Large Irredundant Sets in Operator Algebras

Clayton Suguio Hida and Piotr Koszmider

Abstract. A subset $\mathcal{X}$ of a C*-algebra $\mathcal{A}$ is called irredundant if no $A \in \mathcal{X}$ belongs to the C*-subalgebra of $\mathcal{A}$ generated by $\mathcal{X} \setminus \{A\}$. Separable C*-algebras cannot have uncountable irredundant sets and all members of many classes of nonseparable C*-algebras, *e.g.*, infinite dimensional von Neumann algebras have irredundant sets of cardinality continuum.

There exists a considerable literature showing that the question whether every AF commutative nonseparable C*-algebra has an uncountable irredundant set is sensitive to additional set-theoretic axioms, and we investigate here the noncommutative case.

Assuming $\diamondsuit$ (an additional axiom stronger than the continuum hypothesis), we prove that there is an AF C*-subalgebra of $\mathcal{B}(\ell_2)$ of density $2^\omega = \omega_1$ with no nonseparable commutative C*-subalgebra and with no uncountable irredundant set. On the other hand we also prove that it is consistent that every discrete collection of operators in $\mathcal{B}(\ell_2)$ of cardinality continuum contains an irredundant sub-collection of cardinality continuum.

Other partial results and more open problems are presented.

1 Introduction

Definition 1.1 Let $\mathcal{A}$ be a C*-algebra. A subset $\mathcal{X} \subseteq \mathcal{A}$ is called *irredundant* if and only if for every $A \in \mathcal{X}$, the C*-subalgebra of $\mathcal{A}$ generated by $\mathcal{X} \setminus \{A\}$ does not contain A. We define

$$\mathrm{irr}(\mathcal{A}) := \sup\{|\mathcal{X}| : \mathcal{X} \text{ is an irredundant set in } \mathcal{A}\}.$$

Recall that the density of a C*-algebra $\mathcal{A}$, denoted $d(\mathcal{A})$, is the least cardinality of a norm dense subset of $\mathcal{A}$; *i.e.,* $\mathcal{A}$ is separable if and only if $d(\mathcal{A})$ is countable. It is easy to see that $\mathrm{irr}(\mathcal{A}) \leq d(\mathcal{A})$ for every C*-algebra, as irredundant sets must be norm discrete. When $\mathcal{A}$ is an infinite dimensional C*-algebra, $\mathrm{irr}(\mathcal{A})$ is infinite, because $\mathcal{A}$ then contains an infinite dimensional abelian C*-subalgebra ([41]), and locally compact infinite Hausdorff spaces contain pairwise disjoint infinite collections of open sets that yield infinite irredundant sets (Proposition 3.12). In this article, we are interested in uncountable irredundant sets in (C*-subalgebras of) the algebra $\mathcal{B}(\ell_2)$ of all linear bounded operators on a separable Hilbert space.

Irredundant sets have been considered in the context of other structures. For example, a subset of a Boolean algebra is called irredundant if none of its elements belong to the Boolean subalgebra generated by the remaining elements. We call such sets *Boolean irredundant* (Definition 3.8). In Banach spaces irredundant sets, *i.e.,*

where no element belongs to the closed subspace spanned by the remaining elements correspond exactly to biorthogonal systems ([20], see [30] for some comparisons between this type of notions). Examples of Boolean irredundant sets include independent families, ideal independent families, or (almost) disjoint families, but there are Boolean algebras of uncountable irredundance with no uncountable families of the above-mentioned classes (see Remark 3.7). A collection $(x_\alpha, x_\alpha^*)_{\alpha < \kappa} \subseteq \mathcal{B} \times \mathcal{B}^*$ of a Banach space $\mathcal{B}$ is *biorthogonal* if $x_\alpha^*(x_\alpha) = 1$ and $x_\alpha^*(x_\beta) = 0$ for all $\alpha < \beta < \kappa$. As linear functionals on a C*-algebra $\mathcal{A}$ are not usually multiplicative, there are many more biorthogonal systems than irredundant sets in $\mathcal{A}$, one can even consistently have a commutative C*-algebra $C(K)$ with countable irredundance but with uncountable biorthogonal systems ([9]).

Among our main motivations are consistent constructions of uncountable Boolean algebras with no uncountable irredundant sets. They were first obtained by Rubin ([46]) under the assumption of $\diamondsuit^1$ and then by Kunen ([40]) under the continuum hypothesis CH (improved further by Todorcevic to a $\mathfrak{b} = \omega_1$ construction from [51, 2.4]). Also some versions of the classical Ostaszewski construction assuming $\diamondsuit$ from [43] have these properties as further constructions assuming $\clubsuit$ from [20] as well as forcing constructions from [6, 9, 29].

Some of the above constructions are of Boolean algebras and others of (locally) compact Hausdorff totally disconnected spaces. Using the Stone duality, one translates one language to the other easily. The fact that the Kunen or Ostaszewski types of constructions mentioned above correspond to superatomic Boolean algebras or equivalently their Stone spaces are scattered spaces (every subset has a relative isolated point) yields the equality between the Boolean irredundance of the Boolean algebra and the irredundance of the commutative C*-algebra of continuous functions (Corollary 3.10). In particular, the corresponding $C(K)$s have no uncountable irredundant sets. In fact, the scatteredness can be exploited further to prove that the Banach spaces $C(K)$ have no uncountable biorthogonal systems [20, 40].

The first question we considered was whether such phenomena can take place if the C*-algebra is made considerably noncommutative. One of our main results is the following theorem.

Theorem 1.2 *Assume* $\diamondsuit$. *There is a fully noncommutative nonseparable scattered C*-algebra (of operators in* $\mathcal{B}(\ell_2)$*) with no nonseparable commutative subalgebra and with no uncountable irredundant set.*

Proof Apply Theorems 2.12 and 6.2. ∎

Here scattered C*-algebras are the noncommutative analogues of the scattered locally compact spaces. The condition of being fully noncommutative means that these algebras are "maximally noncommutative" among scattered algebras. These notions are reviewed in Section 2.1.

---

[1]$\diamondsuit$ is an additional axiom (introduced by R. Jensen), which is true in the universe of constructible sets. It says that there is a sequence $(S_\alpha)_{\alpha < \omega_1}$ that "predicts" all subsets of ω_1 in the sense that for any $X \subseteq \omega_1$ the set $\{\alpha < \omega_1 : X \cap \alpha = S_\alpha\}$ meets every closed and unbounded subset of ω_1; for details see [25] or [33]. Recently, $\diamondsuit$ has been successfully applied in the context of nonseparable C*-algebras by Akemann, Farah, Hirshberg, and Weaver [3, 4, 16]. We will not use the $\diamondsuit$ axiom directly, but will apply its consequence from Theorem 2.12, which was developed by S. Todorcevic in [54].

Another motivation for our project was the result of Todorcevic ([52, 53]) that assuming Martin's axiom MA and the negation of the CH, every uncountable Boolean algebra contains an uncountable irredundant set. Here the following main question remains open.

Question 1.3 Is it consistent that every nonseparable (AF, scattered) C*-algebra (of operators in $\mathcal{B}(\ell_2)$) contains an uncountable irredundant set?

It should be added that even the commutative general case is open, since the result of Todorcevic provides uncountable irredundant sets in $C(K)$s only for Ks totally disconnected, and there can be nonmetrizable compact spaces with no totally disconnected nonmetrizable compact subspace and similar examples (see [31]). So it is natural to initially restrict the attention in the noncommutative problem to C*-algebras corresponding to totally disconnected spaces, namely to approximately finite dimensional C*-algebras (AF), *i.e.*, where there is a dense subset that is the union of a directed family of finite dimensional C*-subalgebras (see [17] for diverse notions of approximate finite-dimensionality in the nonseparable context). Another natural narrowing of the question is to consider only the scattered C*-algebras, since one of the conditions equivalent to being scattered for a C*-algebra of density ω_1 is that each of its C*-subalgebras is AF. Attempting to answer Question 1.3, we obtained several results that shed some light on it. Let us discuss them below.

If $\mathcal{A}$ is AF C*-algebra of density equal to the first uncountable cardinal ω_1, then it can be written as $\mathcal{A} = \bigcup_{\alpha < \omega_1} \mathcal{A}_\xi$ where $\mathcal{A}_\xi \subseteq \mathcal{A}_{\xi'}$ for all $\xi < \xi' < \omega_1$ and each $\mathcal{A}_\xi$ is separable and AF. It follows from the result of Thiel in [48] (*cf.* [42, 49]) that each $\mathcal{A}_\xi$ is singly generated by one element $A_\xi \in \mathcal{A}_\xi$. Hence, in the set $\{A_\xi : \xi < \omega_1\}$ irredundant subsets are at most singletons. So there is no chance to extract (possibly using some additional forcing axioms) an uncountable irredundant set from an arbitrary norm discrete set of cardinality ω_1 of operators in $\mathcal{B}(\ell_2)$.

The AF hypothesis nevertheless allows us to avoid sets of operators as above. Namely, if $\mathcal{A} = \overline{\bigcup_{D \in \mathbb{D}} \mathcal{A}_D}$, where all $\mathcal{A}_D$s are finite-dimensional and $\mathcal{A}_D \subseteq \mathcal{A}_{D'}$ whenever $D \leq D'$ for $D \in \mathbb{D}$ and $(\mathbb{D}, \leq)$ is directed, then given any norm discrete $\{A_\xi : \xi < \omega_1\} \subseteq \bigcup_{D \in \mathbb{D}} \mathcal{A}_D$, which exists by the nonseparability of $\mathcal{A}$, for every finite $F \subseteq \omega_1$ the set

$$X_F = \{\xi < \omega_1 : A_\xi \in \mathcal{A}_F\}$$

is a finite superset of F, where $\mathcal{A}_F$ is the C*-subalgebra generated by $\{A_\eta : \eta \in F\}$. So, the search for an uncountable irredundant set among $\{A_\xi : \xi < \omega_1\}$ is equivalent to the search for an uncountable $X \subseteq \omega_1$ such that $X_F \cap X = F$ for every $F \subseteq X$.

However, this combinatorial problem for a general function from finite subsets of ω_1 to themselves has a negative solution.[2] Nevertheless, passing to the second uncountable cardinal ω_2 allows for the following very general consistency result, which is a consequence of Theorem 3.20.

---

[2]It is enough to take X_F to be of the form $Y \cap [(\max F) + 1]$, where $Y \in \mu$ is of minimal rank that contains F and where μ is an (ω, ω_1)-cardinal as in [32]. This μ is originally due to Velleman ([55]). A positive result for general functions is that given $n \in \mathbb{N}$ and a function ϕ from finite subsets of the n-th uncountable cardinal ω_n into countable subsets of ω_n, there is an n-element set $X \subseteq \omega_n$ such that $\xi \notin \phi(X \setminus \{\xi\})$ for any $\xi \in X$. In particular, this gives that any norm discrete subset of cardinality ω_n in any C*-algebra has an irredundant subset of cardinality n.

Theorem 1.4 *It is consistent that $2^\omega = \omega_2$ and for every norm discrete collection of operators $(A_\xi : \xi < \omega_2)$ in $\mathcal{B}(\ell_2)$, there is a subset $X \subseteq \omega_2$ of cardinality ω_2 such that $(A_\xi : \xi \in X)$ is irredundant.*

This is not a mere consequence of $\mathcal{B}(\ell_2)$ having density ω_2, because by a result of Brech and Koszmider [8], it is consistent that there exists a commutative C*-subalgebra of ℓ_∞ of density $2^\omega = \omega_2$ with no uncountable irredundant set. The cardinal ω_2 in Theorem 1.4 can be replaced by any regular cardinal bigger than ω_1, but it is not known if the result of [8] can be generalized to bigger cardinals than ω_2. Combining Theorems 1.4 and 1.2 and knowing that $\diamondsuit$ implies CH, we obtain the following corollary.

Corollary 1.5 *It is independent of ZFC whether there is a norm discrete collection of operators (projections) $(A_\xi : \xi < 2^\omega)$ in $\mathcal{B}(\ell_2)$ with no uncountable (of cardinality 2^ω) irredundant subcollection of size 2^ω.*

It is independent of ZFC whether there is C-subalgebra of $\mathcal{B}(\ell_2)$ of density 2^ω with no uncountable (of size 2^ω) irredundant set.*

The commutative results mentioned above are in fact most often of topological nature, where the compact Hausdorff space under the consideration is the Stone space $K_\mathcal{A}$ of a Boolean algebra $\mathcal{A}$. For example, the reason the above-mentioned Boolean algebras have countable irredundance is that the spread[3] of $K_\mathcal{A} \times K_\mathcal{A}$ is countable, as the finite powers of the mentioned $K_\mathcal{A}$s are hereditarily separable. Namely, in general, we have $\mathrm{irr}(\mathcal{A}) \le s(K_\mathcal{A} \times K_\mathcal{A})$, which was first noted in [23] and easily follows from the characterization of irredundant sets in the commutative case (Lemma 3.4). Also, the Urysohn Lemma gives the inequality $s(K) \le \mathrm{irr}(C(K))$ for any locally compact Hausdorff K. This argument cannot be transferred to the noncommutative setting, since noncommutative versions of the Urysohn Lemma are not so general (for the noncommutative Urysohn Lemma, see [2]). That is, for constructing an irredundant set of cardinality κ in a C*-algebra $\mathcal{A}$ it is enough to construct a sequence of states $(\tau_\alpha : \alpha < \kappa)$ and a sequence of positive elements $(A_\alpha : \alpha < \kappa)$ of $\mathcal{A}$ such that $\tau_\alpha(A_\alpha) > 0$ for all $\alpha < \kappa$ and $\tau_\alpha(A_\beta) = 0$ for all distinct $\alpha, \beta < \kappa$ (Lemma 3.14); however, a weak* discrete set of pure states does not produce the elements A_α as above due to the lack of the Urysohn Lemma for nonorthogonal closed projections. In fact, assuming the Proper Forcing Axiom, PFA, every nonseparable scattered C*-algebra has an uncountable weak* discrete set of pure states (Corollary 3.17), but this does not help us in the scattered case in constructing an uncountable irredundant set and answering Question 1.3 in the positive.

