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ON SS-SUPPLEMENTED SUBGROUPS OF FINITE GROUPS AND THEIR PROPERTIES

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Abstract. A subgroup H of a finite group G is called SS-supplemented in G if there exists a subgroup K of G such that $HK = G$ and $H \cap K$ is S-quasinormal in K. In this paper, we characterize the finite groups in which every subgroup is SS-supplemented and the influence of SS-supplementation of some subgroups on the structure of finite groups is considered. Some recent results on SS-quasinormal subgroups and C-supplemented subgroups are strengthened and enriched.

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1. Introduction. All groups considered in this paper are finite.

A group G is said to be factorized into its subgroups A and B if G is the product of A and B. Obviously, the structure of the factorized group $G = AB$ is restricted by its subgroups A and B. There has been interest in the past in investigating the structure of the factorized group $G = AB$ by means of the structure of A and B. For instance, in 1955, Itô found an impressive and very satisfying theorem. He proved in [**17**] that $G = AB$ is a metabelian group if A and B are abelian. The most famous theorem of this type was due to Kegel (see [**18**]) and Wielandt (see [**28**, **29**]) as they stated the solvability of the factorized group $G = AB$ if A and B are both nilpotent. It is also well-known that the group $G = AB$ is nilpotent if A and B are both normal nilpotent subgroups of G. However, it is known that the factorized group $G = AB$ is not necessary supersolvable if both A and B are normal supersolvable subgroups of G (see [**3**]). Thus, the following interesting question arises:

Let $\mathcal{F}$ be a formation (may be a saturated formation). What will be the conditions needed for the subgroups A and B so that the factorized group $G = AB \in \mathcal{F}$ when both A and B are in the formation $\mathcal{F}$?

In answering the above question, Asaad and Shaalan first proved a theorem in 1989 [**1**] by showing that if $G = HK$ is a factorized group with supersolvable subgroups H and K such that every subgroup of H is permutable with every subgroup of K, then G is supersolvable. In 1992, Maier in [**23**] further proved that the above result can also be obtained by considering the general completeness property of all saturated formations containing the class of supersolvable groups. Along this direction, Ballester-Bolinches and some others have investigated the totally permutable products and the mutually permutable products of finite groups, and consequently many interesting results have been given (for example, see [**4**, **6**]).

Motivated by the above results, we now call a subgroup H of a group G SS-supplemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K$ is an S-quasinormal subgroup in K. In this case, the subgroup K is said to be an SS-supplement of H in G.

Recall that a subgroup H of a group G is S-quasinormal in G if H permutes with every Sylow subgroup of G. After the introduction of the above concept by Kegel (see [**19**]), the structure of a group has been extensively investigated under some additional assumptions on the subgroups of a given group (see [**2**, **24**]). On the other hand, a subgroup H of a group G is called a complemented subgroup of G if there exists another subgroup K of G such that $G = HK$ and $H \cap K = 1$. By using the concept of complemented subgroups, Hall established a fundamental theorem for solvable groups in [**14**] by proving that a group G is solvable if and only if every Sylow subgroup is complemented. Recently, the authors have also investigated the finite p-nilpotent groups with some subgroups c-supplemented in [**13**]. Research on the complemented subgroups of a given group still continues and many related results have been recently obtained (see [**5**, **11**, **12**]).

In this paper, we first describe the relationship between the SS-supplemented subgroups and the complemented subgroups or S-quasinormal subgroups of a given group G. Next, we study the structure of the finite groups whose subgroups are SS-supplemented. Some applications of our results are considered so that a number of related results in the literature are extended and generalized.

2. Preliminaries. In this section, we first discuss the properties of SS-supplemented subgroups and give some lemmas which will be used in the sequel. For the sake of convenience, we recall that a subgroup H of a group G is C-supplemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K \leq H_G$ (see [**7**]), where H_G is the core of H in G. It is obvious that a subgroup H of a group G is C-supplemented in G if and only if there exists a subgroup K_1 of G such that $G = HK_1$ and $H \cap K_1 = H_G$. Hence, the concept of C-supplemented subgroups can be regarded as a generalization of both C-normal subgroups and complemented subgroups; therefore, it is worthwhile to investigate the structure of a group by considering its C-supplemented subgroups. On the other hand, we recall a new concept (see [**21**]), which is a generalization of S-quasinormality. A subgroup H of a group G is called to be SS-quasinormal in G if there is a subgroup K of G such that $G = HK$ and H permutes with every Sylow subgroup of K. Many interesting results on SS-quasinormality of a group have been recently given by Li and others (for instance, see [**21**, **22**]).

Definition 2.1. A subgroup H of a group G is said to be SS-supplemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K$ is S-quasinormal in K. In this case, we say that K is an SS-supplement of H in G.

