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Heat Kernels of Lorentz Cones

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Abstract. We obtain an explicit formula for heat kernels of Lorentz cones, a family of classical symmetric cones. By this formula, the heat kernel of a Lorentz cone is expressed by a function of time t and two eigenvalues of an element in the cone. We obtain also upper and lower bounds for the heat kernels of Lorentz cones.

The irreducible symmetric cones, or equivalently the corresponding simple formally real Jordan algebras, were classified in 1934 by Jordan, von Neumann, and Wigner [7] into four families of classical cones together with a single exceptional cone. They are $\Pi_r(\mathrm{R})$, $\Pi_r(\mathrm{C})$, $\Pi_r(\mathrm{H})$, the cones of all $r \times r$ positive definite matrices over R, C, H, the Lorentz cones Λ_n, and $\Pi_3(\mathrm{O})$, the cone of all 3×3 positive definite matrices over the algebra O of octonions (*cf.* [4] or [3]). As summarized in a comprehensive research monograph [4], one important analysis problem on symmetric cones is to construct various kernels, among which the heat kernels provide important analytic and geometric informations of these cones.

[8] and [13] give an explicit formula for heat kernels of symmetric cones $\Pi_r(\mathrm{C})$. [12] gives an explicit formula for heat kernels of $\Pi_r(\mathrm{H})$. [9] gives an explicit formula for heat kernels of $\Pi_r(\mathrm{R})$. [10] and [11] prove the Anker's conjecture [1] about the growth of the heat kernels on symmetric spaces of noncompact type for $\Pi_r(\mathrm{H})$, $\Pi_3(\mathrm{O})$ and $\Pi_r(\mathrm{R})$. To complete this study, we give in this note an explicit formula for heat kernels of Lorentz cones Λ_n, another family of classical symmetric cones mentioned in the first paragraph, and prove the Anker's conjecture for these cones.

As well known (*cf.* [4]), the Lorentz cone Λ_n is defined by

$$\Lambda_n = \{x \in \mathrm{R}^n : x_1^2 - x_2^2 - \cdots - x_n^2 > 0,\ x_1 > 0\}, \tag{1}$$

where $n \geq 2$. $G = \mathrm{R}_+ \times \mathrm{SO}_0(1, n-1)$ is the automorphism group of Λ_n and $K = \mathrm{SO}(n-1)$ is the maximal compact subgroup of G. Let $p = (p_1, \ldots, p_n) \in \Lambda_n$, $x = (x_1, \ldots, x_n) \in \Lambda_n$. Since G acts on Λ_n transitively, there is $g \in G$ such that $gp = I = (1, 0, \ldots, 0)$. Denote $z = gx$. Since the rank of Λ_n is 2, z has a spectral decomposition $z = \lambda_1 c_1 + \lambda_2 c_2$, where the eigenvalues λ_1 and λ_2 of z depend only on $p, x \in \Lambda_n$ and are independent of choices of $g \in G$. Setting $\lambda_1 = \exp r_1$ and $\lambda_2 = \exp r_2$, $r_1 = r_1(p, x)$ and $r_2 = r_2(p, x)$ are unique in the sense $r_1 \geq r_2$.

Recall that the *heat kernel* of Λ_n is a function $H\colon \mathrm{R} \times \mathrm{R} \times \mathrm{R}_+ \to \mathrm{R}$ which satisfies the following conditions:

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(H_1) H is continuous in all three variables, is of class C^2 in the first two variables, and is of class C^1 in the third variable.

(H_2)

$$\frac{\partial H}{\partial t} = LH, \tag{2}$$

where L is the radial part of the Laplace-Beltrami operator

$$LH = \frac{1}{\omega^d}\left\{\frac{\partial}{\partial r_1}\left(\omega^d \frac{\partial H}{\partial r_1}\right) + \frac{\partial}{\partial r_2}\left(\omega^d \frac{\partial H}{\partial r_2}\right)\right\} \tag{3}$$

with $d = n-2$, and

$$\omega = \sinh \frac{1}{2}(r_1 - r_2). \tag{4}$$

(H_3) For any continuous function f on Λ_n with compact support,

$$\lim_{t\to 0^+} c\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H\big(r_1(I,x), r_2(I,x), t\big) f(r_1, r_2)|\omega|^d \, dr_1\, dr_2 = f(0,0), \tag{5}$$

where $c = \frac{2^{n-3}\pi^{\frac{n}{2}-1}\Gamma(\frac{n}{2})}{(n-2)!}$ (*cf.* [4, Sect. VI.2 and Exercise VI.3]).

