

COMMUTATORS IN BANACH ALGEBRAS

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In a recent paper (6) the present author has shown that, for an element a of a Banach algebra A , the condition

$$|ax|_{\sigma} \leq \alpha |x|_{\sigma}$$

for all $x \in A$ and some constant α is equivalent to $[x, a] \in \text{Rad } a$ for all $x \in A$; it turns out that α may be replaced by $|a|_{\sigma}$. It is the purpose of the present note to investigate a related condition

$$|xa - xa|_{\sigma} \leq \alpha |x|_{\sigma}.$$

Here a is a fixed element of a Banach algebra A , α is a constant and the inequality is supposed to be satisfied for all $x \in A$. We intend to describe here the relation of this condition to the behaviour of $[[x, a]a]$. This note is divided into three parts. In the first section we collect some known facts to be used in Section two. Section three is purely algebraic and the main tool is the Jacobson density theorem for strictly irreducible representations of Banach algebras.

The idea of applying the density theorem to the study of commutativity belongs to C. Le Page (3). Recently a number of papers have appeared devoted to spectral characterisations of commutativity (1, 2, 5, 6, 12, 14).

1. Preliminaries

Let A be a Banach algebra. The spectral radius of an element $x \in A$ will be denoted by $|x|_{\sigma}$. An element $x \in A$ for which $|x|_{\sigma} = 0$ will be called quasinilpotent. The set of all quasinilpotent elements of A will be denoted by N . For $u, v \in A$ we write $[u, v]$ for $uv - vu$. Let $a \in A$ be fixed and denote by f the operator $f(x) = [a, x]$. Then f is a derivation on A , i.e. a linear operator which satisfies the following relation

$$f(xy) = f(x)y + xf(y). \tag{1}$$

It follows from this relation that the n -th iterate $f^{(n)}$ satisfies

$$f^{(n)}(xy) = \sum_{k=0}^n \binom{n}{k} f^{(n-k)}(x)f^{(k)}(y).$$

In particular, if $f^{(2)}(x) = 0$, we have, for $n \geq 2$

$$\begin{aligned} f^{(n)}(x^n) &= f^{(n)}(xx^{n-1}) = \sum_{k=0}^n \binom{n}{k} f^{(n-k)}(x)f^{(k)}(x^{n-1}) = \\ &= nf(x)f^{(n-1)}(x^{n-1}) + xf^{(n)}(x^{n-1}). \end{aligned}$$

This relation makes it possible to prove, by induction, the formula

$$f^{(n)}(x^n) = n!f(x)^n \tag{2}$$

(under the assumption $f^{(2)}(x) = 0$, of course).

Assuming (2) for $n - 1$, we have

$$f^{(n)}(x^{n-1}) = f(f^{(n-1)}(x^{n-1})) = (n - 1)!f(f(x)^{n-1}) = 0$$

since each summand in the expression for $f(f(x)^{n-1})$ contains one factor $f^{(2)}(x)$. It follows from this formula that

$$f^{(2)}(x) = 0 \text{ implies } |f(x)|_\sigma = \lim |f(x)^n|^{1/n} = \lim (1/n!)^{1/n} \\ |f^{(n)}(x^n)|^{1/n} \leq \limsup (1/n!)^{1/n} (|f^n|)^{1/n} = 0.$$

This is the classical result of Širokov (10) obtained also by Kleinecke (10). Another equivalent approach to these questions consists of using the exponential function (1, 2, 7, 8). This is based on the formula

$$f^{(n)}(x) = \sum_{r=0}^n \binom{n}{r} (-1)^r a^r x a^{n-r}. \tag{3}$$

Using this formula, we obtain

$$e^{-\lambda a} x e^{\lambda a} = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{1}{r!} \frac{1}{s!} \lambda^r (-1)^r a^r x \lambda^s a^s = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \sum_{r+s=n} \frac{n!}{r!s!} (-1)^r a^r x a^s = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n f^{(n)}(x)$$

in other words

$$\exp(-\lambda a) x \exp(\lambda a) = (\exp \lambda f)(x). \tag{4}$$

The result of Širokov-Kleinecke says that $f^{(2)}(x) = 0$ implies $f(x) \in N$. In the present note we shall be concerned with the weaker condition $f^{(2)}(x) \in N$.

2. The second commutator

In this section we intend to prove the following

Proposition 2.1 *Let A be a Banach algebra and let a be fixed element of A . Suppose there exists a number α such that*

$$|xa - ax|_\sigma \leq \alpha|x|_\sigma$$

for each $x \in A$. Then $[[x, a], a]_\sigma = 0$ for each $x \in A$.

Proof. Let $y \in A$ be given. Let D be the derivation on A defined by the formula $D(x) = [y, x]$. According to our assumption, we have the estimate

$$|[(\exp \lambda D)a, a]_\sigma \leq \alpha|(\exp \lambda D)a|_\sigma = \alpha|\exp(-\lambda y)a \exp(\lambda y)|_\sigma = \alpha|a|_\sigma.$$

Since

$$|(\exp \lambda D)a, a] = \left[a + \lambda Da + \frac{1}{2!} \lambda^2 D^2 a + \dots, a \right] = \lambda [Da, a] + \frac{1}{2!} \lambda^2 [D^2 a, a] + \dots$$

we have

$$[(\exp \lambda D)a, a] = \lambda g(\lambda)$$

where

$$g(\lambda) = [Da, a] + \frac{1}{2!} \lambda [D^2a, a] + \dots$$

is an entire function for which

$$|g(\lambda)|_\sigma \leq \frac{\alpha |a|_\sigma}{|\lambda|} \quad \text{for } \lambda \neq 0.$$

Since $\lambda \rightarrow g(\lambda)|_\sigma$ is a subharmonic function by Vesentini's theorem, $|g(\lambda)|_\sigma = 0$ for all λ . In particular $|[[y, a], a]|_\sigma = |[Da, a]|_\sigma = 0$. Since y was an arbitrary element of A , the proof is complete.