It is clear that a C-supplemented subgroup of a group G must be SS-supplemented in G. We now assume that H is a SS-quasinormal subgroup of a group G. Then, there exists a subgroup K of G such that $G = HK$ and H permutes with every Sylow subgroup of K. Let P be a Sylow subgroup of K. Then, by $HP = PH$, we deduce that $(H \cap K)P = P(K \cap H)$. This shows that H must be SS-supplemented in G. On the other hand, a SS-quasinormal subgroup of a group may not be C-supplemented and a

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C-supplemented subgroup of a group may not be SS-quasinormal (see Example 2.2). Furthermore, the following Example 2.3 illustrates that a SS-supplemented subgroup of a group may be neither C-supplemented nor SS-quasinormal. Hence the class of all SS-supplemented subgroups in a group contains properly both the class of all C-supplemented subgroups and the class of all SS-quasinormal subgroups in the group.

EXAMPLE 2.2. Let $G = S_4$ be the symmetric group of degree 4 and let $H = \langle(34)\rangle$. Then, H is C-supplemented in G since $G = HA_4$ and $H \cap A_4 = 1$. However, H is not SS-quasinormal in G because $HP \neq PH$ when $P = \langle(123)\rangle$.

Let $P = \langle x, y : x^{16} = y^4 = 1, x^y = x^3\rangle$. Then, $\Phi(P) = \langle x^2, y^2\rangle = \langle x^2\rangle \times \langle y^2\rangle$. It is easy to see that $H = \langle y^2\rangle$ is S-quasinormal in P and so SS-quasinormal in P. However, H is not C-supplemented in P.

EXAMPLE 2.3. Let G be the direct product of S_4 and P with S_4 and P as in Example 2.2. Now let $H = C_2 \times P_1$, $K = A_4 \times P$, where $C_2 = \langle(34)\rangle$, $P_1 = \langle y^2\rangle$ and A_4 is the alternating group of degree 4. Then, $G = HK$ and $H \cap K$ is S-quasinormal in K since $H \cap K \cong P_1$. Hence, H is SS-supplemented in G. However, H is neither C-supplemented nor SS-quasinormal in G.

We now give some basic properties of SS-supplemented subgroups.

LEMMA 2.4. *Let G be a group and H an SS-supplemented subgroup of G. Then, the following statements hold:*

(1) *If M is a subgroup of G and $H \leq M$, then H is SS-supplemented in M.*

(2) *If N is a normal subgroup of G and $N \leq H$, then H/N is SS-supplemented in G/N.*

(3) *Let π be a set of primes. If H is a π-subgroup of G and N is a normal π'-subgroup of G, then HN/N is SS-supplemented in G/N.*

(4) *If L is a subgroup of G and $H \leq \Phi(L)$, then H is S-quasinormal in G.*

Proof. By the hypotheses, there exists $K \leq G$ such that $HK = G$ and $H \cap K$ is S-quasinormal in K. Let $K_1 = M \cap K$. Then, $M = HK_1$ and $H \cap K_1 = H \cap K$ is S-quasinormal in K_1. This shows that H is SS-supplemented in M and thus (1) is proved.

It follows from $G = HK$ that $H/N \cdot KN/N = G/N$. By using the well-known Dedekind identity, we have $H/N \cap KN/N = N(H \cap K)/N$. For any prime number p, it is well known that any Sylow p-subgroup of KN/N has the form K_pN/N, where K_p is a Sylow p-subgroup of K. Thus, $(H/N \cap KN/N)(K_pN/N)$ is a subgroup of G/N since $(H \cap K)K_p$ is a subgroup of G. This implies that $H/N \cap KN/N$ is S-quasinormal in KN/N. Therefore, H/N is SS-supplemented in G/N and (2) is proved.

Since $(|N|, |H|) = 1$, $N \leq K$ and $NH \cap K = N(H \cap K)$. This shows that $NH \cap K$ is S-quasinormal in K, and hence, NH is SS-supplemented in G. By (2), HN/N is SS-supplemented in G/N and (3) follows.

Since $H \leq \Phi(L)$, $L = H(L \cap K) = L \cap K$. It follows that $K = G$ and H is S-quasinormal in G. Thus, (4) holds and the proof is completed. □

The following lemmas are known results of S-quasinormal subgroups of a given group G.

LEMMA 2.5. ([**19**]) *Let G be a group and $H \leq G$. If H is S-quasinormal in G, then H is subnormal in G.*

LEMMA 2.6. ([**24**]) *If H is a p-subgroup of a group G for some prime p, then H is S-quasinormal in G if and only if $O^p(G) \leq N_G(H)$.*

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LEMMA 2.7. ([**8**]) *If A is subnormal in G and B is a minimal normal subgroup of G, then $B \leq N_G(A)$.*

Recall that a class $\mathcal{F}$ of groups is called a formation if $G \in \mathcal{F}$ and $N \trianglelefteq G$ then $G/N \in \mathcal{F}$, and if $G/N_i \in \mathcal{F}$, $i = 1, 2$, then $G/N_1 \cap N_2 \in \mathcal{F}$. In addition, if $G/\Phi(G) \in \mathcal{F}$ implies $G \in \mathcal{F}$, then we call $\mathcal{F}$ a saturated formation. A well-known example of saturated formations is the class $\mathcal{U}$ of supersolvable groups.