The following lemma gives a heat kernel formula for a real hyperbolic space, and can be found in Section 5.7 of [2], or Theorem 1 and its corollary of [6].

Lemma *The function $H\colon \mathrm{R}\times\mathrm{R}\times\mathrm{R}_+ \to \mathrm{R}$ given by*

$$H(x,y,t) = (2\pi)^{-m}\exp(-m^2 t)\left(-\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^m \left((4\pi t)^{-\frac{1}{2}}\exp\left(-\frac{r^2}{4t}\right)\right) \tag{6}$$

for $d = 2m$ or

$$\begin{aligned} H(x,y,t) = (2\pi)^{-m}\exp\left(-\left(m-\frac{1}{2}\right)^2 t\right) 2\sqrt{2}\left(-\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^m \\ \int_0^{\infty}\left((4\pi t)^{-\frac{1}{2}}\exp\left(-\frac{s^2}{4t}\right)\right)_{\cosh s = \cosh r + u^2} du \end{aligned} \tag{7}$$

for $d = 2m-1$ satisfies the differential equation

$$\frac{\partial u(r,t)}{\partial t} = \frac{1}{\sinh^d r}\frac{\partial}{\partial r}\left(\sinh^d r \frac{\partial u(r,t)}{\partial r}\right), \tag{8}$$

where $r = x - y$ in (6) and (7). Moreover,

$$\lim_{t\to 0^+}\int_{-\infty}^{\infty} H(x,y,t) f(y)|\sinh^d y|\, dy = f(x). \tag{9}$$

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We now state and prove the heat kernel formula for Lorentz cones.

Theorem 1 *The heat kernel of the Lorentz cone* Λ_n *is*

$$H(p,x,t) = \frac{1}{c}(2\pi)^{-m}(4\pi t)^{-1}\exp\left(-\frac{1}{2}m^2 t\right)\exp\left(-\frac{(r_1+r_2)^2}{8t}\right) \times\left(-\frac{1}{\sinh\frac{1}{2}(r_1-r_2)}\left(\frac{\partial}{\partial r_1}-\frac{\partial}{\partial r_2}\right)\right)^m\left(\exp\left(-\frac{(r_1-r_2)^2}{8t}\right)\right) \tag{10}$$

for $n = 2m+2$ *or*

$$H(p,x,t) = \frac{1}{c}(2\pi)^{-m}(4\pi t)^{-1}\exp\left(-\frac{1}{2}\left(m-\frac{1}{2}\right)^2 t\right) \times\exp\left(-\frac{(r_1+r_2)^2}{8t}\right)2\sqrt{2}\left(-\frac{1}{\sinh\frac{1}{2}(r_1-r_2)}\left(\frac{\partial}{\partial r_1}-\frac{\partial}{\partial r_2}\right)\right)^m \int_0^\infty \exp\left(-\frac{y^2}{2t}\right)_{\cosh y=\cosh\frac{1}{2}(r_1-r_2)+u^2} du \tag{11}$$

for $n = 2m+1$, *where* $r_1 = r_1(p,x)$ *and* $r_2 = r_2(p,x)$ *are discussed above.*