A bolder approach to Question 1.3 would be to try to answer the following question in the positive.

Question 1.6 Is it consistent (with MA and the negation of CH) that every nonseparable scattered (or even AF) C*-algebra has a nonseparable commutative subalgebra in one of its quotients?

---

[3]The spread of a topological space K, denoted by $s(K)$, is the supremum of the cardinalities of discrete subspaces of K.

Note that the class of scattered C*-algebras is closed under quotients and subalgebras and every locally compact scattered Hausdorff space is totally disconnected, so a positive answer to the question above and the MA result of Todorcevic mentioned above would give a positive answer to Question 1.3 in the scattered case.

Known ZFC examples of nonseparable C*-algebras with no nonseparable commutative subalgebras are the reduced group C*-algebra of an uncountable free group as shown by Popa in [45] and the algebras of Akemann and Doner as shown in [7]. However, the former is not AF (and has an uncountable irredundant set corresponding to the free generators of the group), and the latter has a nonseparable commutative quotient $c_0(\omega_1)$ (which also has an obvious uncountable irredundant set). Perhaps the algebra of [19] could provide a negative answer to Question 1.6.

The reason our algebra from Theorem 1.2 does not contain a nonseparable commutative C*-subalgebra is that, given any discrete sequence of projections in a certain dense subalgebra, there are two of them that have maximal commutator equal to 1/2 (the fact that 1/2 is the maximal value is proved in [47]). However, in such an arbitrary sequence there are also two projections that almost commute (see Theorem 6.2), so in this sense our algebra is quite random; that is, no pattern repeats on any uncountable norm discrete subset of elements. In fact, such behaviour is already sensitive to infinitary combinatorics beyond ZFC determined by $\Diamond$ and the Open Coloring Axiom, (OCA),[4] namely, we have the following theorem.

Theorem 1.7 *Assume OCA. For every $0 < \varepsilon < 1/2$ among any sequence of operators $(A_\xi : \xi < \omega_1)$ in $\mathcal{B}(\ell_2)$ there is an uncountable $X \subseteq \omega_1$ such that*

- *for every distinct $\xi_1, \xi_2 \in X$ we have $[A_{\xi_1}, A_{\xi_2}] > 1/2 - \varepsilon$, or*
- *for every $\xi_1, \xi_2 \in X$ we have $[A_{\xi_1}, A_{\xi_2}] < \varepsilon$.*

However, assuming $\Diamond$ there is a scattered C-algebra $\mathcal{A} \subseteq \mathcal{B}(\ell_2)$ (it is in particular AF) such that for every $0 < \varepsilon < 1/2$ among any discrete sequence of projections $(P_\xi : \xi < \omega_1)$ in $\mathcal{A}$*

- *there are $\xi_1 < \xi_2 < \omega_1$ such that $[P_{\xi_1}, P_{\xi_2}] > 1/2 - \varepsilon$,*
- *there are $\xi_1 < \xi_2 < \omega_1$ such that $[P_{\xi_1}, P_{\xi_2}] < \varepsilon$.*

Proof   Apply Corollary 4.6 and Theorem 6.2   ∎

Another natural question related to uncountable irredundant sets in general C*-algebras is the following.

Question 1.8

(i)   Is it true that $d(\mathcal{A}) \le 2^{\mathrm{irr}(\mathcal{A})}$ holds for every C*-algebra (every C*-algebra of type I)?

(ii) Can there be arbitrarily large C*-algebras with no uncountable irredundant sets?

This is motivated by a Boolean result of McKenzie (see [28], 4.2.3]), which says that a Boolean algebra has a dense subalgebra not bigger than its irredundance. This result was generalized by Hida [24] to all commutative C*-algebras, which implies

---

[4]For the statement of OCA, see Definition 4.4.

that $d(\mathcal{A}) \leq 2^{\mathrm{irr}(\mathcal{A})}$ holds for commutative $\mathcal{A}$. We prove this inequality answering Question 1.8 for scattered C⋆-algebras in Theorem 3.18.

In Section 2 we review scattered C⋆-algebras and construction schemes, which provide an elegant framework to deal with some constructions using ◇ recently introduced by Todorcevic in [54]. It has been applied in several functional analytic, topological and combinatorial contexts in [36, 37, 54]. In Section 3 we prove basic facts concerning irredundant sets in commutative and noncommutative settings. In Section 4 we prove the OCA part of Theorem 1.7. Section 5 is devoted to defining and investigating the partial order of finite dimensional approximations to our algebra from Theorem 1.2. In the final Section 6 we use the appropriate construction schemes described in Section 2 to construct the algebra from Theorem 1.2.

The notation and terminology of this paper should be standard; however, it draws from diverse parts of mathematics like Boolean algebras, operator theory, set-theory, logic, and general topology. When in doubt one could refer to textbooks like [14, 25, 28, 33, 39]. In particular, by an embedding (isomorphism onto its image) we mean a ∗-monomorphism (∗-isomorphism) of C^*-algebras that is not necessarily unital; $\ell_2(X)$ denotes the Hilbert space of square summable complex functions defined on a set X; $\mathcal{B}(\ell_2(X))$ denotes the C⋆-algebra of all bounded operators on $\ell_2(X)$; $\ell_2 = \ell_2(\mathbb{N})$; $\langle \cdot, \cdot \rangle$ denotes the scalar product; $\mathcal{A}_+$ denotes the set of positive elements of a C⋆-algebra $\mathcal{A}$; $1_\mathcal{A}$ denotes the unit of $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ the unitization of $\mathcal{A}$; $B_{\mathcal{A}^*}$ denotes the dual ball of the algebra $\mathcal{A}$; $[A, B] = AB - BA$ for $A, B \in \mathcal{B}(\ell_2)$; M_n denotes the C⋆-algebra of $n \times n$ matrices for $n \in \mathbb{N}$; $C(K)$ denotes the C⋆-algebra of complex valued continuous functions on a compact K and $C_0(X)$ the C⋆-algebra of complex valued continuous functions vanishing at infinity on a locally compact X; χ_U denotes the characteristic function of a set U; $\mathrm{Clop}(K)$ denotes the family of clopen subsets of a space K; ω_n denotes the n-th uncountable cardinal for $n \in \mathbb{N}$; $[X]^n$ denotes the family of all n-element subsets of a set X; $[X]^{<\omega}$ denotes the family of all finite subsets of a set X; $X < Y$ means that $x < y$ for all $x \in X$ and $y \in Y$ where X, Y are sets of ordinals.

2 Preliminaries

2.1 Scattered C*-algebras

The reason that scattered C⋆-algebras play an important role in our investigation of irredundant sets is that in such algebras irredundant sets can easily be replaced by irredundant sets of projections (Proposition 3.3); in particular, the Boolean results pass to the C⋆-algebraic ones (Corollary 3.10). Moreover, all commutative results culminate around the scattered case, which seems most basic.

Recall that a topological space is called *scattered* if it does not contain any perfect subset, in other words, if each (closed) nonempty subset has a relative isolated point. The phenomena related to the scatteredness were analysed by Cantor, which resulted in the notion of the Cantor–Bendixson derivative of a topological space ([14]). The Boolean algebra manifestation of these phenomena was discovered by Mostowski and Tarski in [38] as what is today known as superatomic Boolean algebras. The importance of the class of Banach spaces of the form $C(K)$, where K is scattered, already

implicitly known in the thirties, was first systematically revealed in [44]. Its generalization, Asplund Banach spaces, started to play an important role in Banach space theory in the sixties. It was Jensen [26] who first defined a scattered C^*-algebra, but they were considered earlier by Tomiyama [50] and Wojtaszczyk [56]. A recent survey [18] underlines the links of scattered C*-algebras with its Boolean algebraic and commutative predecessors. Recall that a projection p in a C*-algebra is called *minimal* if and only if $p\mathcal{A}p = \mathbb{C}p$; *i.e.*, minimal projections generalize isolated points. The *-subalgebra of $\mathcal{A}$ generated by the minimal projections of $\mathcal{A}$ will be denoted $\mathcal{J}^{At}(\mathcal{A})$. We have the following observation from [18].

Proposition 2.1   *Suppose that $\mathcal{A}$ is a C^*-algebra.*

(i)   *$\mathcal{J}^{At}(\mathcal{A})$ is an ideal of $\mathcal{A}$.*

(ii) *$\mathcal{J}^{At}(\mathcal{A})$ is isomorphic to a subalgebra of the algebra $\mathcal{K}(\mathcal{H})$ of all compact operators on a Hilbert space $\mathcal{H}$.*

(iii) *$\mathcal{J}^{At}(\mathcal{A})$ contains all ideals of $\mathcal{A}$ that are isomorphic to a subalgebra of $\mathcal{K}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

(iv) *if an ideal $\mathcal{J} \subseteq \mathcal{A}$ is essential and isomorphic to a subalgebra of $\mathcal{K}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, then $\mathcal{J} = \mathcal{J}^{At}(\mathcal{A})$.*

A selected list of conditions equivalent to being scattered and relevant to our paper is given below. Any of these conditions can be taken as the definition of a scattered algebra.

Theorem 2.2 ([18,26,27,34,35,50,56])   *Suppose that $\mathcal{A}$ is a C^*-algebra. The following conditions are equivalent:*

(i)   *Every non-zero *-homomorphic image of $\mathcal{A}$ has a minimal projection.*

(ii) *There is an ordinal $\mathrm{ht}(\mathcal{A})$ and a continuous increasing sequence of closed ideals $(\mathcal{J}_\alpha^{At}(\mathcal{A}))_{\alpha \le \mathrm{ht}(\mathcal{A})}$ called the Cantor–Bendixson composition series for $\mathcal{A}$ such that $\mathcal{J}_0 = \{0\}$, $\mathcal{J}_{\mathrm{ht}(\mathcal{A})} = \mathcal{A}$ and*

$$\mathcal{J}^{At}(\mathcal{A}/\mathcal{J}_\alpha^{At}(\mathcal{A})) = \{[a]_{\mathcal{J}_\alpha^{At}(\mathcal{A})}: a \in \mathcal{J}_{\alpha+1}^{At}(\mathcal{A})\},$$

for every $\alpha < \mathrm{ht}(\mathcal{A})$.

(iii) *Every non-zero subalgebra of $\mathcal{A}$ has a minimal projection.*

(iv) *Every non-zero subalgebra has a projection.*

(v)   *Every subalgebra of $\mathcal{A}$ has real rank zero.*

(vi) *$\mathcal{A}$ does not contain a copy of the C^*-algebra*

$$C_0((0,1]) = \left\{ f \in C((0,1]): \lim_{x \to 0} f(x) = 0 \right\}.$$

(vii) *The spectrum of every self-adjoint element is countable.*

Definition 2.3 ([18])   A scattered C*-algebra is called *thin-tall* if and only if $\mathrm{ht}(\mathcal{A})$ from Theorem 2.2(ii) is equal ω_1 and $\mathcal{J}_{\alpha+1}^{At}(\mathcal{A})/\mathcal{J}_\alpha^{At}(\mathcal{A})$ is separable for each $\alpha < \omega_1$.

In the nonseparable context we are especially interested in condition (ii), which was introduced in [18] and gives an essential composition series corresponding to the Cantor–Bendixson derivative. A scattered C*-algebra is called *fully noncommutative*

if and only if for all $\alpha < \mathrm{ht}(\mathcal{A})$ the algebra $\mathcal{I}^{At}(\mathcal{A}/\mathcal{I}_\alpha)$ is *-isomorphic to the algebra of all compact operators on a Hilbert space. We have the following two observations from [18].