Concerning saturated formations, we have the following known results.

LEMMA 2.8. ([**25**]) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, let G be a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If H is cyclic, then $G \in \mathcal{F}$.*

LEMMA 2.9. ([**26**]) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If for every maximal subgroup M of G, either $F(H) \leq M$ or $F(H) \cap M$ is a maximal subgroup of $F(H)$, then $G \in \mathcal{F}$.*

3. SS-supplemented subgroups of a group. A group G is said to be SS-supplemented if every subgroup of G is SS-supplemented in G. In this section, we first investigate the solvability of groups by using SS-supplemented subgroups and then the SS-supplemented group will hence be characterized.

THEOREM 3.1. *Let G be a group. Then, G is solvable if and only if every Sylow subgroup of G is SS-supplemented in G.*

Proof. If the given group G is solvable, then every Sylow subgroup of G is complemented and hence G is SS-supplemented.

Conversely, we assume that every Sylow subgroup P of G is SS-supplemented in G. Then, by definition, there exists $K \leq G$ such that $PK = G$ and $P \cap K$ is S-quasinormal in K. By Lemma 2.5, $P \cap K$ is subnormal in K. Note that since $P \cap K$ is a Sylow subgroup of K, we can easily see that $P \cap K$ is also a normal Sylow subgroup of K. By applying the Schur–Zassenhaus theorem [**9**, Theorem 6.2.1], we have $K = (P \cap K)K_{p'}$, where $K_{p'}$ is a Hall p'-subgroup of K. Now $G = PK = PK_{p'}$ and $P \cap K_{p'} = 1$. Hence P is complemented in G. The theorem is proved. □

By using the same arguments as in Theorem 3.1, we deduce the following corollary.

COROLLARY 3.2. *Let G be a group and H a Hall subgroup of G. Then H is complemented in G if and only if H is SS-supplemented in G.*

If we only assume that all maximal subgroups are SS-supplemented in a group G, then G need not be solvable. In fact, $L_2(7)$, $L_2(11)$ and $L_5(2)$ are nonabelian simple groups in which every maximal subgroup is complemented (see [**20**], main theorem). However, we have the following result.

THEOREM 3.3. *Let G be a group. Then, G is solvable if and only if every maximal subgroup of G has a subnormal SS-supplement in G.*

Proof. Let G be a solvable group and H a maximal subgroup of G. We now proceed to show that H has a subnormal SS-supplement in G. Assume that $H_G \neq 1$. Consider G/H_G. By using induction on $|G|$, we know that H/H_G has a subnormal SS-supplement K/H_G in G/H_G. Clearly, K is a subnormal SS-supplement of H in G. Assume that $H_G = 1$. Let N be a minimal normal subgroup of G. Then, $HN = G$ and $H \cap N \leq H_G = 1$. Hence, N is a normal SS-supplement of H in G.

Conversely, assume that the result is not true so that we can let G be a counterexample of minimal order. Consider a maximal subgroup H of G. Then there exists a subnormal subgroup K of G such that $HK = G$ and $H \cap K$ is S-quasinormal in K. If G is a nonabelian simple group, then $K = G$ since $H \neq G$. By Lemma 2.5, we know that H is subnormal in G and hence $H = 1$. It follows that G is solvable, which is a contradiction. Now, we let N be a minimal normal subgroup of G. Then, it is easy to see that the hypothesis is still true for the quotient group G/N. By the minimality of G, we infer that G/N is solvable. Furthermore, we may assume that N is the unique minimal normal subgroup of G and N is not contained in $\Phi(G)$. Then, in this case, we can let M be a maximal subgroup of G with $M_G = 1$. By our hypothesis, there exists a subnormal subgroup K of G such that $MK = G$ and $M \cap K$ is S-quasinormal in K. Since K is subnormal in G, Lemma 2.5 implies that $M \cap K$ is subnormal in G. Assume $M \cap K \neq 1$, then we may take a minimal subnormal subgroup L of G contained in $M \cap K$. Since $L \cap N \trianglelefteq L$, either $L \cap N = 1$ or $L \leq N$. By Lemma 2.7, N normalizes L. If $L \cap N = 1$, it follows that $NL = N \times L$ and $L \leq C_G(N) = 1$. Suppose $L \leq N$, then $L^G = L^{NM} = L^M \leq M_G = 1$. We also get $L = 1$, a contradiction. Hence $M \cap K = 1$. By using the same arguments, we can similarly prove that all minimal subnormal subgroups of G are contained in N. Let $N = N_1 \times \cdots \times N_r$, where each N_i is isomorphic to a fixed nonabelian simple group. Then, it is easy to see that $N_1, \ldots, N_r$ coincide with all minimal subnormal subgroups of G. Without loss of generality, we may assume that $N_1 \leq K$. Then, there exists a prime p such that p divides $|G : M| = |K|$. By applying [**3**, Lemma 3, P.121], we obtain that N is solvable, a contradiction. The proof is now completed. □

