \bar{k}_2, \quad k_6 = 0.$$

This, together with our previous expression for $a_0^{(4)}$, yields

$$a_2^{(4)} = -k_2, \qquad \bar{k}_2(k_2^2 - k_4 - 1) = 0.$$

Thus, either $k_2 = 0$ or $k_4 = k_2^2 - 1$. We already know that either $k_2 + \bar{k}_2 = 0$ or $k_2 + \bar{k}_2 = -2$. When $k_2 + \bar{k}_2 = -2$, we cannot have $k_2 = 0$ and, therefore, must have that $k_4 = k_2^2 - 1$. This holds for Type 3, for which we already know that $k_4 = -2k_2 - 2$. Equating these two values of k_4, we find that $k_2 = -1$ and $k_4 = 0$. For Type 2, we already have that $k_2 + \bar{k}_2 = k_4 = 0$. Now $k_4 = k_2^2 - 1 = 0$ would mean that $k_2 = \pm 1$, which would violate $k_2 + \bar{k}_2 = 0$. Thus, for Type 2, we must have that $k_2 = 0$.

$$\text{(Type 2)} \qquad k_1 = k_2 = k_3 = k_4 = k_5 = k_6 = 0, \\ a_0^{(4)} = a_1^{(4)} = a_2^{(4)} = 0.$$

$$\text{(Type 3)} \qquad k_2 = -1, \qquad k_1 = k_3 = k_4 = k_5 = k_6 = 0, \\ a_0^{(4)} = a_2^{(4)} = 1, \qquad a_1^{(4)} = a_3^{(4)}.$$

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The last orthogonality condition involving q_4 is

$$\int \bar{q}_3 q_4 = a_0^{(4)} I_{30} + a_1^{(4)} I_{31} + a_2^{(4)} I_{32} + a_3^{(4)} I_{33} + I_{34} = 0, \tag{11}$$

where we find that

$$I_{33} = \rho^3 - 3\rho, \quad I_{34} + \bar{I}_{34} = \frac{k_7 + \bar{k}_7}{\rho^3 - \rho^2 - 2\rho + 1},$$

$$I_{34} - \bar{I}_{34} = \frac{k_7 - \bar{k}_7}{\rho^3 + \rho^2 - 2\rho - 1}.$$

For functions of Type 2, (11) reduces to

$$a_3^{(4)}(\rho^3 - 3\rho) + I_{34} = 0,$$

and the real and imaginary parts of this equation give

$$\begin{aligned} &(a_3^{(4)} + \overline{a_3^{(4)}})(\rho^6 - \rho^5 - 5\rho^4 + 4\rho^3 + 6\rho^2 - 3\rho) + k_7 + \bar{k}_7 = 0, \\ &(a_3^{(4)} - \overline{a_3^{(4)}})(\rho^6 + \rho^5 - 5\rho^4 - 4\rho^3 + 6\rho^2 + 3\rho) + k_7 - \bar{k}_7 = 0. \end{aligned} \tag{12}$$

For two distinct values of ρ for which orthogonality holds, the first of these equations gives rise to a system of two linear, homogeneous, equations in the two unknowns $a_3^{(4)} + \overline{a_3^{(4)}}$ and $k_7 + \bar{k}_7$. The determinant of this system has the value $f(\rho_1) - f(\rho_2)$, where

$$f(\rho) = \rho^6 - \rho^5 - 5\rho^4 + 4\rho^3 + 6\rho^2 - 3\rho.$$

The derivative of this function can be written as

$$f'(\rho) = (6\rho^4 + 8\rho^2)(\rho - 2) + 7\rho^2(\rho - 2)^2 + 12(\rho - 2) + 21.$$

For $\rho > 2$, this derivative remains positive. Hence, $f(\rho)$ is a strictly monotonic increasing function of ρ and cannot assume the same value for two distinct values of ρ. Thus, the determinant of the system cannot vanish, and the only solution is the trivial one, namely, $a_3^{(4)} + \overline{a_3^{(4)}} = k_7 + \bar{k}_7 = 0$. We can apply the same type of argument to the system corresponding to the second equation of (12). Using as $f(\rho)$ the function appearing in this equation, we find that

$$f'(\rho) = (6\rho^4 + 17\rho^3 + 14\rho^2)(\rho - 2) + 16\rho^2 + 12\rho + 3.$$

Since this derivative also remains positive for $\rho > 2$, the only solution is again the trivial one. It follows at once that $a_3^{(4)} = k_7 = 0$.