Proposition 2.4 *Suppose that $\mathcal{A}$ is a scattered C*-algebra. The following are equivalent:*

  (i)   *$\mathcal{A}$ is fully noncommutative;*
 (ii)   *the ideals of $\mathcal{A}$ form a chain;*
(iii)   *the centers of the multiplier algebras of any quotient of $\mathcal{A}$ are all trivial.*

Proposition 2.5 *Every scattered C*-algebra $\mathcal{A}$ is atomic, i.e., the ideal $\mathcal{I}^{At}(\mathcal{A})$ is essential.*

Recall that in a topological space a sequence of points $\{x_\xi : \xi < \kappa\}$ is called *right-separated* (resp. *left-separated*) if and only if $x_\xi \notin \overline{\{x_\eta : \eta > \xi\}}$ for all $\xi < \kappa$ (resp. $x_\xi \notin \overline{\{x_\eta : \eta < \xi\}}$ for all $\xi < \kappa$). Left and right separated sequences play an important role in commutative set-theoretic topology, because a regular space is hereditarily Lindelöf (resp. hereditarily separable) if it has no uncountable right-separated (resp. left-separated) sequences. Additional axioms like $\diamondsuit$, CH, MA, PFA[5] have substantial impact on the existence of right or left separated sequences in regular topological spaces, for example PFA implies that there are no regular S-spaces, *i.e.*, every regular topological space which has an uncountable right-separated sequence has an uncountable left-separated sequence as well ([51, Theorem 8.9]).

Proposition 2.6 *Suppose that $\mathcal{A}$ is a thin-tall C*-algebra. Then the dual ball $B_{\mathcal{A}^*}$ of $\mathcal{A}^*$ contains an uncountable right-separated sequence of pure states in the weak* topology. In particular, under the Proper Forcing Axiom (PFA), the dual ball $B_{\mathcal{A}^*}$ of $\mathcal{A}^*$ contains an uncountable discrete set consisting of pure states.*

Proof   Let $(\mathcal{I}_\alpha)_{\alpha < \omega_1}$ be the Cantor–Bendixson composition series of Theorem 2.2(iii). As $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$ is an essential ideal of $\mathcal{A}/\mathcal{I}_\alpha$, which is *-isomorphic with the algebra of all compact operators on ℓ_2, we can embed $\mathcal{A}/\mathcal{I}_\alpha$ into $\mathcal{B}(\ell_2)$ with the range containing all compact operators. Take τ_α to be a vector pure state on $\mathcal{B}(\ell_2)$ composed with the quotient map and the embedding. So τ_α is a pure state on $\mathcal{A}$ that is zero on $\mathcal{I}_\alpha$, and there is $A_\alpha \in \mathcal{I}_{\alpha+1}$ such that $\tau_\alpha(A_\alpha) = 1$. Denote the set of all pure states on $\mathcal{A}$ by $P(\mathcal{A})$. Now consider

$$U_\alpha = \{\tau \in P(\mathcal{A}) : \tau(A_\alpha) > 0\}.$$

Note that if $\tau_\beta \in U_\alpha$, then $\beta \le \alpha$, so $\{\tau_\alpha : \alpha < \omega_1\}$ is right-separated in the weak* topology. So $\{\tau_\alpha : \alpha < \omega_1\}$ contains and uncountable left-separated sequence by PFA ([51, Theorem 8.9]). It is clear that a sequence that is both left and right separated is discrete. ∎

---

[5]For the statement of the Proper Forcing Axiom (PFA) or Martin's Axiom (MA), we refer the reader to, for example, [25] or [51]. PFA implies among others MA, OCA, and $2^\omega = \omega_2$.

2.2 Construction Schemes

In this section, we recall some definitions and results from [54].

Definition 2.7 Let $E, F \in [\omega_1]^{<\omega}$.

  (i) $F < E$ whenever $\alpha < \beta$ for all $\alpha \in F$ and $\beta \in E$.
  (ii) $F \sqsubseteq E$ whenever there is $\alpha \in \omega_1$ such that $E \cap \alpha = F$ (we say that F is an initial fragment of E or that E end-extends F).
  (iii) $F \sqsubset E$ whenever $F \sqsubseteq E$ and $E \smallsetminus F \neq \varnothing$.

Definition 2.8 Let η be an ordinal and let $(F_\xi : \xi < \eta) = \mathcal{F} \subseteq [\omega_1]^{<\omega}$.

  (i) $\mathcal{F}$ is cofinal if for all $E \in [\omega_1]^{<\omega}$ there is $F \in \mathcal{F}$ such that $E \subseteq F$.
  (ii) $(F_\xi : \xi < \eta)$ is a Δ-system of length η with root Δ whenever $F_\xi \cap F_{\xi'} = \Delta$ for all $\xi < \xi' < \eta$,
  (iii) A Δ-system $(F_\xi : \xi < \eta)$ with root Δ is increasing whenever $F_\xi \smallsetminus \Delta < F_{\xi'} \smallsetminus \Delta$ for all $\xi < \xi' < \eta$.
  (iv) A subset of a Δ-system is called a subsystem.
  (v) $\mathcal{F}|F = \{E \in \mathcal{F} : E \not\subseteq F\}$ for $F \subseteq \omega_1$.

Definition 2.9 A pair of sequences $(n_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ and $(r_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ are called allowed parameters if and only if

  (i) $r_0 = r_1 = n_0 = 0$;
  (ii) $n_k \geq 2$ for all $k \in \mathbb{N}$;
  (iii) each natural value appears in the sequence $(r_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ infinitely many times;
  (iv) $r_{k+1} < m_k$ where $m_0 = 1$, $m_{k+1} = r_{k+1} + n_{k+1}(m_k - r_{k+1})$ for $k > 0$.

Definition 2.10 A construction scheme with a pair of allowed parameters $(n_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ and $(r_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ is a cofinal family $\mathcal{F} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ satisfying the following:

  (i) $\mathcal{F}_0 = [\omega_1]^1$.
  (ii) If $k > 0$ and $E, F \in \mathcal{F}_k$, then $|E| = |F|$ and $E \cap F \sqsubseteq E, F$ and

$$\{\phi_{F,E}[G] : G \in \mathcal{F}|E\} = \mathcal{F}|F,$$

  where $\phi_{F,E} : E \to F$ is the order preserving bijection between E and F.
  (iii) If $k \geq 0$ and $F \in \mathcal{F}_{k+1}$, then the maximal elements of $\mathcal{F}|F$ are in $\mathcal{F}_k$ and they form an increasing Δ-system of length n_{k+1} such that F is its union. The family of all these maximal elements is called the *canonical decomposition* of F.

Definition 2.11 Given a construction scheme $\mathcal{F}$, we say that an $F \in \mathcal{F}_k$ for $k > 0$ captures a Δ-system $(s_i : i < n)$ of finite subsets of ω_1 with root s if the canonical decomposition $(F_i : i < n_k)$ of F with root Δ has the following properties:

  (i) $n_k \geq n$, $s \subseteq \Delta$, and $s_i \smallsetminus s \subseteq F_i \smallsetminus \Delta$ for all $i < n$.
  (ii) $\phi_{F_i, F_j}[s_i] = s_j$ for all $i < j < n$.
When $n = n_k$, we say that F fully captures the Δ-system.

Theorem 2.12 ([54]) *Assume $\diamondsuit$. For any pair of allowed parameters $(n_k)_{k \in \mathbb{N}}$ and $(r_k)_{k \in \mathbb{N}}$, there is a construction scheme $\mathcal{F}$ with these parameters and there is a partition*

$(P_n)_{n \in \mathbb{N}}$ *of $\mathbb{N}$ into infinitely many infinite sets such that for every $n \in \mathbb{N}$ and every uncountable Δ-system T of finite subsets of ω_1 there exist arbitrarily large $k \in P_n$ and $F \in \mathcal{F}_{k+1}$ that fully captures a subsystem of T.*

3 Irredundant Sets

3.1 Reducing Irredundant Sets to Special Ones

Because of the Weierstrass–Stone theorem for unital commutative C*-algebras, sometimes it is useful to consider a strengthening of being irredundant, where the subalgebras we generate are unital. However, as can be seen in the following lemma, this does not affect the cardinalities of irredundant sets much.

Lemma 3.1 *Suppose that $\mathcal{A}$ is a unital C*-algebra and that $\mathcal{X} \subseteq \mathcal{A}$ is its nonempty irredundant set. Then there is $x_0 \in \mathcal{X}$ such that no element x of $\mathcal{X} \smallsetminus \{x_0\}$ belongs to the unital C*-subalgebra generated by $\mathcal{X} \smallsetminus \{x_0, x\}$.*

Proof If no element x of $\mathcal{X}$ belongs to the unital C*-subalgebra generated by $\mathcal{X} \smallsetminus \{x\}$, we are done by taking any element of $\mathcal{X}$ as x_0.

Otherwise let $x_0 \in \mathcal{X}$ belong to the unital C*-subalgebra generated by $\mathcal{X} \smallsetminus \{x_0\}$ so $x_0 = \lambda 1 + y$ where y is in the subalgebra generated by $\mathcal{X} \smallsetminus \{x_0\}$ and $\lambda \in \mathbb{C} \smallsetminus \{0\}$. Suppose that there is $x \in \mathcal{X} \smallsetminus \{x_0\}$ in the unital C*-subalgebra generated by $\mathcal{X} \smallsetminus \{x_0, x\}$, i.e., $x = \lambda' 1 + z$ where z is in the subalgebra generated by $\mathcal{X} \smallsetminus \{x_0, x\}$ and $\lambda' \in \mathbb{C} \smallsetminus \{0\}$. So 1 is in the algebra generated by $\mathcal{X} \smallsetminus \{x_0\}$, but this shows that $x_0 = \lambda 1 + y$ is in the subalgebra generated by $\mathcal{X} \smallsetminus \{x_0\}$, a contradiction to the fact that $\mathcal{X}$ is irredundant. ∎

Clearly any two orthogonal one-dimensional projections in M_2 form an irredundant set; however, each of them is in the unital C*-algebra generated by the other projection.

Proposition 3.2 *Suppose that $\mathcal{A}$ is a C*-algebra, κ is an infinite cardinal and $\{A_\xi : \xi < \kappa\}$ is an irredundant set in $\mathcal{A}$. Then there is an irredundant set $\{B_\xi : \xi < \kappa\}$ consisting of positive elements of $\mathcal{A}$.*

Proof Given $X \subseteq \kappa$, let $\mathcal{A}_X$ be the C*-subalgebra of $\mathcal{A}$ generated by $\{A_\xi : \xi \in X\}$. Clearly, $(A_\eta + A_\eta^*)/2, (A_\eta - A_\eta^*)/2i \in \mathcal{A}_{\kappa \smallsetminus \{\xi\}}$ for every $\eta \neq \xi$ and $(A_\xi + A_\xi^*)/2$ and $(A_\xi - A_\xi^*)/2i$ cannot both belong to $\mathcal{A}_{\kappa \smallsetminus \{\xi\}}$. So $\{B_\xi : \xi < \kappa\}$ is an irredundant set consisting of self-adjoint elements, where $B_\xi \in \{(A_\xi + A_\xi^*)/2, (A_\xi - A_\xi^*)/2i\}$ is such that $B_\xi \notin \mathcal{A}_{\kappa \smallsetminus \{\xi\}}$.

To prove that we can obtain the same cardinality irredundant set consisting of all positive elements, by the above we can assume that the original A_ξs are self-adjoint. We have $A_\xi = A_{\xi_+} - A_{\xi_-}$. Note that there is $B_\xi \in \{A_{\xi_+}, A_{\xi_-}\}$, which does not belong to $\mathcal{A}_{\kappa \smallsetminus \{\xi\}}$. But

$$A_{\eta_+} = (|A_\eta| + A_\eta)/2 \quad \text{and} \quad A_{\eta_-} = (A_\eta - |A_\eta|)/2,$$

where $|A_\eta| = \sqrt{A_\eta^2}$, belong to $\mathcal{A}_\xi$ for all $\eta \in \kappa \smallsetminus \{\xi\}$. So $\{B_\xi : \xi < \kappa\}$ is irredundant, as required. ∎

The following proposition shows the role of being scattered while extracting irredundant sets consisting of projections.

Proposition 3.3   *Suppose that $\mathcal{A}$ is a scattered C*-algebra, κ is an infinite cardinal, and $\{A_\xi : \xi < \kappa\}$ is an irredundant set in $\mathcal{A}$. Then there is an irredundant set $\{P_\xi : \xi < \kappa\}$ consisting of projections.*

Proof   By Lemma 3.2 we can assume that A_ξs are self-adjoint. Let us adopt the notation $\mathcal{A}_X$ for $X \subseteq \kappa$ from the proof of Lemma 3.2.

Since subalgebras of scattered algebras are scattered, $\mathcal{A}_{\{\xi\}}$s are scattered for each $\xi < \kappa$ and so of the form $C_0(K_{\{\xi\}})$ for some locally compact scattered $K_{\{\xi\}}$ that must be totally disconnected. It follows that linear combinations of projections of $\mathcal{A}_{\{\xi\}}$s are norm dense in $\mathcal{A}_{\{\xi\}}$s. Hence, for each $\xi < \kappa$, there is a projection $P_\xi \in \mathcal{A}_{\{\xi\}}$ such that $P_\xi \notin \mathcal{A}_{\kappa \setminus \{\xi\}}$. It follows that $\{P_\xi : \xi < \kappa\}$ is irredundant.   ∎

3.2   Irredundant Sets in Commutative C*-algebras

The following two lemmas characterize irredundant sets in commutative C*-algebras.

