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ON PARTITIONS OF NONNEGATIVE INTEGERS AND REPRESENTATION FUNCTIONS

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Abstract

Let $\mathbb{N}$ be the set of all nonnegative integers. For any set $A \subset \mathbb{N}$, let $R(A, n)$ denote the number of representations of n as $n = a + a'$ with $a, a' \in A$. There is no partition $\mathbb{N} = A \cup B$ such that $R(A, n) = R(B, n)$ for all sufficiently large integers n. We prove that a partition $\mathbb{N} = A \cup B$ satisfies $|R(A, n) - R(B, n)| \le 1$ for all nonnegative integers n if and only if, for each nonnegative integer m, exactly one of $2m + 1$ and $2m$ is in A.

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1. Introduction

Let $\mathbb{N}$ be the set of all nonnegative integers. For any set $A \subset \mathbb{N}$, let

$$R_1(A, n) = |\{(a, a') \in A \times A : n = a + a'\}|,$$
$$R_2(A, n) = |\{(a, a') \in A \times A : n = a + a', a < a'\}|,$$
$$R_3(A, n) = |\{(a, a') \in A \times A : n = a + a', a \le a'\}|.$$

In each case $i \in \{1, 2, 3\}$, Sárközy asked if there exist two subsets A, B of $\mathbb{N}$ with $|(A \cup B)\backslash(A \cap B)| = \infty$ such that $R_i(A, n) = R_i(B, n)$ for all sufficiently large integers n. Using the properties of the Thue–Morse sequence, the following results have been proved.

THEOREM A [2]. *The set of positive integers can be partitioned into two subsets A and B such that $R_2(A, n) = R_2(B, n)$ for all $n \ge 0$.*

THEOREM B [1]. *The set of positive integers can be partitioned into two subsets A and B such that $R_3(A, n) = R_3(B, n)$ for all $n \ge 3$.*

Hence the answer is positive for $i \in \{2, 3\}$. For $i = 1$, however, Dombi [2] showed that the answer is negative. It is clear that, for any integer n, $R_1(A, 2n)$ is odd

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if $n \in A$; otherwise, $R_1(A, 2n)$ is even. Thus $R_1(A, 2n) \neq R_1(B, 2n)$ for all integers $n \in (A \cup B) \backslash (A \cap B)$. There are many other related results (see [3–6] and the references therein).

In this paper, we say that $\mathbb{N} = A \cup B$ is a partition if $\mathbb{N} = A \cup B$ and $A \cap B = \emptyset$ and simply write $R_1(A, n) = R(A, n)$. We obtain the following result.

THEOREM 1.1. *Let $\mathbb{N} = A \cup B$ be a partition. The inequality $|R(A, n) - R(B, n)| \leq 1$ holds for all nonnegative integers n if and only if, for each nonnegative integer m, exactly one of $2m + 1$ and $2m$ is in A.*

2. Proof of Theorem 1.1

Let $\mathbb{N} = A \cup B$ be a partition. Without loss of generality, we may assume that $0 \in A$. Define

$$d(x) = \sum_{n=0}^{\infty} (R(A, n) - R(B, n))x^n = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{Z}[x].$$

Then $|R(A, n) - R(B, n)| \leq 1$ is equivalent to $a_n \in \{-1, 0, 1\}$. Let $\chi(n) = 1$ if $n \in A$; otherwise, $\chi(n) = 0$. Let

$$f(x) = \sum_{a \in A} x^a = 1 + \sum_{n=1}^{\infty} \chi(n)x^n. \tag{2.1}$$

Then

$$\sum_{n=0}^{\infty} R(A, n)x^n = f^2(x)$$

and

$$\sum_{n=0}^{\infty} R(B, n)x^n = \left(\frac{1}{1 - x} - f(x)\right)^2.$$

It follows that

$$d(x) = f^2(x) - \left(\frac{1}{1 - x} - f(x)\right)^2 = \frac{2f(x)}{1 - x} - \frac{1}{(1 - x)^2}.$$

Hence

$$f(x) = \frac{1}{2}\left(d(x)(1 - x) + \frac{1}{1 - x}\right) = \frac{1}{2}\left(1 + a_0 + \sum_{n=1}^{\infty} (a_n - a_{n-1} + 1)x^n\right). \tag{2.2}$$

Comparing (2.1) and (2.2),

$$a_0 = 1 \tag{2.3}$$

and

$$\chi(n) = \frac{a_n - a_{n-1} + 1}{2} \quad \text{for all } n \geq 1. \tag{2.4}$$

Thus

$$2 \nmid a_n - a_{n-1} \quad \text{for all } n \geq 1. \tag{2.5}$$

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Let $a_{-1} = 0$. By (2.3) and (2.4),

$$\chi(2m+1) + \chi(2m) = \frac{a_{2m+1} - a_{2m-1}}{2} + 1 \quad \text{for all } m \geq 0.$$

Hence $\chi(2m+1) + \chi(2m) = 1$ is equivalent to $a_{2m+1} - a_{2m-1} = 0$. Further, note that $|R(A,n) - R(B,n)| \leq 1$ is equivalent to $a_n \in \{-1, 0, 1\}$. Hence it is enough to prove that $a_n \in \{-1, 0, 1\}$ for $n \geq 0$ is equivalent to $a_{2m+1} - a_{2m-1} = 0$ for $m \geq 0$.

Suppose that $a_n \in \{-1, 0, 1\}$ for $n \geq 0$. It follows from (2.3) and (2.5) that $a_{2m+1} = 0$ for $m \geq 0$. Then $a_{2m+1} - a_{2m-1} = 0$ for $m \geq 0$.

Suppose that $a_{2m+1} - a_{2m-1} = 0$ for $m \geq 0$. Since $a_{-1} = 0$,

$$a_{2m+1} = 0 \quad \text{for all } m \geq 0. \tag{2.6}$$

But $\chi(n) \in \{0, 1\}$ and it follows from (2.4) that

$$a_n - a_{n-1} \in \{-1, 1\} \quad \text{for all } n \geq 1.$$

Then

$$-a_{2m} = a_{2m+1} - a_{2m} \in \{-1, 1\} \quad \text{for all } m \geq 0.$$

Hence

$$a_{2m} \in \{-1, 1\} \quad \text{for all } m \geq 0. \tag{2.7}$$

It follows from (2.6) and (2.7) that $a_n \in \{-1, 0, 1\}$ for $n \geq 0$. This completes the proof.

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