The following corollary is a direct consequence of Theorem 3.3.

Corollary 3.4. ([**19**]) *A group G is solvable if and only if for every maximal subgroup M of G, there exists a subnormal subgroup K of G such that $G = MK$ and $M \cap K \leq M_G$.*

Remark. From the proof it can be noted that Theorem 3.3 is also valid if 'subnormal' is replaced by 'normal'. The same is valid for Corollary 3.4.

If a group G has a solvable maximal subgroup M such that M is SS-supplemented in G, then G need not be solvable, for instance, A_5. However, we have the following result.

Theorem 3.5. *Let G be a group. Then, G is solvable if and only if G has a solvable maximal subgroup H such that H has a normal SS-supplement K in G.*

Proof. If G is solvable, then G has a normal maximal subgroup H. It is easy to see that H has a normal SS-supplement K in G, namely G. Conversely, assume that the theorem is not true. Then, we let G be a counterexample of minimum order. If $H_G \neq 1$, then H/H_G is a solvable maximal subgroup of G/H_G and KH_G/H_G is a normal SS-supplement of H/H_G in G/H_G. The choice of G implies that G/H_G is solvable and therefore G is solvable, a contradiction. Hence, $H_G = 1$. Let N be a minimal normal subgroup of G and $C = C_G(N)$. Then, it follows from [**8**, A, 17.2] that either N is the unique minimal normal subgroup of G and $C \leq N$ or G has precisely two minimal normal subgroups N and R so that $N \simeq R$ is nonabelian, and hence, $R = C$ and $N \cap H = 1 = R \cap H$. By our hypotheses, we deduce that $H \cap K$ is S-quasinormal in K and therefore, by Lemma 2.5, we know that $H \cap K$ is subnormal in K and is hence in G. Now, assume that $H \cap K \neq 1$ and let L be a minimal subnormal subgroup of G contained in $H \cap K$. If $L \leq N$, then $L^G = L^{NH} = L^H \leq H_G = 1$, a contradiction.

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This shows that L is not contained in N and L is analogously not contained in R. It hence follows that $N \cap L = 1 = R \cap L$. On the other hand, by Lemma 2.7, we have $NL = N \times L$ and therefore $L \leq C$, which contradicts $C \leq N$ or $C = R$. Hence, we conclude that $H \cap K = 1$. This implies that $G = [K]H$ and K is a minimal normal subgroup of G.

Now, we let T be a minimal normal subgroup of H. Then, T is clearly an elementary abelian p-group for some $p \in \pi(H)$. Since $C_K(T)$ is normalized by both H and K, we know that $C_K(T) \trianglelefteq G$. If $C_K(T) = K$, then $T \leq H_G$, a contradiction. Hence, $C_K(T) = 1$. It now follows from [**9**, Theorem 6.2.2] that K is a p'-group. By [9, Theorem 6.2.3], K contains a unique T-invariant Sylow q-subgroup Q for every prime $q \in \pi(K)$. For any $h \in H$, we have $(Q^h)^T = (Q^T)^h = Q^h$, that is, Q^h is also a T-invariant Sylow q-subgroup of K, and thereby $Q = Q^h$. Consequently, we have $[Q]H = G = [K]H$ and so $K = Q$ is a q-group. This implies that G is a solvable group, a contradiction. Thus, the proof is completed. □

We now characterize the SS-supplemented groups.

THEOREM 3.6. *Let G be a group. Then, the following statements are pairwise equivalent.*

(1) *G is an SS-supplemented group.*

(2) *G is supersolvable, every Sylow subgroup of $G/\Phi(G)$ is elementary abelian and every subgroup of $\Phi(G)$ is S-quasinormal in G.*

(3) *every subgroup of $G/\Phi(G)$ is complemented and every subgroup of $\Phi(G)$ is S-quasinormal in G.*