$$\text{(Type 2)} \qquad \begin{aligned} k_1 = k_2 = k_3 = k_4 = k_5 = k_6 = k_7 = 0, \\ a_0^{(4)} = a_1^{(4)} = a_2^{(4)} = a_3^{(4)} = 0. \end{aligned}$$

For Type 3, equation (11) reduces to

$$a_3^{(4)}(\rho^3 - 4\rho + 2\rho^{-1}) + I_{34} = 0,$$

giving rise to the pair of equations

$$\begin{aligned} &(a_3^{(4)} + \overline{a_3^{(4)}})(\rho^6 - \rho^5 - 6\rho^4 + 5\rho^3 + 10\rho^2 - 6\rho - 4 + 2\rho^{-1}) + k_7 + \bar{k}_7 = 0, \\ &(a_3^{(4)} - \overline{a_3^{(4)}})(\rho^6 + \rho^5 - 6\rho^4 - 5\rho^3 + 10\rho^2 + 6\rho - 4 - 2\rho^{-1}) + k_7 - \bar{k}_7 = 0. \end{aligned}$$

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We treat these by the same method used for equations (12). For the first equation, we find that

$$f'(\rho) = (6\rho^4 + 3\rho + 10)(\rho - 2) + (7\rho^2 + 4\rho)(\rho - 2)^2 + 14(\rho^2 - 4)\rho^{-2} + 54\rho^{-2},$$

while for the second equation,

$$f'(\rho) = (6\rho^4 + 17\rho^3 + 10\rho^2)(\rho - 2) + 5\rho^2 + 20\rho + 6 + 2\rho^{-2}.$$

Both derivatives remain positive for $\rho > 2$, and we have that $a_3^{(4)} = k_7 = 0$.

(Type 3) $\qquad k_2 = -1, \qquad k_1 = k_3 = k_4 = k_5 = k_6 = k_7,$

$$a_0^{(4)} = a_2^{(4)} = 1, \qquad a_1^{(4)} = a_3^{(4)} = 0.$$

We now collect our information to date for all three types of functions. For convenience, we set $k = k_1$.

(Type 1) $\qquad k_1^2 = 1, \qquad k_2 = k_3 = k_4 = k_5 = 0,$

$$q_0 = 1, \quad q_1 = -k + \zeta_1, \quad q_2 = 1 - k\zeta_1 + \zeta_2, \quad q_3 = -k + \zeta_1 - k\zeta_2 + \zeta_3.$$

(Type 2) $\qquad k_1 = k_2 = k_3 = k_4 = k_5 = k_6 = k_7 = 0,$

$$q_0 = 1, \quad q_1 = \zeta_1, \quad q_2 = \zeta_2, \quad q_3 = \zeta_3, \quad q_4 = \zeta_4.$$

(Type 3) $\qquad k_2 = -1, \qquad k_1 = k_3 = k_4 = k_5 = k_6 = k_7 = 0,$

$$q_0 = 1, \quad q_1 = \zeta_1, \quad q_2 = 1 + \zeta_2, \quad q_3 = \zeta_1 + \zeta_3, \quad q_4 = 1 + \zeta_2 + \zeta_4.$$

5. The induction when $k^2 = 1$. Let us suppose that for a given n we have established that

$$k^2 = k_1^2 = 1, \quad k_2 = \ldots = k_{2n-1} = 0,$$
$$q_j = \zeta_j - k\zeta_{j-1} + \zeta_{j-2} - k\zeta_{j-3} + \ldots - k\zeta_1 + 1 \qquad (j \text{ even}, j \leqq n),$$
$$q_j = \zeta_j - k\zeta_{j-1} + \zeta_{j-2} - k\zeta_{j-3} + \ldots + \zeta_1 - k \qquad (j \text{ odd}, j \leqq n).$$

We shall show that $k_{2n} = k_{2n+1} = 0$ and that q_{n+1} is of the same form as the q_j for $j \leqq n$. To do this, we use the orthogonality conditions involving q_{n+1}. The only integrals involved which do not vanish because of the k_j which are already known to vanish are