Lemma 3.4   *Suppose that K is a compact Hausdorff space and $\mathcal{X} \subseteq C(K)$ is such that no $f \in \mathcal{X}$ belongs to the unital C*-subalgebra of $C(K)$ generated by $\mathcal{X} \setminus \{f\}$. Then for each $f \in \mathcal{X}$, there are $x_f, y_f \in K$ such that $f(x_f) \neq f(y_f)$ but $g(x_f) = g(y_f)$ for any $g \in \mathcal{X} \setminus \{f\}$.*

Consequently, if $\mathcal{X}$ is a nonempty irredundant set in $C(K)$, then there is $h \in \mathcal{X}$ such that $\mathcal{X} \setminus \{h\}$ has the above property.

Proof   By the Gelfand representation we can assume that $C(K)$ is the unital C*-algebra generated by $\mathcal{X}$. By the complex Stone–Weierstrass theorem the proper C*-subalgebra generated by $\mathcal{X} \setminus \{f\}$ does not separate a pair of points of K, say x_f, y_f. But they must be separated by f, by the fact that $\mathcal{X}$ generated $C(K)$.

The last part of the lemma follows from Lemma 3.1.   ∎

Lemma 3.5   *Suppose that X is a locally compact noncompact Hausdorff space and $\mathcal{X} \subseteq C_0(X)$ is irredundant then for every $f \in \mathcal{X}$ there are $x_f, y_f \in X$ such that either*

- *$f(x_f) \neq 0$ and $g(x_f) = 0$ for all $g \in \mathcal{X} \setminus \{f\}$ or*
- *$f(x_f) \neq f(y_f)$ but $g(x_f) = g(y_f)$ for all $g \in \mathcal{X} \setminus \{f\}$.*

Points x_f satisfying the first case form a discrete subspace of X.

Proof   Let $K = X \cup \{\infty\}$ be the one-point compactification of X. We will identify $C_0(K)$ with a C*-subalgebra of $C(K)$. Note that $\mathcal{X}$ satisfies the hypothesis of Lemma 3.4, because if $f = \lambda 1 + g$ for $f, g \in C_0(X)$, the unit would be in $C_0(X)$, which contradicts the hypothesis that X is noncompact. So we obtain the pairs of points $x_f, y_f \in K$ as in Lemma 3.4. The first case of the lemma corresponds to the situation when one of the points x_f, y_f is ∞, say y_f. But then $h(y_f) = 0$ for all $h \in C_0(X)$, which implies $f(x_f) \neq 0$ and $g(x_f) = 0$ for all $g \in \mathcal{X} \setminus \{f\}$.

Considering open sets $U_f = \{x \in X_0 : f(x) \neq 0\}$, we obtain neighbourhoods witnessing the discreteness of the set of x_fs satisfying the first case. ∎

In fact, as can be seen in the following remark, discrete subsets of K provide strong irredundant subsets in $C(K)$.

Remark 3.6 Suppose that K is a compact Hausdorff space and $D \subseteq K$ is discrete. For each $d \in D$, consider $f_d \in C(K)$ such that $f_d(d) \neq 0$ and $f_d(d') = 0$ for all $d' \in D \setminus \{d\}$. Then f_d does not belong to the ideal generated by $\{f_{d'} : d' \in D \setminus \{d\}\}$. In particular, $\{f_d : d \in D\}$ is irredundant.

However, we note in the next remark that there could be a dramatic gap between the sizes of discrete subsets and the sizes of irredundant sets.

Remark 3.7 Let K be the split interval, *i.e.*, $\{0,1\}^{\mathbb{N}} \times \{0,1\}$ with the order topology induced by the lexicographical order. Then K has no uncountable discrete subset (in fact, K is hereditarily separable and hereditarily Lindelöf), but $C(K)$ has an irredundant set $\{\chi_{[0^{\mathbb{N}}{}^\frown 0, x^\frown 0]} : x \in \{0,1\}^{\mathbb{N}}\}$ of cardinality continuum.

Most of the literature concerning implicitly or explicitly irredundant sets is related to Boolean algebras. As shown in the following lemma, the relationship between Boolean irredundance and irredundance for C*-algebras is very close in the light of Lemma 3.1.

Definition 3.8 A subset $\mathcal{X}$ of a Boolean algebra $\mathcal{A}$ is called Boolean irredundant if for every $x \in \mathcal{X}$ the element x does not belong to the Boolean subalgebra generated by $\mathcal{X} \setminus \{x\}$.

Lemma 3.9 *Suppose that $\mathcal{A}$ is a unital C*-algebra and $\mathcal{B} \subseteq \mathcal{A}$ is a Boolean algebra of projections in $\mathcal{A}$ and $\mathcal{X} \subseteq \mathcal{B}$ is Boolean irredundant. Then $\mathcal{X}$ is irredundant in $\mathcal{A}$.*

Suppose that K is a totally disconnected space and $\mathcal{X} \subseteq C(K)$ consists of projections where no element of $x \in \mathcal{X}$ belongs to the unital C-algebra generated by $\mathcal{X} \setminus \{x\}$. Then $\mathcal{X}$ is Boolean irredundant in the Boolean algebra $\{\chi_U : U \in \mathrm{Clop}(K)\}$.*

Proof Let $\mathcal{C}$ be the C*-subalgebra of $\mathcal{A}$ generated by $\mathcal{B}$. It is abelian, so it is of the form $C(K)$ where K is the Stone space of $\mathcal{B}$. It is enough to prove that $\mathcal{X}$ is irredundant in $\mathcal{C}$. But given a proper Boolean subalgebra, there are distinct ultrafilters on the superalgebra that coincide on the subalgebra. These ultrafilters are the points of K witnessing the irredundance of $\mathcal{F}$ as in Lemma 3.15. ∎

For commutative scattered C*-algebras the relationship between Boolean and C*-algebraic irredundance is even closer, as can be seen in the following corollary.

Corollary 3.10 *Suppose that $\mathcal{A}$ is an infinite superatomic Boolean algebra. Then the Boolean irredundance of $\mathcal{A}$ is the same as $\mathrm{irr}(C(K_{\mathcal{A}}))$, where $K_{\mathcal{A}}$ is the Stone space of $\mathcal{A}$.*

Proof As $\mathcal{A}$ is infinite, its Boolean irredundance is infinite (just take an infinite pairwise disjoint collection). By the first part of Lemma 3.9, the Boolean irredundance of

$\mathcal{A}$ is not bigger than $\text{irr}(C(K_\mathcal{A}))$. On the other hand consider any infinite irredundant subset $\mathcal{X}$ of $C(K_\mathcal{A})$. Then $K_\mathcal{A}$ is scattered as $\mathcal{A}$ is superatomic, and so $C(K_\mathcal{A})$ is a scattered C*-algebra. By Proposition 3.3, there is an irredundant subset $\mathcal{Y}$ of $C(K_\mathcal{A})$ of the same cardinality as $\mathcal{X}$ and consisting of projections in $C(K_\mathcal{A})$. By removing at most one element of $\mathcal{Y}$, by the second part of Lemma 3.9 and Lemma 3.1, it is a Boolean irredundant set in the Boolean algebra $\{\chi_U : U \in \text{Clop}(K_\mathcal{A})\}$, and this yields a Boolean irredundant set in $\mathcal{A}$. ∎

The above positively answers [12, Question 3.10 (3)] in the case of a scattered space.

Corollary 3.11   *Suppose that K is an infinite Hausdorff compact space. Then $\text{irr}^\#(C_\mathbb{R}(K)) = \text{irr}(C(K))$ where $\text{irr}^\#(C_\mathbb{R}(K))$ is the supremum over the cardinalities of sets $\mathcal{X}$ of real-valued continuous functions on K such that no $f \in \mathcal{X}$ belongs to the real unital Banach algebra generated by $\mathcal{X} \smallsetminus \{f\}$. In particular, the π-weight of K is bounded by $\text{irr}(C(K))$ and the density of $C(K)$ is bounded by $2^{\text{irr}(C(K))}$.*

Proof   Let $X \subseteq C(K)$ be an infinite irredundant set. By Lemma 3.2, we can assume that it consists of real-valued (non-negative) functions. As in the proof of Lemma 3.1, by removing at most one element, we can assume that no $f \in \mathcal{X}$ belongs to the real unital C*-algebra generated by $\mathcal{X} \smallsetminus \{f\}$. So $\text{irr}^\#(C_\mathbb{R}(K)) \geq \text{irr}(C(K))$.

Now, given a set $\mathcal{X}$ as in the lemma, by the real unital Weierstrass–Stone theorem there are pairs of points $x_f, y_f \in K$ such that $f(x_f) \neq f(y_f)$ but $g(x_f) = g(y_f)$ for any $g \in \mathcal{X} \smallsetminus \{f\}$ (*cf.* [30]). Hence, $\mathcal{X}$ is an irredundant set in the C*-algebra $C(K)$ by Lemma 3.4.

The last part of the corollary follows from [24, Theorem 10] where $\pi(K) \leq \text{irr}^\#(C_\mathbb{R}(K))$ is proved and from the fact that the weight of a regular space is bounded by the exponent of its π-weight ([21, Theorem 3.3]). ∎

3.3   Irredundant Sets in General C*-algebras

Having developed the motivations in the previous section, we now move to the irredundant sets in general, possibly noncommutative C*-algebras.

Proposition 3.12   *Every infinite pairwise orthogonal collection of self-adjoint elements in a C*-algebra is irredundant. In particular, every infinite dimensional C*-algebra contains an infinite irredundant set.*

Proof   This follows from the fact that given a self-adjoint element A of a C*-algebra $\mathcal{A}$, the set $\{B \in \mathcal{A} : AB = BA = 0\}$ is a C*-subalgebra of $\mathcal{A}$. ∎

Proposition 3.13   *Suppose that an infinite dimensional C*-algebra $\mathcal{A}$ is a von Neumann algebra. Then $\mathcal{A}$ has an irredundant set of cardinality continuum.*

Proof   An infinite dimensional von Neumann algebra has an infinite pairwise orthogonal collection of projections, and so it contains the commutative C*-algebra ℓ_∞ that is *-isomorphic to $C(\beta\mathbb{N})$. The Boolean algebra $\wp(\mathbb{N})$ is isomorphic to $\{\chi_U : U \in \text{Clop}(\beta\mathbb{N})\}$, and so the Boolean irredundance of $\wp(\mathbb{N})$ is equal to the irredundance

of ℓ_∞ by Lemma 3.10. By considering an almost disjoint family (or an independent family) of cardinality continuum of subsets of $\mathbb{N}$, we obtain an irredundant set of cardinality continuum. ∎

The following lemma corresponds to Remark 3.6.

Lemma 3.14 *Suppose that $\mathcal{A}$ is a C*-algebra, κ a cardinal $\{A_\xi : \xi < \kappa\} \subseteq \mathcal{A}_+$ and $\{\tau_\alpha : \alpha < \kappa\}$ a family of states such that*

- $\tau_\alpha(A_\alpha) > 0$,
- $\tau_\alpha(A_\xi) = 0$ *for* $\xi \neq \alpha$.

Then $\{A_\xi : \xi < \kappa\}$ is irredundant.

Proof As in the GNS construction, one proves that

$$L_\alpha = \{A \in \mathcal{A} : \tau_\alpha(A^*A) = 0\}$$

is a left-ideal in $\mathcal{A}$, and, in particular, a C*-subalgebra. So $X_\xi \in L_\alpha$, where $A_\xi = X_\xi^* X_\xi$ and so $A_\xi \in L_\alpha$ for all $\xi \neq \alpha$. However, by [39, Theorem 3.3.2.] we have

$$0 < \tau_\alpha(A_\alpha) \leq \|\tau_\alpha\| \tau_\alpha(A_\alpha^* A_\alpha),$$

so $A_\alpha \notin L_\alpha$. ∎

In the noncommutative case, for pure states $\{\tau_\alpha : \alpha < \kappa\}$, being discrete in the weak* topology does not yield in general the existence of positive elements A_α as in the lemma above, as the noncommutative Urysohn lemmas require extra hypotheses ([2]).

The following proposition is a version of the commutative characterizations in Lemmas 3.4 and 3.5. It is interesting to note that a version of the following proposition where "representations" are replaced by "irreducible representations" implies the noncommutative Stone–Weierstrass theorem, which remains a well-known open problem. One should note that below, one of the possibilities of the representation is to be constantly zero.

Proposition 3.15 ([22]) *Suppose that $\mathcal{A}$ is a C*-algebra and $\mathcal{X} \subseteq \mathcal{A}$ is an irredundant set. Then for all $a \in \mathcal{X}$, there are Hilbert spaces H_a and representations $\pi_a^1, \pi_a^2 : \mathcal{A} \to \mathcal{B}(H_a)$ such that $\pi_a^1(a) \neq \pi_a^2(a)$, but*

$$\mathcal{X} \smallsetminus \{a\} \subseteq \{b \in A : \pi_a^1(b) = \pi_a^2(b)\}.$$

3.4 Irredundance in Scattered C*-algebras

The following proposition shows that thin-tall algebras play a special role in the context of uncountable irredundant sets.