Proof. (1) $\Rightarrow$ (2). We first prove that G is supersolvable. By the hypotheses and Theorem 3.1, G is solvable. Let N be a minimal normal subgroup of G. Then, N is an elementary abelian p-group for some prime p. By Lemma 2.4(2), it is known that G/N is SS-supplemented and hence G/N is supersolvable by induction. It follows that in order to prove that G is supersolvable, it suffices to prove that $N = \langle x \rangle$ is cyclic. Let P be a Sylow p-subgroup of G and let $x \in N \cap Z(P)$ with $|x| = p$. Then, there exists $K \leq G$ such that $\langle x \rangle K = G$ and $\langle x \rangle \cap K$ is S-quasinormal in K. Since $\langle x \rangle \cap K$ is nomalized by all p'-elements of K and centralized by P, It follows that $\langle x \rangle \cap K$ is a normal subgroup of G. By minimality of N, $\langle x \rangle \cap K = 1$ or $N \leq K$. Assume that $\langle x \rangle \cap K = 1$. By order considerations, it follows that $N = \langle x \rangle$. Assume now that $N \leq K$. Then $\langle x \rangle = \langle x \rangle \cap K \leq N$ and so $N = \langle x \rangle$.

Let P be a Sylow p-subgroup of G and H is a subgroup of $\Phi(P)$. Then by Lemma 2.4(4), H is S-quasinormal in G. By Lemma 2.6, we deduce that $\Phi(P)$ is normal in G. Hence, $\Phi(P) \leq \Phi(G)$ and, therefore every Sylow subgroup of $G/\Phi(G)$ is elementary abelian. The last argument follows from Lemma 2.4(4).

(2)$\Rightarrow$ (3). This part follows from [**15**, Theorem 2].

(3)$\Rightarrow$ (1). Assume that every subgroup of $G/\Phi(G)$ is complemented and every subgroup of $\Phi(G)$ is S-quasinormal in G. Let H be a subgroup of G. Then, there exists a subgroup $K/\Phi(G)$ of $G/\Phi(G)$ such that $(H\Phi(G)/\Phi(G))(K/\Phi(G)) = G/\Phi(G)$ and $(H\Phi(G)/\Phi(G)) \cap (K/\Phi(G)) = (H \cap K)\Phi(G)/\Phi(G) = 1$. It follows that $HK = G$ and $H \cap K \leq \Phi(G)$. Hence, $H \cap K$ is S-quasinormal in G. By definition, H is SS-supplemented in G and hence G is an SS-supplemented group. The proof of theorem is now complete. □

4. Applications. In this section, we concentrate on the structure of a group under the assumption that some subgroups of Sylow subgroups are SS-supplemented. Many known results will be generalized. In our first result, the p-nilpotency of a group is studied.

THEOREM 4.1. *Let G be a group and let p be the smallest prime divisor of $|G|$. Let P be a Sylow p-subgroup of G. If every maximal subgroup of P is SS-supplemented in G, then G is p-nilpotent.*

Proof. Assume that the theorem is false and let G be a counterexample of minimal order. Then, it follows from [**16**, IV, 2.8] that P is not cyclic. Let P_1 be a maximal subgroup of P. Then, there exists $K \leq G$ such that $P_1K = G$ and $P_1 \cap K$ is S-quasinormal in K. It follows from Lemma 2.6 and $|P \cap K : P_1 \cap K| \leq p$ that $P_1 \cap K$ is normal in K. Applying [**16**, IV, 2.8] again, $K/P_1 \cap K$ is p-nilpotent with normal Hall p'-subgroup $H/P_1 \cap K$. Then, by the Schur-Zassenhaus theorem [**9**, Theorem 6.2.1], we know that $P_1 \cap K$ has a p-complement M in H. By using the Frattini argument, we deduce that $K = HN_K(M) = (P_1 \cap K)N_K(M)$ and hence $G = P_1N_G(M)$. By the choice of G, it implies that $N_G(M) < G$ and $P \cap N_G(M) < P$. Now, choose a maximal subgroup P_2 of P such that $P \cap N_G(M) \leq P_2$. By repeating the above argument once again, we can show that there also exists $K_1 \leq G$ such that $P_2K_1 = G$ and $P_2 \cap K_1$ is S-quasinormal in K_1 and $G = P_2N_G(M_1)$, where M_1 is a Hall p'-subgroup of G. If $p = 2$, then by applying the Gross theorem [**10**, main theorem], we obtain that $M_1^g = M$ for some $g \in P$. If $p > 2$, then the odd order theorem implies the same conclusion. Therefore, $G = P_2N_G(M_1) = (P_2N_G(M_1))^g = P_2N_G(M)$. It follows that $P = P_2(P \cap N_G(M)) = P_2$, a contradiction. The proof is completed. □

THEOREM 4.2. *Let $\mathcal{F}$ be a saturated formation containing the class $\mathcal{U}$ of all supersoluble groups and H a normal subgroup of a group G such that $G/H \in \mathcal{F}$. If all maximal subgroups of every non-cyclic Sylow subgroup of H are SS-supplemented in G, then $G \in \mathcal{F}$.*

Proof. Let p be the smallest prime divisor of $|H|$ and P a Sylow p-subgroup of H. If P is cyclic, then by [**16**, IV, 2.8], H is p-nilpotent. If P is non-cyclic, then by Lemma 2.4 (1) and Theorem 4.1, we deduce that H is p-nilpotent. By using the same argument and induction, we may conclude that H is a Sylow tower group.