$$I_{j,j-1} = I_{j-1,j} = k\,\frac{R^{2j-1} + R^{-2j+1}}{R + R^{-1}} \qquad (1 \leqq j \leqq n),$$
$$I_{j,j} = R^{2j} + R^{-2j} \qquad (1 \leqq j \leqq n - 1),$$
$$I_{n-1,n+1} + \bar{I}_{n-1,n+1} = (k_{2n} + \bar{k}_{2n})\,\frac{R^2 + R^{-2}}{R^{2n} + R^{-2n}},$$
$$I_{n-1,n+1} - \bar{I}_{n-1,n+1} = (k_{2n} - \bar{k}_{2n})\,\frac{R^2 - R^{-2}}{R^{2n} - R^{-2n}},$$
$$I_{n,n} = R^{2n} + R^{-2n} + \frac{k_{2n} + \bar{k}_{2n}}{R^{2n} + R^{-2n}},$$
$$I_{n,n+1} + \bar{I}_{n,n+1} = 2k\,\frac{R^{2n+1} + R^{-2n-1}}{R + R^{-1}} + (k_{2n+1} + \bar{k}_{2n+1})\,\frac{R + R^{-1}}{R^{2n+1} + R^{-2n-1}},$$
$$I_{n,n+1} - \bar{I}_{n,n+1} = (k_{2n+1} - \bar{k}_{2n+1})\,\frac{R - R^{-1}}{R^{2n+1} - R^{-2n-1}}.$$

Furthermore, we find that for $1 \leqq j \leqq n-1$ the relationship $I_{j,j} - kI_{j,j-1} = k^{-1}I_{j,j+1}$ holds. Using the first orthogonality condition involving q_{n+1} (that is, (1) with $s = n+1$), we have that $a_0^{(n+1)} = -ka_1^{(n+1)}$. For $1 \leqq j \leqq n-2$, we have that

$$\int \bar{q}_j q_{n+1} = a_{j-1}^{(n+1)} I_{j,j-1} + a_j^{(n+1)} I_{j,j} + a_{j+1}^{(n+1)} I_{j,j+1} = 0.$$

If we know that $a_{j-1}^{(n+1)} = -ka_j^{(n+1)}$, this becomes

$$a_j^{(n+1)}(I_{j,j} - kI_{j,j-1}) + a_{j+1}^{(n+1)} = I_{j,j+1}(a_{j+1}^{(n+1)} + k^{-1}a_j^{(n+1)}) = 0.$$

Since $I_{j,j+1} \neq 0$, this means that $a_j^{(n+1)} = -ka_{j+1}^{(n+1)}$. Thus, since we already have that $a_0^{(n+1)} = -ka_1^{(n+1)}$, it follows that $a_j^{(n+1)} = -ka_{j+1}^{(n+1)}$ holds for $0 \leqq j \leqq n-2$. We now have that

$$\int \bar{q}_{n-1} q_{n+1} = a_{n-2}^{(n+1)} I_{n-1,n-2} + a_{n-1}^{(n+1)} I_{n-1,n-1} + a_n^{(n+1)} I_{n-1,n} + I_{n-1,n+1} = AI_{n-1,n} + I_{n-1,n+1} = 0,$$

where $A = k^{-1}a_{n-1}^{(n+1)} + a_n^{(n+1)}$. Using real and imaginary parts, we have that

$$(13)\qquad \begin{aligned} (A + \bar{A})k\frac{(R^{2n-1} + R^{-2n+1})(R^{2n} + R^{-2n})}{(R + R^{-1})(R^2 + R^{-2})} + k_{2n} + \bar{k}_{2n} &= 0, \\ (A - \bar{A})k\frac{(R^{2n-1} + R^{-2n+1})(R^{2n} - R^{-2n})}{(R + R^{-1})(R^2 - R^{-2})} + k_{2n} - \bar{k}_{2n} &= 0. \end{aligned}$$

If we can show that the functions of R appearing above are monotone for $R > 1$, we shall know that both systems of two equations in two unknowns, which are furnished by two distinct values of R, have no solution other than the trivial one. Now, the function

$$f(x) = \frac{x^m + x^{-m}}{x + x^{-1}}$$

has the derivative

$$\frac{(m-1)(x^m - x^{-m-2}) + (m+1)(x^{m-2} - x^{-m})}{(x + x^{-1})^2},$$

which is clearly positive for $x > 1$ and $m > 1$. Thus, $f(x)$ is strictly monotone increasing. The function of R in the first equation of (13) is a product of two functions of this form; hence, it is monotone increasing. The function of R in the second equation is the product of a function of this same form and of a function which is a sum of functions of the form $g(x) = x^m + x^{-m}$ (with the constant 1 added when m is odd). But $g'(x) = m(x^{m-1} - x^{-m-1})$ is also positive for $x > 1$, so that $g(x)$ is monotone increasing. Thus, the function of R in the second equation is also monotone. Hence, the two systems arising from (13) both have only the trivial solution. It follows at once that $A =$