Proposition 3.16 *If there is a nonseparable scattered C*-algebra with no uncountable irredundant set, then it contains a thin-tall scattered C*-algebra.*

Proof First note that by the characterization of subalgebras of the algebra of compact operators ([5]), a C*-algebra that is isomorphic to a subalgebra of the algebra of all compact operators on a Hilbert space $\mathcal{H}$ but not isomorphic to a subalgebra of the

algebra of all compact operators on the separable Hilbert space $\mathcal{H}$ must contain an uncountable pairwise orthogonal set that is irredundant by Proposition 3.12. So if a scattered $\mathcal{A}$ has no uncountable irredundant set, then all the algebras $\mathcal{I}^{At}(\mathcal{A}/\mathcal{I}_\alpha)$ are $*$-isomorphic to a subalgebra of the algebra of all compact operators on the separable or finite dimensional Hilbert space, but as $\mathcal{A}$ is nonseparable, $\mathrm{ht}(\mathcal{A}) \geq \omega_1$, and so $\mathcal{I}_{\omega_1}$ is the required thin-tall subalgebra of $\mathcal{A}$.   ∎

Corollary 3.17   *Assume PFA. Suppose that $\mathcal{A}$ is a nonseparable scattered $C^\star$-algebra. Then there is an uncountable weak* discrete set of pure states of $\mathcal{A}$.*

Proof   First suppose that $\mathcal{A}$ has a quotient that contains an uncountable orthogonal set of projections. Then it is clear that we can find pure states that form a weak* discrete set. Otherwise, using the argument as in the proof of Proposition 3.16, we can assume that $\mathcal{A}$ is thin-tall. By Proposition 2.6, $\mathcal{A}$ has an uncountable weak* discrete set of pure states.   ∎

Below we prove a simple noncommutative version of a theorem of McKenzie (see [28, 4.2.3]).

Theorem 3.18   *If $\mathcal{A}$ is a scattered $C^\star$-algebra, then $d(\mathcal{A}) \leq 2^{\mathrm{irr}(\mathcal{A})}$.*

Proof   Let κ be the minimal cardinal such that $\mathcal{I}^{At}(\mathcal{A})$ is a subalgebra of the algebra of all compact operators on $\ell_2(\kappa)$. By the characterization of subalgebras of the algebra of compact operators ([5]) $\mathcal{A}$ must contain pairwise orthogonal set of cardinality κ, which is irredundant by Proposition 3.12. So $\kappa \leq \mathrm{irr}(\mathcal{A})$. By the essentiality of $\mathcal{I}^{At}(\mathcal{A})$ which follows from Proposition 2.5 we can embed $\mathcal{A}$ into $\mathcal{B}(\ell_2(\kappa))$, so $d(\mathcal{A}) \leq 2^\kappa \leq 2^{\mathrm{irr}(\mathcal{A})}$ as required.   ∎

3.5 Extracting Irredundant Sets from a Given Collection of Operators

Proposition 3.19   *There is a collection of operators $(A_\xi : \xi < \omega_1)$ in $\mathcal{B}(\ell_2)$ that generates a nonseparable $C^\star$-subalgebra of $\mathcal{B}(\ell_2)$ with no two-element irredundant subset. Any fully noncommutative thin-tall $C^\star$-algebra is generated by such a sequence.*

Proof   Construct a fully noncommutative thin-tall $C^\star$-algebra $\mathcal{A}$ as in [18, Theorem 7.6], in particular with the Cantor–Bendixson decomposition $(\mathcal{I}_\alpha^{At}(\mathcal{A}))_{\alpha < \omega_1}$ (see 2.2 (2)), where $\mathcal{I}_{\alpha+1}^{At}(\mathcal{A})$ is $*$-isomorphic to $\widetilde{\mathcal{I}_\alpha^{At}(\mathcal{A})} \otimes \mathcal{K}(\ell_2)$.

By [42, Theorem 8], any $C^\star$-algebra of the form $\mathcal{B} \oplus \mathcal{K}(\ell_2)$ is singly generated if $\mathcal{B}$ is separable and unital. So for each $\alpha < \omega_1$, pick A_α to be a single generator of $\mathcal{I}_{\alpha+1}^{At}(\mathcal{A})$.

An alternative approach that gives the final statement of the proposition is to use the fact that scattered $C^\star$-algebras are locally finite dimensional (see [17] for more on these notions in the nonseparable context) in the sense that each of its finite subsets can be approximated from a finite dimensional $C^\star$-subalgebra ([34, 35]). So $\mathcal{I}_\alpha^{At}(\mathcal{A})$ is locally finite dimensional and separable for each $\alpha < \omega$ and so AF. Thus the result of [48] implies that $\mathcal{I}_\alpha^{At}(\mathcal{A})$ is singly generated for each $\alpha < \omega_1$. So pick $A_{\alpha+1}$ as before. This completes the proof of the theorem.   ∎

Using the free set lemmas as in [13], one can prove that given a discrete set of operators $(\mathcal{A}_\alpha)_{\alpha<\omega_n}$ for $n \in \mathbb{N}$, there is an n-element irredundant set. However, there is the following much stronger consistent extraction principle.

Theorem 3.20   *It is relatively consistent that whenever $(A_\xi : \xi < 2^\omega)$ is a collection of operators in $\mathcal{B}(\ell_2)$ that generates a C*-algebra of density continuum, then there is a set $I \subseteq 2^\omega$ of cardinality continuum such that $(A_\xi : \xi \in I)$ is irredundant.*

Proof   To obtain the relative consistency, we will use the method of forcing (see [33]). We start with the ground model V satisfying the generalized continuum hypothesis (GCH), and we will consider the generic extension $V[G]$ where G is a generic set in the forcing $\mathbb{P} = Fn(\omega_2, 2)$ for adding ω_2 Cohen reals (see [33, Chapter VIII §2]).

Fix a ground model orthonormal basis $(e_n : n \in \mathbb{N})$ for ℓ_2 in V. In $V[G]$ let $(A_\xi : \xi < 2^\omega)$ be as in the theorem. By passing to a subset of cardinality $2^\omega = \omega_2$ and using the hypothesis that $(A_\xi : \xi < 2^\omega)$ is a collection of operators in $\mathcal{B}(\ell_2)$ that generates a C*-algebra of density continuum, we may assume that A_ξ does not belong to the C*-algebra generated by the operators $(A_\eta : \eta < \xi)$ for each $\xi < \omega_1$. Moreover, by passing to a subsequence, we can assume that there is a rational $\varepsilon > 0$ such that $\|A - A_\xi\| > \varepsilon$ for every A in the C*-algebra generated by the operators $(A_\eta : \eta < \xi)$ for each $\xi < \omega_1$.

Each A_ξ can be identified with an $\mathbb{N} \times \mathbb{N}$ complex valued matrix $(\langle A_\xi(e_n), e_m \rangle)_{m,n \in \mathbb{N}}$. Let $\dot{A}_\xi$ be $\mathbb{P}$-names in V for these matrices. Using the standard argument of nice names, the countable chain condition for $\mathbb{P}$ and passing to a subsequence using the Δ-system lemma for countable sets that follows from the GCH, we can assume that there are permutations $\sigma_{\xi,\eta} : \omega_2 \to \omega_2$ that lift to the automorphisms of $\mathbb{P}$ and the permutations $\sigma'_{\xi,\eta}$ of $\mathbb{P}$ names such that

$$\sigma'_{\xi,\eta}(\dot{A}_\eta) = \dot{A}_\xi,$$

and for every $\xi, \eta \in \omega_2$, we have that

$$\mathbb{P} \Vdash \phi(\dot{x}_1, \dots, \dot{x}_k) \text{ if and only if } \mathbb{P} \Vdash \phi(\sigma'_{\xi,\eta}(\dot{x}_1), \dots, \sigma'_{\xi,\eta}(\dot{x}_k))$$

for any formula ϕ in $k \in \mathbb{N}$ free variables and any sequence $\dot{x}_1, \dots, \dot{x}_k$ of $\mathbb{P}$-names for $k \in \mathbb{N}$ ([33, 7.13]). Using this for the formulas that say that the distance of A_ξ from any element of the C*-algebra generated by the operators $(A_\eta : \eta \in F)$ for any finite $F \subseteq \xi$ is bigger than ε, we conclude that $\mathbb{P}$ forces that no $\dot{A}_\xi$ belongs the C*-algebra generated by any countable collection from $\{\dot{A}_\eta : \eta \neq \xi\}$ (by considering a permutation of ω_2 that moves ξ above the countable set). This means that $\mathbb{P}$ forces that no $\dot{A}_\xi$ belongs the C*-algebra generated by the remaining operators $\{\dot{A}_\eta : \eta \neq \xi\}$, *i.e.*, that the collection is irredundant as required.   ∎

The above is a version of applying a standard argument as in [52] in the context of Boolean irredundance.

4 Commutators Under OCA

The main consistent construction of this paper presented in the following sections has a strong randomness property. In this section, we show that this randomness does not take place for any uncountable collection of operators in $\mathcal{B}(\ell_2)$ under the assumption

of Open Coloring Axiom (OCA). We will follow the approach to the strong operator topology from the book [11] of Davidson. Thus, we have the following definition.

Definition 4.1 Let H be a Hilbert space. The *strong operator topology* (SOT) on $\mathcal{B}(H)$ is defined as the weakest topology such that the sets

$$S(a, x) := \{b \in B(H) : \|(b - a)(x)\| < 1\}$$

are open for each $a \in \mathcal{B}(H)$ and $x \in H$. We denote by $(\mathcal{B}(H), \tau_{\text{sot}})$ and $(\mathcal{B}(H)_1, \tau_{\text{sot}})$, respectively the space $\mathcal{B}(H)$ and the unit ball of $\mathcal{B}(H)$ with the strong operator topology.

Proposition 4.2 If H is a separable Hilbert space, then $(\mathcal{B}(H)_1, \tau_{\text{sot}})$ is metrizable and separable in the strong operator topology.

Proof For metrizability, see [11, Proposition I.6.3.]. For the separability fix some orthonormal basis $(e_n)_{n \in \mathbb{N}}$ and consider finite rank operators in the linear span of one dimensional operators of the form $v \otimes w$ where v, w have finitely many nonzero rational coordinates with respect to $(e_n)_{n \in \mathbb{N}}$. It is clear that such operators are SOT dense in $\mathcal{B}(l_2)_1$, as required. ∎

By the remarks on [11, pp. 16–17], we have the following lemma.

Lemma 4.3 The multiplication on $\mathcal{B}(H)_1$ is jointly continuous in the SOT topology and so every polynomial [6] is SOT continuous on $\mathcal{B}(H)_1$.

We will follow the approach to the Open Coloring Axiom (OCA) from [15, p. 55]. Its weaker version was discovered by Abraham, Rubin, and Shelah [1], and the final form was introduced by Todorcevic [51]. It is consistent with ZFC. In fact, it is a consequence of the Proper Forcing Axiom (PFA); see [51, Theorem 8]. Recall that

$$[X]^2 = \{\{x, y\} \subseteq X : x \neq y\}.$$

It is well known that the original form of OCA from [51] for subsets of the reals is equivalent to the version for separable metric spaces as in [15].

Definition 4.4 (Todorcevic [51]) OCA denotes the following statement:

If X is a separable metric Hausdorff space and $[X]^2 = K_0 \cup K_1$ is a partition with K_0 open[7], then either there is an uncountable $Y \subseteq X$ such that $[Y]^2 \subseteq K_0$, or else $X = \bigcup_{n \in \mathbb{N}} X_n$ where $[X_n]^2 \subseteq K_1$ for each $n \in \mathbb{N}$.

Theorem 4.5 (OCA) Let $(A_\alpha)_{\alpha < \omega_1}$ be an uncountable family in $\mathcal{B}(l_2)$ and let $P(x, y)$ be a polynomial satisfying $\|P(A, B)\| = \|P(B, A)\|$ for all $A, B \in \mathcal{B}(l_2)$. Then given $\varepsilon > 0$, either there is an uncountable $\Gamma_0 \subset \omega_1$ such that $\|P(A_\alpha, A_\beta)\| \leq \varepsilon$ for every distinct $\alpha, \beta \in \Gamma_0$ or else there is an uncountable $\Gamma_1 \subset \omega_1$ such that $\|P(A_\alpha, A_\beta)\| > \varepsilon$ for every distinct $\alpha, \beta \in \Gamma_1$.

---

[6] By a polynomial $P(x, y)$, we mean a expression in the form $P(x, y) = \sum_i a_i x^i + \sum_i b_i y^i + \sum_{i,j} c_{i,j} x^i y^j + \sum_{i,j} d_{i,j} y^i x^j + e_0$.

[7] We call $K_0 \subseteq [X]^2$ open if the symmetric set $\{(x, y) \in X \times X : \{x, y\} \in K_0\}$ is open in $(K \times K) \smallsetminus \Delta$ in the product topology, where Δ denotes the diagonal of $X \times X$.

Proof By passing to an uncountable subset, we can assume that there is $M > 0$ such that $\|A_\alpha\| \leq M$ for all $\alpha < \omega_1$. Let $X = \{A_\alpha : \alpha < \omega_1\} \subseteq M\mathcal{B}(\ell_2)_1$ and note that $M\mathcal{B}(\ell_2)_1$ is metric and separable by Proposition 4.2. Define

$$K_0 = \left\{ \{A, B\} \in [X]^2 : \|P(A, B)\| > \varepsilon \right\}$$

and $K_1 = [X]^2 \setminus K_0$.