Now, let q be the largest prime dividing $|H|$ and Q a Sylow q-subgroup of H. Then, Q is normal in G. If Q_1 is a normal subgroup of G with $1 \neq Q_1 \leq Q$, then, by Lemma 2.4 (2) or (3), G/Q_1 satisfies the hypotheses of the theorem and therefore we have $G/Q_1 \in \mathcal{F}$, by induction. If $Q_1 \leq \Phi(G)$, then it follows from $G/Q_1 \in \mathcal{F}$ that $G \in \mathcal{F}$. Hence, in this case, we may assume that Q is not contained in $\Phi(G)$ and Q is a minimal normal subgroup of G. If Q is not a cyclic group, then we let $\{N_1, \ldots, N_t\}$ be the set of all maximal subgroups of Q. For each N_i, by the hypotheses, there exists $K_i \leq G$ such that $N_iK_i = G$ and $N_i \cap K_i$ is S-quasinormal in K_i. Hence, we have $Q = N_i(Q \cap K_i)$ and $Q \cap K_i \trianglelefteq G$. By the minimality of Q, we deduce that $Q \cap K_i = 1$ or $Q \leq K_i$. If $Q \cap K_i = 1$, then $Q = N_i$, a contradiction. Thus, $Q \leq K_i$ and so N_i is S-quasinormal in G. Now, Lemma 2.6 implies that $|G : N_G(N_i)| = q^k$ for some nonnegative integer k. It hence follows from [**16**, III, 8.5(d)] that some maximal subgroup of N is normal in G, which is a contradiction. This shows that Q is a cyclic group of order q. By Lemma 2.8, we conclude that $G \in \mathcal{F}$. The proof is completed. □

The following corollary follows immediately from Theorem 4.2.

COROLLARY 4.3. *Let N be a normal subgroup of a group G such that G/N is supersolvable. If every maximal subgroup of every Sylow subgroup of N is c-supplemented in G, then G is supersolvable.*

THEOREM 4.4. *Let $\mathcal{F}$ be a saturated formation containing the formation $\mathcal{U}$ of all supersoluble groups and H a solvable normal subgroup of a group G such that $G/H \in \mathcal{F}$. If all maximal subgroups of every Sylow subgroup of the Fitting subgroup $F(H)$ of H are SS-supplemented in G, then $G \in \mathcal{F}$.*

Proof. Let M be a maximal subgroup of G not containing $F(H)$. Then, by Lemma 2.9, it suffices to prove that $F(H) \cap M$ is maximal in $F(H)$. To proceed with the proof, let P be a Sylow p-subgroup of $F(H)$ not contained in M and let G_p be a Sylow p-subgroup of G. Then, $PM = G$ and $G_p \cap M < G_p$. Choose a maximal subgroup G_1 of G_p such that $G_p \cap M \leq G_1$ and let $P_1 = G_1 \cap P$. Then, P_1 is a maximal subgroup of P and $P_1 \cap M = P \cap M$. Now, we suppose that $P \cap \Phi(G) \neq 1$. Then, we can let N be a minimal normal subgroup of G contained in $P \cap \Phi(G)$. In this case, we have $F(H)/N = F(H/N)$ and G/N satisfies the hypotheses. By using induction, we know that $G/N \in \mathcal{F}$ and therefore $G \in \mathcal{F}$. Hence, we may assume that $P \cap \Phi(G) = 1$ and therefore $\Phi(P) = 1$. Thus, $P \cap M \trianglelefteq G$ and $P \cap M \leq (P_1)_G$. It hence follows that $(P_1)_G M < G$ and so $P \cap M = (P_1)_G$. By the hypotheses, there exists $K_1 \leq G$ such that $P_1 K_1 = G$ and $P_1 \cap K_1$ is S-quasinormal in K_1. If Q is a Sylow q-subgroup of K_1 with $q \neq p$ then it is clear that Q normalizes $P_1 \cap K_1$. On the other hand, since $PK_1 = G$ and P is abelian, we have that $P \cap K_1$ is normal in G. It follows from $G_p = PG_1$ that $P_1 \cap K_1 = G_1 \cap P \cap K_1$ is normalized by G_p. Therefore, we have $P_1 \cap K_1 \trianglelefteq G$ and $P_1 \cap K_1 \leq (P_1)_G$. Let $K = K_1(P_1)_G$. Then, $P_1 \cap K = (P_1)_G$. The maximality of M implies that $(P \cap K)M = M$ or $(P \cap K)M = G$. If $(P \cap K)M = M$, then $P \cap K \leq P \cap M = (P_1)_G$ and therefore $P \cap K = (P_1)_G = P_1 \cap K$. It follows that $P_1 = P$, a contradiction. Hence, $(P \cap K)M = G$. It follows that $P \cap K = P$ by order considerations and so $P \leq K$. This proves that $P_1 = P_1 \cap K = (P_1)_G = P \cap M$. Consequently, $|F(H) : F(H) \cap M| = |P : P \cap M| = p$ and $F(H) \cap M$ is maximal in $F(H)$, as required. □