$k_{2n} = 0$. We now have that $a_{n-1}^{(n+1)} = -ka_n^{(n+1)}$. Proceeding to the final orthogonality condition involving q_{n+1}, we have that

$$\int \bar{q}_n q_{n+1} = a_{n-1}^{(n+1)} I_{n-1,n} + a_n^{(n+1)} I_{n,n} + I_{n,n+1} = a_n^{(n+1)} \frac{R^{2n+1} + R^{-2n-1}}{R + R^{-1}} + I_{n,n+1} = 0.$$

This furnishes the two equations

$$(A + \bar{A})\left(\frac{R^{2n+1} + R^{-2n-1}}{R + R^{-1}}\right)^2 + k_{2n+1} + \bar{k}_{2n+1} = 0,$$

$$(A - \bar{A}) \frac{(R^{2n+1} + R^{-2n-1})(R^{2n+1} - R^{-2n-1})}{(R + R^{-1})(R - R^{-1})} = 0,$$

where $A = a_n^{(n+1)} + k$. The same arguments which we applied above show that the two resulting systems have only the trivial solution. Thus, we have that $a_n^{(n+1)} = -k$ and $k_{2n+1} = 0$. This completes the induction for functions of Type 1.

6. The induction when $k_1 = k_2 = 0$. Next, let us suppose that for a given n we have established that

$$k_1 = k_2 = \ldots = k_{2n-1} = 0, \quad q_j = \zeta_j \qquad (j = 0, \ldots, n).$$

We want to show that $k_{2n} = k_{2n+1} = 0$ and that $q_{n+1} = \zeta_{n+1}$. The only integrals in the orthogonality conditions involving q_{n+1} which are not known to vanish are $I_{j,j}$ for $1 \leqq j \leqq n$, $I_{n-1,n+1}$, $I_{n,n+1}$. Their values are as listed in the preceding section, where we use here the fact that $k = 0$. Equation (1) with $s = n + 1$ gives $a_0^{(n+1)} = 0$. For $1 \leqq j \leqq n - 2$, we have that

$$\int \bar{q}_j q_{n+1} = a_j^{(n+1)} I_{j,j} = a_j^{(n+1)} (R^{2j} + R^{-2j}) = 0.$$

Since $R^{2j} + R^{-2j} \neq 0$, we must have that $a_j^{(n+1)} = 0$ for $1 \leqq j \leqq n - 2$. Next, we have that

$$\int \bar{q}_{n-1} q_{n+1} = a_{n-1}^{(n+1)} I_{n-1,n-1} + I_{n-1,n+1} = a_{n-1}^{(n+1)} (R^{2n-2} + R^{-2n-2}) + I_{n-1,n+1} = 0,$$

which gives rise to the two equations

$$(A + \bar{A}) \frac{(R^{2n-2} + R^{-2n+2})(R^{2n} + R^{-2n})}{R^2 + R^{-2}} + k_{2n} + \bar{k}_{2n} = 0,$$

$$(A - \bar{A}) \frac{(R^{2n-2} + R^{-2n+2})(R^{2n} - R^{-2n})}{R^2 - R^{-2}} + k_{2n} - \bar{k}_{2n} = 0,$$

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where $A = a_{n-1}^{(n+1)}$. Our previous arguments apply again, and we have that $a_n^{(n+1)} = k_{2n} = 0$. Finally,

$$\int \bar{q}_n q_{n+1} = a_n^{(n+1)} I_{n,n} + I_{n,n+1} = a_n^{(n+1)}(R^{2n} + R^{-2n}) + I_{n,n+1} = 0,$$

$$(A + \bar{A})\frac{(R^{2n} + R^{-2n})(R^{2n+1} + R^{-2n-1})}{R + R^{-1}} + k_{2n+1} + \bar{k}_{2n+1} = 0,$$

$$(A - \bar{A})\frac{(R^{2n} + R^{-2n})(R^{2n+1} - R^{-2n-1})}{R - R^{-1}} + k_{2n+1} - \bar{k}_{2n+1} = 0,$$

where $A = a_n^{(n+1)}$. As before, the two systems of equations provided by two distinct values of R have only the trivial solution. Thus, $a_n^{(n+1)} = k_{2n+1} = 0$, and the induction in this case is complete.