First note that the separability is hereditary for metric spaces, so X is metric separable as a subspace of $(M\mathcal{B}(l_2)_1, \tau_{sot})$.

Now note that K_0 is open. Indeed if $\|P(A, B)\| > \varepsilon$, then there is $x \in \ell_2$ of norm one and $\delta > 0$ such that $\|P(A, B)(x)\| > \varepsilon + \delta$. Now if $P(A', B') \in S(P(A, B), x/\delta)$, we have $\|P(A', B')(x) - P(A, B)(x)\| < \delta$ and so $\|P(A', B')\| > \varepsilon$. Hence, $\{\{A', B'\} \in [X]^2 : P(A', B') \in S(P(A, B), x/\delta)\} \subseteq K_0$. But $(A, B) \in P^{-1}[S(P(A, B), x/\delta)]$ is open in $X \times X$ with the product SOT topology, by the continuity of P (Lemma 4.3).

So we are in the position of applying the OCA. From Definition 4.4 we obtain the required uncountable set Γ_0 or Γ_1. ∎

Corollary 4.6 (OCA) *Let $(A_\alpha)_{\alpha < \omega_1}$ be an uncountable family in $B(l_2)$. Then given $\varepsilon > 0$, either there is an uncountable $\Gamma_0 \subset \omega_1$ such that $\|[A_\alpha, A_\beta]\| \leq \varepsilon$ for every $\alpha, \beta \in \Gamma_0$ or else there is an uncountable $\Gamma_1 \subset \omega_1$ such that $\|[A_\alpha, A_\beta]\| > \varepsilon$ for every $\alpha, \beta \in \Gamma_1$.*

Proof Consider $P(x, y) = xy - yx$ and apply Theorem 4.5. ∎

Remark 4.7 Let us remark on two trivial versions of the above results. First, let $(A_n)_{n \in \mathbb{N}}$ be an infinite family in $B(l_2)$. Then given $\varepsilon > 0$, either there is an infinite $\Gamma_0 \subset \mathbb{N}$ such that $\|[A_n, A_m]\| \leq \varepsilon$ for every $n, m \in \Gamma_0$ or else there is an infinite $\Gamma_1 \subset \mathbb{N}$ such that $\|[A_n, A_m]\| > \varepsilon$ for every $n, m \in \Gamma_1$. This follows from the Ramsey theorem whose consistent generalization is the OCA.

Secondly, note that if $(A_\alpha)_{\alpha < \omega_1}$ is an uncountable family in a separable C*-subalgebra of $B(\ell_2)$, then by its second countability in the norm topology, it follows that for every $\delta > 0$, there is an uncountable $\Gamma_0 \subseteq \omega_1$ such that $\|A_\alpha - A_\beta\| < \delta$ for every $\alpha, \beta \in \Gamma_0$, and so given any polynomial P satisfying $P(x, x) = 0$ and $\varepsilon > 0$, by the norm continuity of P there is an uncountable $\Gamma_0 \subseteq \omega_1$ such that $\|P(A_\alpha, P_\beta)\| < \varepsilon$ for every $\alpha, \beta \in \Gamma_0$.

In fact, in the nontrivial cases of Theorem 4.5 and Corollary 4.6 when $(A_\alpha)_{\alpha < \omega_1}$ generates a nonseparable C*-subalgebra of $\mathcal{B}(\ell_2)$, we can assume that $(A_\alpha)_{\alpha < \omega_1}$ forms a norm discrete set.

5 The Partial Order of Finite Dimensional Approximations

5.1 Notation

The C*-algebras that we consider in the rest of this paper are subalgebras of $\mathcal{B}(\ell_2(\omega_1 \times \mathbb{N}))$. In fact, the subspaces $\ell_2(\{\xi\} \times \mathbb{N})$ of $\ell_2(\omega_1 \times \mathbb{N})$, which we call columns, will be invariant for all our algebras, so our algebras could be identified with subalgebras of $\Pi_{\xi < \omega_1} \mathcal{B}(\ell_2(\{\xi\} \times \mathbb{N}))$. Also, the map

$$\pi_\alpha \colon \Pi_{\alpha \leq \xi < \omega_1} \mathcal{B}\big(\ell_2(\{\xi\} \times \mathbb{N})\big) \longrightarrow \mathcal{B}\big(\ell_2(\{\alpha\} \times \mathbb{N})\big),$$

applied to the appropriate quotients, will be faithful (see Lemma 5.24(iii)). Thus, the purpose of this presentation of the algebras is related to the transparent structure of the Cantor–Bendixson composition series (see Proposition 5.25(iii)).

For $X \subseteq \omega_1 \times \mathbb{N}$, we introduce the following notation:

- $(e_{\xi,n} : \xi < \omega_1, n \in \mathbb{N})$ is the canonical orthonormal basis of $\ell_2(\omega_1 \times \mathbb{N})$.
- Denote by $\mathcal{B}_X$ the family of all operators A in $\mathcal{B}(\ell_2(\omega_1 \times \mathbb{N}))$ such that
  - $\ell_2(X \cap (\{\xi\} \times \mathbb{N}))$ is A-invariant for all $\xi < \omega_1$,
  - $A(e_{\xi,n}) = 0$ whenever $(\xi, n) \notin X$.
- The unit of the C*-algebra $\mathcal{B}_X$ will be denoted by P_X.
- $1_{\xi,m,n}$ is the operator in $\mathcal{B}_{\omega_1 \times \mathbb{N}}$ satisfying

$$1_{\xi,m,n}(e_{\eta,k}) = \begin{cases} e_{\xi,m} & \text{if } k = n, \xi = \eta, \\ 0 & \text{otherwise.} \end{cases}$$

- If $A \in \mathcal{B}_{\omega_1 \times \mathbb{N}}$ we define $A|X = AP_X$.
- If $A \in \mathcal{B}_{\omega_1 \times \mathbb{N}}$ and $a \subseteq \omega_1$ we define $A|a$ as $A|(a \times \mathbb{N})$.
- $\mathcal{A}|X = \{A|X : A \in \mathcal{A}\}$ for $\mathcal{A} \subseteq \mathcal{B}_{\omega_1 \times \mathbb{N}}$ and $X \subseteq \omega_1 \times \mathbb{N}$.

5.2 The Definition of the Partial Order of Finite-dimensional Approximations

Definition 5.1 We define a partial order $\mathbb{P}$ consisting of elements

$$p = \left(a_p, \{n_\xi^p : \xi \in a_p\}, \{A_{\xi,m,n}^p : \xi \in a_p, n, m \in [0, n_\xi^p)\} \right),$$

where

(i) a_p is a finite subset of ω_1;
(ii) $n_\xi^p \in \mathbb{N}$ for each $\xi \in a^p$;
(iii) $A_{\xi,m,n}^p \in \mathcal{B}_{X_p}$ for each $\xi \in a_p$ and $n, m \in [0, n_\xi^p)$, where

$$X_p = \{(\xi, n) : \xi \in a_p; n \in [0, n_\xi^p)\};$$

(iv) $A_{\xi,m,n}^p = (A_{\xi,m,n}^p|\xi) + 1_{\xi,m,n}$ for each $\xi \in a_p$ and $n, m \in [0, n_\xi^p)$.

The order $\leq_{\mathbb{P}} = \leq$ on $\mathbb{P}$ is defined by declaring $p \leq q$ if and only if the following hold:

(a) $a_p \supseteq a_q$;
(b) $n_\xi^p \geq n_\xi^q$ for $\xi \in a_q$;
(c) there is a (nonunital) *-embedding $i_{pq} : \mathcal{B}_{X^q} \to \mathcal{B}_{X^p}$ such that $i_{pq}(A_{\xi,m,n}^q) = A_{\xi,m,n}^p$ for all $\xi \in a_q$ and $m, n \in [0, n_\xi^q)$;
(d) $i_{p,q}(A)|X_q = A$ for all $A \in \mathcal{B}_{X^q}$.

Definition 5.2 Suppose that $p \in \mathbb{P}$ and $X \subseteq X_p$. Then the C*-subalgebra of $\mathcal{B}_{X_p}$ generated by $\{A_{\xi,m,n}^p : (\xi, m), (\xi, n) \in X\}$ is denoted by $\mathcal{A}_X^p$.

Lemma 5.3 *For every $\alpha \in \omega_1$ and every $p \in \mathbb{P}$, we have*

$$\mathcal{A}_{X_p \cap (\alpha \times \mathbb{N})}^p = \mathcal{B}_{X_p \cap (\alpha \times \mathbb{N})}.$$

In particular, $\mathcal{A}_{X_p}^p = \mathcal{B}_{X_p}$.

Proof We will prove this by induction on $|a_p \cap \alpha|$. If $a_p \cap \alpha = \varnothing$, then both of the algebras are $\{0\}$. Suppose $|a_p \cap \alpha| = n + 1$ and we have proved the Lemma for every $q \in \mathbb{P}$ and $\alpha < \omega_1$ such that $|a_q \cap \alpha| = n$. Let $\xi = \max(a_p \cap \alpha)$. By the definition of $\mathcal{B}_{X_p}$ we have that $\mathcal{B}_{X_p \cap (\alpha \times \mathbb{N})}$ is $*$-isomorphic to $\mathcal{B}_{X_p \cap (\xi \times \mathbb{N})} \oplus \mathcal{B}_{\{\xi\} \times [0, n_\xi^p)}$. By the inductive hypothesis, $\mathcal{B}_{X_p \cap (\xi \times \mathbb{N})}$ is generated by $\{A_{\eta,m,n}^p : \eta \in a_p \cap \xi; m, n \in [0, n_\xi^p)\}$. But by Definition 5.1(iv), we have that $1_{\xi,m,n} = A_{\xi,m,n}^p - A$ for some $A \in \mathcal{B}_{X_p \cap (\xi \times \mathbb{N})}$ and all $m, n \in [0, n_\xi^p)$. In particular, $\mathcal{B}_{\{\xi\} \times [0, n_\xi^p)}$ is included in the algebra generated by $\{A_{\eta,m,n}^p : \eta \in a_p \cap \alpha; m, n \in [0, n_\eta^p)\}$. This together with the inductive hypothesis completes the proof. ∎

Lemma 5.4 *Suppose that $\alpha < \omega_1$ and $p, q \in \mathbb{P}$ satisfy $p \leq q$ and $A = i_{p,q}(B)$, where $B \in \mathcal{A}_{X_q}^q$. Then*

$$\|A|[\alpha, \omega_1)\| = \|B|[\alpha, \omega_1)\|.$$

Proof Since $B|\alpha$ and $B|[\alpha, \omega_1)$ are in $\mathcal{A}_{X_q}^q$, by Lemma 5.3, we have

$$\|A|[\alpha, \omega_1)\| = \|i_{p,q}(B)|[\alpha, \omega_1)\| = \|i_{p,q}(B|\alpha)|[\alpha, \omega_1) + i_{p,q}(B|[\alpha, \omega_1))|[\alpha, \omega_1)\|.$$

But $B|\alpha \in \mathcal{A}_{X_q \cap (\alpha \times \mathbb{N})}^q$ by Lemma 5.3 and this generation must be preserved by the isomorphism $i_{p,q}$, *i.e.*, $i_{p,q}(B|\alpha)|[\alpha, \omega_1) = 0$, and so

$$\|A|[\alpha, \omega_1)\| = \|i_{p,q}(B|[\alpha, \omega_1))|[\alpha, \omega_1)\| \leq \|i_{p,q}(B|[\alpha, \omega_1))\|.$$

Since $i_{p,q}$ is an embedding (in particular an isometry), we conclude that

$$\|A|[\alpha, \omega_1)\| \leq \|B|[\alpha, \omega_1)\|.$$

The other inequality follows from Definition 5.1(iii)–(iv). ∎

5.3 Density Lemmas

In the terminology related to partial orders occurring in the theory of forcing a subset $\mathbb{D}$ of a partial order $\mathbb{Q}$ is said to be dense if for every $p \in \mathbb{Q}$ there is $d \in \mathbb{D}$ satisfying $d \leq p$. In what follows, we usually need stronger information for $\mathbb{Q} = \mathbb{P}$, namely that $a_d = a_p$.

Lemma 5.5 *Suppose that $\xi < \omega_1$. Then $\mathbb{D}_\xi = \{p \in \mathbb{P} : \xi \in a_p\}$ is a dense subset of $\mathbb{P}$.*

Proof Let $q \in \mathbb{P}$ be such that $\xi \notin a_q$. Define p as follows:

- $a_p = a_q \cup \{\xi\}$;
- $n_\eta^p = n_\eta^q$ for $\eta \in a_q$ and $n_\xi^p = 1$;
- $A_{\eta,m,n}^p = A_{\eta,m,n}^q$ for $\eta \in a_q$ and $A_{\xi,0,0}^p = 1_{\xi,0,0}$.