COROLLARY 4.5. ([**12**]) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Let H be a solvable normal subgroup of a group G such that $G/H \in \mathcal{F}$. If all maximal subgroups of every Sylow subgroup of $F(H)$ are complemented in G, then $G \in \mathcal{F}$.*

Now we want to delete the solvability of H in the assumption of Theorem 4.4 by replacing $F(H)$ by $F^*(H)$, the generalized Fitting subgroup of H.

THEOREM 4.6. *Let G be a group with a normal subgroup H such that G/H is supersolvable. If every maximal subgroup of every Sylow subgroup of $F^*(H)$ is SS-supplemented in G, then G is supersolvable.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. Then, every proper normal subgroup of G containing $F^*(H)$ is supersolvable. In fact, let N be a proper normal subgroup of G containing $F^*(H)$. Then, $N/N \cap H \cong NH/H$ is supersolvable. Since $F^*(H) = F^*(F^*(H)) \leq F^*(H \cap N) \leq F^*(H)$, we see $F^*(H \cap N) = F^*(H)$. Hence, every maximal subgroup of every Sylow subgroup of $F^*(H \cap N)$ is SS-supplemented in G and therefore in N by Lemma 2.4(1). So, N with the normal subgroup $N \cap H$ satisfies the hypotheses of the theorem. The choice of G implies that N is supersolvable.

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If $H < G$, then H is supersolvable. In this case, $F^*(H) = F(H)$. Theorem 4.4 implies that G is supersolvable, a contradiction. Thus, $H = G$. If $F^*(G) = G$, then G is supersolvable by Theorem 4.2 for the special case $\mathcal{F} = U$, a contradiction. Thus, $F^*(G) < G$. By the above proof, $F^*(G)$ is supersolvable and so $F^*(G) = F(G)$.

Let P be a Sylow p-subgroup of $F(G)$. Suppose that $P \cap \Phi(G) \neq 1$, and let N be a minimal normal subgroup of G contained in $P \cap \Phi(G)$. Then, $F(G)/N = F(G/N)$ and G/N satisfies the hypotheses. By the minimality of G, G/N is supersolvable and so does G. Hence, $P \cap \Phi(G) = 1$, and therefore $\Phi(P) = 1$ and P is abelian.

Let P_1 be a maximal subgroup of P. Then, there exists $K \leq G$ such that $P_1K = G$ and $P_1 \cap K$ is S-quasinormal in K. Thus, $O^p(K) \leq N_G(P_1 \cap K)$ and so $P_1 \cap K \trianglelefteq PO^p(K)$. Obviously, $F(G) \leq PO^p(K)$. Assume that $PO^p(K) < G$. Then, $PO^p(K)$ is supersolvable. Since $PO^p(K) \trianglelefteq PK = G$ and $G/PO^p(K)$ is a p-group, G is solvable. By Theorem 4.4, G is supersolvable, a contradiction. Hence $PO^p(K) = G$ and $P_1 \cap K \trianglelefteq G$. Therefore, P_1 is C-supplemented in G. Now applying [**27**, Theorem 1.1], we get G is supersolvable, the final contradiction. The proof is hence completed. □

THEOREM 4.7. *Let $\mathcal{F}$ be a saturated formation containing the class $\mathcal{U}$ of all supersoluble groups and let G be a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If every maximal subgroup of every Sylow subgroup of $F^*(H)$ is SS-supplemented in G, then $G \in \mathcal{F}$.*

Proof. By Lemma 2.4(1), every maximal subgroup of every Sylow subgroup of $F^*(H)$ is SS-supplemented in H. Thus, H is supersolvable by Theorem 4.6. In particular, H is solvable and so $F^*(H) = F(H)$. Now Theorem 4.4 implies that $G \in \mathcal{F}$, as desired. □

THEOREM 4.8. *Let G be a group and p the smallest prime divisor of $|G|$. If every cyclic subgroup of G with order p and order 4 (if $p = 2$) is SS-supplemented in G, then G is p-nilpotent.*

Proof. Assume that the theorem is false and let G be a counterexample of minimal order. Then, by Lemma 2.4(1), G is a minimal non-p-nilpotent group (that is, G is not p-nilpotent but every proper subgroup of G is p-nilpotent). Now by invoking a known result of Itô [**16**, III, 5.4], we know that G is a minimal non-nilpotent group. According to a result of Schmidt in [**16**, III, 5.2], G has a normal Sylow p-subgroup P such that $G = PQ$ for a Sylow q-subgroup Q ($q \neq p$).