7. The induction when $k_2 = -1$. For functions of Type 3, we assume that we have for a given n

$$k_1 = 0, \qquad k_2 = -1, \qquad k_3 = \ldots = k_{2n-1} = 0,$$
$$q_j = \zeta_j + \zeta_{j-2} + \ldots + \zeta_2 + 1 \qquad (j \text{ even}, j \leqq n),$$
$$q_j = \zeta_j + \zeta_{j-2} + \ldots + \zeta_3 + \zeta_1 \qquad (j \text{ odd}, j \leqq n).$$

The non-vanishing integrals occurring in the orthogonality conditions which involve q_{n+1} are

$$I_{j-2,j} = I_{j,j-2} = -\frac{R^{2j-2} + R^{-2j+2}}{R^2 + R^{-2}} \qquad (2 \leqq j \leqq n),$$
$$I_{j,j} = R^{2j} + R^{-2j} \qquad (1 \leqq j \leqq n-1),$$

$$I_{n-1,n+1} + \bar{I}_{n-1,n+1} = -2\frac{R^{2n} + R^{-2n}}{R^2 + R^{-2}} + (k_{2n} + \bar{k}_{2n})\frac{R^2 + R^{-2}}{R^{2n} + R^{-2n}},$$

$$I_{n-1,n+1} - \bar{I}_{n-1,n+1} = (k_{2n} - \bar{k}_{2n})\frac{R^2 - R^{-2}}{R^{2n} - R^{-2n}},$$

$$I_{n,n} = R^{2n} + R^{-2n} + \frac{k_{2n} + \bar{k}_{2n}}{R^{2n} + R^{-2n}},$$

$$I_{n,n+1} + \bar{I}_{n,n+1} = (k_{2n+1} + \bar{k}_{2n+1})\frac{R + R^{-1}}{R^{2n+1} + R^{-2n-1}},$$

$$I_{n,n+1} - \bar{I}_{n,n+1} = (k_{2n+1} - \bar{k}_{2n+1})\frac{R - R^{-1}}{R^{2n+1} - R^{-2n-1}}.$$

For $2 \leqq j \leqq n-2$, the relationship $I_{j,j-2} + I_{j,j} = -I_{j,j+2}$ holds among these integrals. Since the induction here uses methods similar to those of the preceding section, we simply outline it here. Equation (1) with $s = n+1$ yields $a_0^{(n+1)} = a_2^{(n+1)}$. The equation $\int \bar{q}_1 q_{n+1} = 0$ furnishes $a_1^{(n+1)} = a_3^{(n+1)}$. We then use the equations $\int \bar{q}_j q_{n+1} = 0$ to find that $a_{j-2}^{(n+1)} = a_j^{(n+1)}$ implies that $a_j^{(n+1)} = a_{j+2}^{(n+1)}$ for $0 \leqq j \leqq n-2$. Next, $\int \bar{q}_{n-1} q_{n+1} = 0$ allows us to

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conclude that $a_{n-1}^{(n+1)} = 1$, $k_{2n} = 0$. Finally, $\int \bar{q}_n q_{n+1} = 0$ yields $a_n^{(n+1)} = k_{2n+1} = 0$. This concludes the proof for functions of Type 3.

8. Identification of the results. Substituting the values $A_0 = \frac{1}{2}$, $A_1 = k_1/(R^2 + 1)$, $A_2 = k_2/(R^4 + 1)$, $A_3 = A_4 = \ldots = 0$ and using the appropriate values of k_1 and k_2, we find that functions of Type 1 can be written either as

$$q_h(z) = \sum_{j=0}^{h} \zeta_j = 1 + \sum_{j=1}^{h} (z^j + z^{-j})$$

with norm function

$$m(z) = (z - 1)(R^2 - z)z^{-1}(R^2 + 1)^{-1}$$

or as

$$q_h(z) = \sum_{j=0}^{h} (-1)^{h-j}\zeta_j = (-1)^h + \sum_{j=1}^{h} (-1)^{h-j}(z^j + z^{-j})$$

with norm function

$$m(z) = (z + 1)(R^2 + z)z^{-1}(R^2 + 1)^{-1}.$$

These forms are not essentially distinct, since the second can be obtained from the first by a rotation about the origin through the angle π. For Type 2, we have that

$$q_h(z) = \zeta_h = z^h + z^{-h}, \qquad m(z) = 1.$$

For Type 3,

$$q_h(z) = \sum_{j=0}^{h} (1 + (-1)^{h-j})\zeta_j/2,$$

$$m(z) = (z^2 - 1)(R^4 - z^2)z^{-2}(R^4 + 1)^{-1}.$$

These three sets of functions are readily identified as those already known (**4**) to be the transforms of the three sets of polynomials which are orthogonal on all confocal ellipses.

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