It is clear that $p \in \mathbb{P}$. Also, $p \leq q$ as $\mathrm{Id}_{\mathcal{B}_{X_q}} : \mathcal{B}_{X_q} \to \mathcal{B}_{X_p}$ is a $*$-embedding good for $i_{p,q}$ in Definition 5.1(iii). ∎

Lemma 5.6 *Suppose that $\xi < \omega_1$, $k \in \mathbb{N}$, and $q \in \mathbb{P}$ are such that $\xi \in a_q$. Then there is*

$$p \in \mathbb{E}_{\xi,k} = \{p \in \mathbb{P} : \xi \in a_p, n_\xi^p \geq k\}$$

such that $p \leq q$ and $a_p = a_q$.

Proof

Consider $q \in \mathbb{P}$ such that $\xi \in a_q$ but $n_\xi^q < k$. Define p as follows:

- $a_p = a_q$;
- $n_\eta^p = n_\eta^q$ for $\eta \in a_p \smallsetminus \{\xi\}$ and $n_\xi^p = k$;
- $A_{\eta,m,n}^p = A_{\eta,m,n}^q$ $\eta \in a_q \smallsetminus \{\xi\}$;
- $A_{\xi,m,n}^p = A_{\xi,m,n}^q$ for $n, m \in [0, n_\xi^q)$;
- $A_{\xi,m,n}^p = 1_{\xi,m,n}$ if $n, m \in [0, k)$ and $\{n, m\} \cap [n_\xi^q, k) \ne \varnothing$.

It is clear that $p \in \mathbb{P} \cap \mathbb{E}_{\xi,k}$. Also $p \le q$ as $\mathrm{Id}_{\mathcal{B}_{X_q}} : \mathcal{B}_{X_q} \to \mathcal{B}_{X_p}$ is a $*$-embedding good for $i_{p,q}$ in Definition 5.1(iii). ∎

Lemma 5.7 *Suppose that $q \in \mathbb{P}$ and $X \subseteq X_q$ and that $\alpha \in a_q$. Then there is $p \le q$ such that $p \in \mathbb{F}_{X,\alpha}$, where*

$$\mathbb{F}_{X,\alpha} = \left\{ p \in \mathbb{P} : \alpha \in a_p, X \subseteq X_p, \text{ and } \forall A \in \mathcal{A}_X^p \|A|\{\alpha\}\| \ge \|A|[\alpha, \omega_1)\| \right\}.$$

Moreover, $a_p = a_q$ and $n_\xi^p = n_\xi^q$ whenever $\xi \in a_p \smallsetminus \{\alpha\}$.

Proof   Let $q \in \mathbb{P}$. We can assume that $X = X_q$. If $\alpha = \max(a_q)$, then there is nothing to prove. So let $a_q \smallsetminus (\alpha + 1) = \{\xi_1, \dots, \xi_k\}$ for some $k \in \mathbb{N}$ and put

$$l = \sum\{ n_{\xi_i}^q : 1 \le i \le k \}.$$

Consider $Y = X_q \cap ((\alpha, \omega_1) \times \mathbb{N})$. Let $\phi : Y \to [n_\alpha^q, n_\alpha^q + l)$ be any bijection. We obtain a $*$-homomorphism $i : \mathcal{B}_{X_q} \to \mathcal{B}_{X_q \cup (\{\alpha\} \times [n_\alpha^q, n_\alpha^q + l))}$ given by $i(A) = A + i_r(A)$, where $i_r : \mathcal{B}_{X_q} \to \mathcal{B}_{\{\alpha\} \times [n_\alpha^q, n_\alpha^q + l)}$ satisfies

$$\left\langle i_r(A)(e_{\alpha, n_\alpha^q + \phi(\xi_i, k)}), e_{\alpha, n_\alpha^q + \phi(\xi_{i'}, k')} \right\rangle = \left\langle A(e_{\xi_i, k}), e_{\xi_{i'}, k'} \right\rangle$$

for all $(\xi, k), (\xi', k') \in Y$ and every $A \in \mathcal{B}_{X_q}$. Define p in the following way:

- $a_p = a_q$;
- $n_\xi^p = n_\xi^q$ if $\xi \in a_p \smallsetminus \{\alpha\}$ and $n_\alpha^p = n_\alpha^q + l$;
- $A_{\xi,m,n}^p = i(A_{\xi,m,n}^q)$ for $(\xi, m), (\xi, n) \in X_q$;
- $A_{\alpha,m,n}^p = 1_{\alpha,m,n}$ if $\{m, n\} \cap [n_\alpha^q, n_\alpha^p) \ne \varnothing$.

It is clear from the construction that $p \in \mathbb{P}$, as condition (iv) of Definition 5.1 is satisfied due to the fact that we change only $A_{\xi,m,n}^q$ for $\xi > \alpha$ on $\{\alpha\} \times \mathbb{N}$, and that (a), (b) of Definition 5.1 are satisfied.

   If we put $i_{p,q} = i$, condition (c) follows from the fact that i is a $*$-embedding, since $\{\alpha\} \times [n_\alpha^q, n_\alpha^q + l) \cap X_q = \varnothing$. We also have $i_{p,q}(A_{\xi,m,n}^q) = A_{\xi,m,n}^p$ for $(\xi, m), (\xi, n) \in X_q$. The construction yields (d) of Definition 5.1.

Finally, to check the main assertion of the lemma, note that by Lemma 5.4 for any $A \in \mathcal{B}_{X_q}$, we have

$$
\begin{aligned}
\|i_{p,q}(A)|\{\alpha\}\| &= \max\left(\|i_{p,q}(A)|\{\alpha\}\|, \|i_{p,q}(A)|\{\alpha\} \times [n_\alpha^q, n_\alpha^q + n)\|\right) \\
&= \max\left(\|i_{p,q}(A)|\{\alpha\}\|, \|A|(\alpha, \omega_1)\|\right) \\
&= \max\left(\|i_{p,q}(A)|\{\alpha\}\|, \|i_{p,q}(A)|(\alpha, \omega_1)\|\right) \\
&= \|i_{p,q}(A)|[\alpha, \omega_1)\|
\end{aligned}
$$

for any $A \in \mathcal{B}_{X_q}$ as required, since $X \subseteq X_q$. ∎

Lemma 5.8 *Let $X \subseteq \omega_1 \times \mathbb{N}$ be finite and let $\alpha \in X$. If $q \in \mathbb{F}_{X,\alpha}$ and $p \le q$, then $p \in \mathbb{F}_{X,\alpha}$.*

Proof Let $A \in \mathcal{A}_X^p$. As $X \subseteq X_q$, we have that $A = i_{p,q}(B)$ for some $B \in \mathcal{A}_X^q \subset \mathcal{A}_{X_q}^q$. First note that by Lemma 5.4,

$$
\|A|[\alpha, \omega_1)\| = \|B|[\alpha, \omega_1)\|.
$$

Now $\|B|[\alpha, \omega_1)\| \le \|B|\{\alpha\}\|$ by the hypothesis that $q \in F_{X,\alpha}$. But $\|B|\{\alpha\}\| \le \|A|\{\alpha\}\|$ by the fact that $A|X^q = B$ by Definition 5.1(d). So $\|A|[\alpha, \omega_1)\| \le \|A|\{\alpha\}\|$, as required. ∎

5.4 Basic Amalgamations

Definition 5.9 We say that two elements $p, q \in \mathbb{P}$ are in convenient position (as witnessed by $\sigma\colon a_p \to a_q$) if and only if

$$
\Delta := a_p \cap a_q < a_p \smallsetminus \Delta < a_q \smallsetminus \Delta,
$$

and there is an order preserving bijection $\sigma\colon a_p \to a_q$ such that

- $n_\xi^p = n_{\sigma(\xi)}^q$ for $\xi \in a_p$,

and the $*$-isomorphism of $\mathcal{B}_{X_q}$ onto $\mathcal{B}_{X_p}$ induced by σ, denoted by j_σ, which is given by

$$
\left\langle j_\sigma(A)(e_{\xi,k}), e_{\xi,l} \right\rangle = \left\langle A(e_{\sigma(\xi),k}), e_{\sigma(\xi),l} \right\rangle
$$

for every $(\xi, k), (\xi, l) \in X_p$ and $A \in \mathcal{B}_{X_q}$ satisfies

- $j_\sigma(A_{\sigma(\xi),n,m}^q) = A_{\xi,n,m}^p$ for every $\xi \in a_p$, $n, m \in [0, n_\xi^p)$.

Lemma 5.10 *Suppose that two elements $p, q \in \mathbb{P}$ are in convenient position as witnessed by $\sigma\colon a_p \to a_q$ and that $\xi \in \Delta = a_p \cap a_q$. Then $A_{\xi,n,m}^q = A_{\xi,n,m}^p$ for every $n, m \in [0, n_\xi^p) = [0, n_\xi^q)$.*

Proof Note that in Definition 5.9 the bijection σ must be the identity on Δ, because it is order-preserving and Δ is the initial fragment of both a_p and a_q and so any $\xi \in \Delta$ must have the same position in both a_p and a_q. So $j_\sigma(A_{\xi,n,m}^q) = A_{\xi,n,m}^p$, and it is enough to prove that $j_\sigma(A_{\xi,n,m}^q) = A_{\xi,n,m}^q$.

For $\eta \in a_p \smallsetminus \Delta$ we have

$$
\langle j_\sigma(A_{\xi,n,m}^q)(e_{\eta,k}), e_{\eta,l} \rangle = \langle A_{\xi,n,m}^q(e_{\sigma(\eta),k}), e_{\sigma(\eta),l} \rangle = 0
$$

for every $k, l \in \mathbb{N}$ such that $(\eta, k), (\eta, l) \in X_p$ as $\sigma(\eta) \in a_q \smallsetminus \Delta$.

On the other hand, for $\eta \in \Delta$ we have $\sigma(\eta) = \eta$, and so

$$\left\langle j_\sigma(A^q_{\xi,n,m})(e_{\eta,k}), e_{\eta,l} \right\rangle = \left\langle A^q_{\xi,n,m}(e_{\eta,k}), e_{\eta,l} \right\rangle$$

for every $k, l \in \mathbb{N}$ such that $(\eta, k), (\eta, l) \in X_p$ as $\sigma(\eta) = \eta$ by Definition 5.9. Using Definition 5.1(iv), this proves the required $A^q_{\xi,n,m} = j_\sigma(A^q_{\xi,n,m}) = A^p_{\xi,n,m}$. ∎

Lemma 5.11 *Suppose that $p, q \in \mathbb{P}$ are in convenient position as witnessed by $\sigma \colon a_p \to a_q$. Then there is $r \leq p, q$ such that*

- $a_r = a_p \cup a_q$;
- $n^r_\xi = n^p_\xi$ if $\xi \in a_p$ and $n^r_\xi = n^q_\xi$ if $\xi \in a_q$;
- $i_{r,p} = \mathrm{Id}_{\mathcal{B}_{X_p}}$, $i_{r,q} = \mathrm{Id}_{\mathcal{B}_{X_q}}$.

In particular,

- $A^r_{\xi,m,n} = A^p_{\xi,m,n}$ *for each $\xi \in a_p$ and $n, m \in [0, n^r_\xi)$;*
- $A^r_{\xi,m,n} = A^q_{\xi,m,n}$ *for each $\xi \in a_q$ and $n, m \in [0, n^r_\xi)$.*

The element r will be called the disjoint amalgamation of p and q.

Proof Define r as in the lemma. As $p, q \in \mathbb{P}$, it is easy to see that $r \in \mathbb{P}$. To see that $r \leq p, q$ note that $\mathrm{Id}_{\mathcal{B}_{X_p}}$ and $\mathrm{Id}_{\mathcal{B}_{X_q}}$ are $*$-embeddings into $\mathcal{B}_{X_r}$. ∎

Lemma 5.12 *Suppose that p, q are two elements of $\mathbb{P}$ in convenient position as witnessed by $\sigma_{q,p} \colon a_p \to a_q$. Let $U \in \mathcal{B}_{X_p \cup X_q}$ be a partial isometry satisfying $UU^* = U^*U = P_{X_p \setminus X_q}$, where $P_{X_p \setminus X_q}$ is the projection on the space spanned by $\{e_{\xi,k} : (\xi, k) \in X_p \setminus X_q\}$. Then there is $r_U = r \leq p, q$ such that*

- $a_r = a_p \cup a_q$;
- $n^r_\xi = n^p_\xi$ if $\xi \in a_p$, $n^r_\xi = n^q_\xi$ if $\xi \in a_q$;
- $i_{r,p} = \mathrm{Id}_{\mathcal{B}_{X_p}}$;
- $i_{r,q}(A) = A + U j_{\sigma_{q,p}}(A) U^*$ *for all $A \in \mathcal{B}_{X_q}$.*

In particular,

- $A^r_{\xi,m,n} = A^p_{\xi,m,n}$ *for $\xi \in a_p$ and $m, n \in [0, n^r_\xi)$;*
- $A^r_{\xi,m,n} = U A^p_{\sigma^{-1}_{q,p}(\xi),m,n} U^* + A^q_{\xi,m,n}$ *for $\xi \in a_q \setminus a_p$ and $m, n \in [0, n^r_\xi)$.*

The element r_U will be called the U-including amalgamation of p and q; if $U = P_{X_p \setminus X_q}$, then r_U is called the including amalgamation.