3. Algebraic results

In this section we obtain an algebraic description of the behaviour of the second commutator. For the sake of comparison we also include Proposition 3.1 which is based on ideas of Le Page. We then show that the conditions of Proposition 3.1 and those of 3.2 are not equivalent.

Proposition 3.1 *Let A be a unital Banach algebra and let a be an element of A . Then the following conditions are equivalent:*

1° *for each strictly irreducible representation T of A there exists a scalar λ_T such that*

$$T(a - \lambda_T) = 0$$

2° $xa - ax \in \text{Rad } A$ *for each* $x \in A$

3° $xa - ax \in N$ *for each* $x \in A$

Proof. If 1° is satisfied and $x \in A$ is given we have $T(xa - ax) = 0$ for each strictly irreducible representation T . It follows that $xa - ax \in \text{Rad } A$. This proves 2°. The implication 2° \rightarrow 3° is immediate. Now assume 3° and consider a strictly irreducible representation T . Suppose that 1° is not satisfied. Then there exists a vector u such that u and $T(a)u$ are linearly independent. By the Jacobson density theorem there exists an $x \in A$ such that $T(x)u = 0$ and $T(x)T(a)u = u$; it follows that

$$T(xa - ax)u = u$$

whence

$$|xa - ax|_\sigma \geq |T(xa - ax)|_\sigma \geq 1$$

which contradicts 3°. The proof is complete.

Proposition 3.2 *Let A be a unital Banach algebra and let a be a fixed element of A . Then the following conditions are equivalent*

1° for each strictly irreducible representation T of A there exists a scalar λ_T such that

$$T((a - \lambda_T)^2) = 0$$

2° $[[x, a], a]^2 \in \text{Rad } A$ for each $x \in A$.

3° $[[x, a], a] \in N$ for each $x \in A$.

Proof. Assume 1° and let $x \in A$ be given. Write c for $[[x, a], a]$. We intend to show that $T(c^2) = 0$ for each strictly irreducible representation T of A . Set $b = a - \lambda_T$ so that $T(b^2) = 0$. Since $c = [[x, b], b]$ we have $T(c) = -2T(bxb)$ whence $T(c^2) = 0$. This proves condition 2°.

The implication 2° \rightarrow 3° is immediate.

Now assume 3°. Consider a fixed strictly irreducible representation T of the algebra A . Let us show first that there exists a polynomial p of degree not exceeding 2 such that $p(T(a)) = 0$. Indeed, suppose not. It follows that the operators $T(1)$, $T(a)$, $T(a^2)$ are linearly independent so that there exists a vector u such that the vectors u , $T(a)u$, $T(a^2)u$ are linearly independent. Hence there exists an $x \in A$ for which

$$T(x)T(a^2)u = u, \quad T(x)T(a)u = 0, \quad T(x)u = 0$$

It follows that

$$T(xa^2 - 2axa + a^2x)u = u$$

whence

$$|xa^2 - 2axa + a^2x|_\sigma \geq |T(xa^2 - 2axa + a^2x)|_\sigma \geq 1;$$

this is a contradiction which proves the existence of p . Let p be a polynomial of minimal degree for which $p(T(a)) = 0$. If p is linear, we have $T(a) - \lambda = 0$ for some λ whence $T((a - \lambda)^2) = (T(a) - \lambda)^2 = 0$.

Now consider the case where p is quadratic. Write p in the form

$$p(t) = (t - \lambda)^2 - \alpha$$

and let us prove that $\alpha = 0$. Since p is quadratic, it follows that $T(a) - \lambda \neq 0$ so that there exists a vector u for which u and $T(a)u$ are linearly independent. Let $x \in A$ be such that $T(x)u = 0$ and $T(x)T(a)u = T(a - \lambda)u$. It follows that $T(x(a - \lambda))u = T(a - \lambda)u$.

Now

$$[[x, a], a] = [[x, a - \lambda], a - \lambda] = x(a - \lambda)^2 - 2(a - \lambda)x(a - \lambda) + (a - \lambda)^2x$$

whence

$$T([[x, a], a]) = \alpha T(x) - 2T((a - \lambda)x(a - \lambda)) + \alpha T(x)$$

This implies

$$T([[x, a], a])u = -2T((a - \lambda)x(a - \lambda))u = -2T(a - \lambda)T(a - \lambda)u = -2\alpha u.$$

It follows that $0 = \|[[x, a], a]\|_\sigma \geq |T([[x, a], a])|_\sigma \geq 2|\alpha|$ so that $\alpha = 0$. The proof is complete.

Proposition 3.3. *There exists a Hilbert space H and an element $a \in B(H)$ such that $[[x, a], a] \in N(B(H))$ for all $x \in B(H)$ but $[[x_0, a], a] \neq 0$ for a certain x_0 .*

Proof. A simple example can be constructed.

It follows that condition 3° of Proposition 3.2 does not imply $[[x, a], a] \in \text{Rad } A$ in general. In particular, condition 1° of 3.2 does not imply condition 1° of 3.1.

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