Proof It is clear that the function H given by (10) or (11) satisfies condition (H_1). Let $s_1 = \frac{1}{2}(r_1+r_2)$, $s_2 = \frac{1}{2}(r_1-r_2)$. It follows from (3) that

$$L = \frac{1}{2}\frac{\partial^2}{\partial s_1^2} + \frac{1}{2\omega^d}\frac{\partial}{\partial s_2}\left(\omega^d\frac{\partial}{\partial s_2}\right), \tag{12}$$

where $\omega = \sinh s_2$. By a well known formula for the heat kernel of R, the Lemma, and a variable change, the function $H\colon \mathrm{R}\times\mathrm{R}\times\mathrm{R}_+\to\mathrm{R}$ given by

$$H(s_1,s_2,t) = \frac{1}{2c}(2\pi)^{-m}(2\pi t)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}m^2 t\right)\exp\left(-\frac{s_1^2}{2t}\right) \times\left(-\frac{1}{\sinh s_2}\frac{\partial}{\partial s_2}\right)^m\left((2\pi t)^{-\frac{1}{2}}\exp\left(-\frac{s_2^2}{2t}\right)\right) \tag{13}$$

for $n = 2m+2$ or

$$H(s_1,s_2,t) = \frac{1}{2c}(2\pi)^{-m}(2\pi t)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}\left(m-\frac{1}{2}\right)^2 t\right) \times\exp\left(-\frac{s_1^2}{2t}\right)2\sqrt{2}\left(-\frac{1}{\sinh s_2}\frac{\partial}{\partial s_2}\right)^m \int_0^\infty\left((2\pi t)^{-\frac{1}{2}}\exp\left(-\frac{y^2}{2t}\right)\right)_{\cosh y=\cosh s_2+u^2} du \tag{14}$$

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for $n = 2m + 1$ satisfies the differential equation

$$\frac{\partial H}{\partial t} = LH = \frac{1}{2}\frac{\partial^2 H}{\partial s_1^2} + \frac{1}{2\omega^d}\frac{\partial}{\partial s_2}\left(\omega^d \frac{\partial H}{\partial s_2}\right). \tag{15}$$

Substituting back to r_1 and r_2, we obtain that the function H given by (10) or (11) satisfies (2).

Similarily, (5) follows from (9) and the proof of the theorem is completed. ∎

Remark I would like to thank the referee for the following simplifying observation: Any symmetric cone V is a direct product as a Riemannian space of R and the subspace $V_1 = \{x \in V, \det x = 1\}$. (One has to write $x = e^{s_1} x_1$ with $s_1 \in$ R, $x_1 \in V_1$.) In the case of the Lorentz cone V_1 is a real hyperbolic space; so the equation (12) is easily understood in that way. In (12), the term $\frac{1}{2}\frac{\partial^2}{\partial s_1^2}$ is the component due to R and the term $\frac{1}{2\omega^d}\frac{\partial}{\partial s_2}\left(\omega^d \frac{\partial}{\partial s_2}\right)$ the contribution from the hyperbolic space. Because of (12) or (15), the heat kernel of Λ_n given by (10) or (11) is the product of heat kernels of R and of a real hyperbolic space.

In [1], J.-Ph. Anker gives an upper bound formula for the heat kernels of the symmetric spaces $\mathrm{U}(p,q)/\,\mathrm{U}(p) \times \mathrm{U}(q)$. Anker then conjecture that this upper bound holds for all symmetric spaces of noncompact type. As pointed out in the Remark above, a symmetric cone is a direct product of R and a symmetric space of non-compact type. The following theorem follows from Theorem 1 above and Theorem 5.7.2 of [2] directly, and implies the Anker's conjecture for Lorentz cones.

Theorem 2 *For all $n \geq 2$, there exists a positive constant c_n such that*

$$c_n^{-1} h_n(r_1, r_2, t) \leq H(p, x, t) \leq c_n h_n(r_1, r_2, t), \tag{16}$$

for all $t > 0$, where $H(p,x,t)$ is the heat kernel of the Lorentz cone Λ_n given by (10) or (11), $r_1 = r_1(p,x)$ and $r_2 = r_2(p,x)$ are discussed above, and

(17)
$$h_n(r_1, r_2, t) = (4\pi t)^{-n/2} \exp\bigl(-(n-2)^2 t/4 - (n-2)(r_1 - r_2)/(2\sqrt{2}) - (r_1^2 + r_2^2)/(4t)\bigr) \times \bigl(1 + (r_1 - r_2)/\sqrt{2} + t\bigr)^{n/2-2}(1 + (r_1 - r_2)/\sqrt{2}).$$

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