Let $P_0 \leq P$ with order p. Then, there exists $K \leq G$ such that $P_0K = G$ and $P_0 \cap K$ is S-quasinormal in K. If $P_0 \cap K = 1$, then $K \trianglelefteq G$ and K is nilpotent. Thus, $Q \trianglelefteq G$, which is a contradiction. If $P_0 \leq K$, then P_0 is S-quasinormal in G and therefore P_0Q is a group. By the choice of G, we have $P_0Q < G$ and hence $P_0Q = P_0 \times Q$. It follows that Q centralizes $\Omega_1(P)$. If $C_G(\Omega_1(P)) < G$, then $C_G(\Omega_1(P))$ is nilpotent and so $Q \trianglelefteq G$, again a contradiction. This leads to $C_G(\Omega_1(P)) = G$ and $\Omega_1(P) \leq Z(G)$. If $\exp P = p$, then G is p-nilpotent, a contradiction. Thus, $p = 2$ and $\exp P = 4$. Let $x \in P$ with $|\langle x \rangle| = 4$. Then, there exists $T \leq G$ such that $\langle x \rangle T = G$ and $\langle x \rangle \cap T$ is S-quasinormal in T. If $|G : T| = 4$, then $\langle x^2 \rangle T \trianglelefteq G$ and hence $Q \trianglelefteq G$, again a contradiction. In the case $|G : T| = 2$, we also have $Q \trianglelefteq G$, the same contradiction. Therefore $T = G$ and $\langle x \rangle$ is S-quasinormal in G. By the choice of G, we have $\langle x \rangle Q < G$ and hence $\langle x \rangle$ centralizes Q. Thus, again we have $Q \trianglelefteq G$, a contradiction. The proof is completed. □

Finally, we formulate another new theorem which also gives some other conditions for a finite group to be p-nilpotent.

THEOREM 4.9. *Let G be a group which is A_4-free and let p be the smallest prime divisor of $|G|$. If every subgroup of G having order p^2 is SS-supplemented in G, then G is p-nilpotent.*

Proof. Assume that the theorem is false and let G be a counterexample of minimal order. Let M be a maximal subgroup of G. Assume $|M|_p \leq p$. Then, by [**16**, IV, 2.8], M is p-nilpotent. If $|M|_p > p$, then by Lemma 2.4 (1) and the choice of G we can deduce that M is p-nilpotent. Thus, G is a minimal non-p-nilpotent group, and consequently, G has a normal Sylow p-subgroup P such that $G = PQ$, where Q is a Sylow q-subgroup of G with $q \neq p$.

Let $H \leq G$ with $|H| = p^2$. Then, there exists $K \leq G$ such that $HK = G$ and $H \cap K$ is S-quasinormal in K. Without loss of generality, we may assume that $Q \leq K$. Suppose $H \cap K = 1$, then K is nilpotent. Let K_p be a Sylow p-subgroup of K and P_1 is a maximal subgroup of P containing K_p. Then, $N_K(K_p)$ contains P_1 and Q. It follows that $|G : N_K(K_p)| \leq p$. If $|G : N_K(K_p)| = p$, then $N_K(K_p) \trianglelefteq G$. However, it follows that Q is normal in G, a contradiction. Assume that $K_p \trianglelefteq G$. We consider the group $\overline{G} = G/K_p$. Clearly, $\overline{G}/C_{\overline{G}}(\overline{P})$ is isomorphic to a subgroup of $\mathrm{Aut}(\overline{P})$ so that $q \mid p^2 - 1 = (p-1)(p+1)$. This implies that $p = 2$ and $q = 3$. Hence, $\overline{G}/\Phi(\overline{Q})$ is isomorphic to A_4, a contradiction.

If $|H \cap K| = p$, then $K \trianglelefteq G$. Hence $Q \trianglelefteq G$, again a contradiction.

Now, we have $H \leq K$ and thereby H is S-quasinormal in G. If $HQ = G$, then $P = H$ is not cyclic. Clearly, $C_G(P) < G$. Now, $G/C_G(P)$ is isomorphic to a subgroup of $\mathrm{Aut}(P)$ so that $p = 2$ and $q = 3$. Hence, $G/\Phi(Q)$ is isomorphic to A_4, which is a contradiction. Thus, $HQ < G$ and HQ is nilpotent. It follows that P normalizes Q, which is a contradiction. Thus the proof is completed. □

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