Proof Define r_U as in the lemma. It is clear by Definition 5.1 applied to p and q that $r \in \mathbb{P}$. Then $r_U \leq p$, because $\mathrm{Id}_{\mathcal{B}_{X_p}} \colon \mathcal{B}_{X_p} \to \mathcal{B}_{X_r}$ is a $*$-embedding. For $r_U \leq q$, we note that $A^r_{\xi,m,n}|X_q = A^q_{\xi,m,n}$ as $(U A^p_{\sigma^{-1}_{q,p}(\xi),m,n} U^*)|X_q = 0$, since $UU^* = U^*U = P_{X_p \setminus X_q}$ and that the formula $i_{r,q}(A) = A + U j_\sigma(A) U^*$ for all $A \in \mathcal{B}_{X_q}$ defines a $*$-embedding from $\mathcal{B}_{X_p}$ to $\mathcal{B}_{X_r}$. This follows from the fact that sending A to $U j_\sigma(A) U^*$ is a $*$-homomorphism since $\mathcal{B}_{X_p \setminus X_q}$ is $A^p_{X_p}$-invariant, so $i_{r,q}$ is a $*$-homomorphism. But its kernel is null, since $U j_\sigma(A) U^* = (U j_\sigma(A) U^*)|(X_p \setminus X_q)$ for all $A \in \mathcal{B}_{X_q}$. ∎

Lemma 5.13 *Suppose that v_1, v_2 are two orthogonal unit vectors of $\mathbb{C}^n$ for $n > 1$. Then there is a unitary $U \in M_n$ such that*

$$\|[UAU^*, A]\| = 1/2$$

for every nonexpanding linear $A \in M_n$ satisfying $A(v_1) = v_1$ and $A(v_2) = 0$.

Proof Choose an orthonormal basis $v_1, \ldots, v_n$ of $\mathbb{C}^n$ starting with v_1, v_2 and consider the orthogonal projection $P \in M_n$ onto the line containing v_1, so, in particular, we have $P(v_1) = v_1$ and $P(v_2) = 0$. Let $U = V \oplus I_{n-2}$, $U^* = V^* \oplus I_{n-2}$, where

$$V = V^* = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

So we obtain that

$$UPU^* = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \oplus 0_{n-2} = \begin{pmatrix} \frac{-1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{-1}{2} \end{pmatrix} \oplus 0_{n-2}.$$

Hence,

$$[UPU^*, P] = UPU^*P - PUPU^*$$

$$= \begin{pmatrix} \frac{-1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{-1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus 0_{n-2} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{-1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{-1}{2} \end{pmatrix} \oplus 0_{n-2}$$

$$= \begin{pmatrix} \frac{-1}{2} & 0 \\ \frac{-1}{2} & 0 \end{pmatrix} \oplus 0_{n-2} - \begin{pmatrix} \frac{-1}{2} & \frac{-1}{2} \\ 0 & 0 \end{pmatrix} \oplus 0_{n-2} = \begin{pmatrix} 0 & \frac{-1}{2} \\ \frac{-1}{2} & 0 \end{pmatrix} \oplus 0.$$

And so $\|[UPU^*, P]\| = 1/2$, and, in particular,

- $[UPU^*, P](v_1) = (-1/2)v_2$,
- $[UPU^*, P](v_2) = (1/2)v_1$.

Since P equals A on the space spanned by v_1 and v_2, and U, U^* leave this space invariant, we have the same equalities for A instead of P, hence $\|[UAU^*, A]\| \geq 1/2$. The other inequality follows from the fact that $\|[B, C]\| \leq 1/2$ for any two B, C satisfying $0 \leq B, C \leq 1$ by a result of Stampfli [47, Corollary 2]. ∎

Lemma 5.14 *Suppose that p, q are two elements of $\mathbb{P}$ in convenient position as witnessed by $\sigma\colon a_p \to a_q$ such that $\Delta < a_p \smallsetminus \Delta < a_q \smallsetminus \Delta$. Suppose that $n_\xi^q = n \geq 1$ for every $\xi \in a_q \smallsetminus a_p$ and that $v_1 = (v_0^1, \ldots, v_{n-1}^1)$, $v_2 = (v_0^2, \ldots, v_{n-1}^2)$ are two orthogonal unit vectors of $\mathbb{C}^n$. Then there is $r \leq p, q$ such that*

- $a_r = a_p \cup a_q$,
- $n_\xi^r = n_\xi^p$ *if* $\xi \in a_p$, $n_\xi^r = n_\xi^q$ *if* $\xi \in a_q$,
- $\|[i_{rq}(A), i_{rp}(j_\sigma(A))]\| = 1/2$

for every nonexpanding $A \in \mathcal{B}_{X_q}$ such that there is $\xi \in a_q \smallsetminus a_p$ with

$$A\Big(\sum_{k<n} v_k^1 e_{\xi,k} \Big) = \sum_{k<n} v_k^1 e_{\xi,k} \quad and \quad A\Big(\sum_{k<n} v_k^2 e_{\xi,k} \Big) = 0.$$

We call r the (v_1, v_2)-anticommuting amalgamation of p and q.

Proof By Lemma 5.13, for each $\xi \in a_q \setminus a_p$ for $\eta_\xi = \sigma^{-1}(\xi)$, there is a unitary $U_\xi \in \mathcal{B}_{\{\eta_\xi\} \times [0,n)}$ such that

$$(*) \qquad \left\| [U_\xi(j_\sigma(A)|\{\eta_\xi\})U_\xi^*, j_\sigma(A)|\{\eta_\xi\}] \right\| = 1/2,$$

whenever $A \in \mathcal{B}_{X_q}$ is nonexpanding such that

$$A\left(\sum_{k<n} v_k^1 e_{\xi,k}\right) = \sum_{k<n} v_k^1 e_{\xi,k} \quad \text{and} \quad A\left(\sum_{k<n} v_k^2 e_{\xi,k}\right) = 0.$$

Let $U \in \mathcal{B}_{X^p \cup X^q}$ be a partial isometry such that $U|(\{\eta_\xi\} \times [0,n)) = U_\xi$ and U is zero on the columns not in in $X^p \setminus X^q$, and $UU^* = P_{X^p \setminus X^q}$. Consider the U-including amalgamation $r_U \leq p, q$ as in Lemma 5.12.

We claim that $r = r_U$ satisfies the lemma we are proving. Let $A \in \mathcal{B}_{X_q}$ be non-expanding, and let $\xi \in a_q \setminus a_p$ be such that $A(\sum_{k<n} v_k^1 e_{\xi,k}) = \sum_{k<n} v_k^1 e_{\xi,k}$ and $A(\sum_{k<n} v_k^2 e_{\xi,k}) = 0$. Since $\ell_2(\{\xi\} \times [0, n_\xi^q))$ for $\xi \in a_q \setminus a_p$ are invariant for $\mathcal{B}_{X^q}$, the operator $A|\{\xi\}$ is nonexpanding as well, and so is $j_\sigma(A)|\{\eta_\xi\}$. By Lemma 5.12, we have $i_{r,q}(A) = A + Uj_\sigma(A)U^*$ and $i_{r,p}(j_\sigma(A)) = j_\sigma(A)$, so for $\eta_\xi = \sigma^{-1}(\xi)$ we have

$$[i_{r,q}(A), i_{r,p}(j_\sigma(A))]|(\{\eta_\xi\} \times [0,n)) = [U_\xi(j_\sigma(A)|\{\eta_\xi\})U_\xi^*, j_\sigma(A)|\{\eta_\xi\}],$$

So by $(*)$ we have $\|[i_{r,q}(A), i_{r,p}(j_\sigma(A))]\| \geq 1/2$. The other inequality follows from the maximality of $1/2$ [47, Corollary 2]. ∎

5.5 Types of 3-amalgamations

Lemma 5.15 *Suppose that p_1, p_2, p_3 are distinct elements in $\mathbb{P}$ that are pairwise in convenient position as witnessed by $\sigma_{j,i} : a_{p_i} \to a_{p_j}$ for $1 \leq i < j \leq 3$ such that $\Delta < a_{p_1} \setminus \Delta < a_{p_2} \setminus \Delta < a_{p_3} \setminus \Delta$. Then there is $r \leq p_1, p_2, p_3$ satisfying the following:*

- *$a_r = a_{p_1} \cup a_{p_2} \cup a_{p_3}$;*
- *there is $n \in \mathbb{N}$ such that for each $\xi \in a_r$ we have*

$$n = n_\xi^r > n' = \max\{n_\xi^{p_i} : \xi \in a_{p_i}, 1 \leq i \leq 3\};$$

- *$r \in \bigcap\{\mathbb{F}_{X,\alpha} : X \in \{X_{p_1}, X_{p_2}, X_{p_3}\}, \alpha \in X\}$.*

The element r is called the amalgamation of p_1, p_2, p_3 of type 1.

Proof Let $a_{p_1} = \{\alpha_1, \ldots, \alpha_k\}$ in the increasing order. Using Lemma 5.7 find $p_1 \geq p_1^1 \geq \cdots \geq p_1^k$ such that $a_{p_1} = a_{p_1^k}$ and $a_{p_1^j} \in \mathbb{F}_{X_{p_1}, \alpha_j}$ for $1 \leq j \leq k$. Now using Lemma 5.6 several times, find $q_1 \leq p_1^k$ such that $a_{q_1} = a_{p_1}$ and $n_\xi^{q_1} = n > n'$ for every $\xi \in a_{q_1}$.

Now find $q_2, q_3 \in \mathbb{P}$ such that $q_2 \leq p_2$ and $q_3 \leq p_3$ and "isomorphic" with q_1, i.e., with $a_{q_2} = a_{p_2}$, $a_{q_3} = a_{p_3}$ and where q_1, q_2, q_3 are pairwise in convenient position as witnessed by $\sigma_{j,i} : a_{q_i} \to a_{q_j}$ for $1 \leq i < j \leq 3$. Note that by Lemma 5.8 we have

$$q_i \in \bigcap\{\mathbb{F}_{X_{p_i}, \alpha} : \alpha \in X_{p_i}\}.$$

Now let $s_1 \leq q_1, q_2$ and $s_2 \leq q_1, q_3$ be the disjoint amalgamations as in Lemma 5.11. Note that s_1 and s_2 are in convenient position as witnessed by $\text{Id}_{a_{p_1}} \cup \sigma_{3,2} : a_{p_1} \cup a_{p_2} \to a_{p_1} \cup a_{p_3}$ where $a_{s_1} \cap a_{s_2} = a_{p_1}$. So now let $r \leq s_1, s_2$ be the disjoint amalgamation

of s_1 and s_2 as in Lemma 5.11. Note that we have the final statement of the lemma by Lemma 5.8. ∎

Lemma 5.16 *Suppose that p_1, p_2, p_3 are distinct elements in $\mathbb{P}$ that are pairwise in convenient position as witnessed by $\sigma_{j,i} \colon a_{p_i} \to a_{p_j}$ for $1 \le i < j \le 3$ such that $\Delta < a_{p_1} \smallsetminus \Delta < a_{p_2} \smallsetminus \Delta < a_{p_3} \smallsetminus \Delta$. Then there is $r \le p_1, p_2, p_3$ satisfying*

- $a_r = a_{p_1} \cup a_{p_2} \cup a_{p_3}$;
- $n_\xi^r = n_\xi^{p_i}$ *if $\xi \in a_{p_i}$ for $1 \le i \le 3$;*
- $i_{r,p_1} = \mathrm{Id}_{\mathcal{B}_{X_{p_1}}}$;
- $i_{r,p_2}(A) = A + j_{\sigma_{2,1}}(A)|_{X_{p_1} \smallsetminus X_{p_2}}$ *for all $A \in \mathcal{B}_{X_{p_2}}$;*
- $i_{r,p_3}(A) = A + j_{\sigma_{3,1}}(A)|_{X_{p_1} \smallsetminus X_{p_3}}$ *for all $A \in \mathcal{B}_{X_{p_3}}$.*

In particular,

$$i_{r,p_3}(A) i_{r,p_2}(j_{\sigma_{3,2}}(A)) = i_{r,p_1}(j_{\sigma_{3,1}}(A))^2$$

for every $A \in \mathcal{B}_{X_{p_3}}$. The element r is called the amalgamation of p_1, p_2, p_3 of type 2.

Proof First consider $s_2 \le p_1, p_2$ and $s_3 \le p_1, p_3$, which are the including amalgamations of p_1, p_2 and p_1, p_3 as in Lemma 5.12. It is clear that s_1 and s_2 are in convenient position as witnessed by $\mathrm{Id}_{a_{p_1}} \cup \sigma_{3,2} \colon a_{p_1} \cup a_{p_2} \to a_{p_1} \cup a_{p_3}$. Now let r be the disjoint amalgamation of s_1 and s_2 as in Lemma 5.11. The properties of r follow from Lemma 5.12 and Definition 5.1.

To prove the last statement of the lemma, note that

$$\begin{aligned}
i_{r,p_3}&(A) i_{r,p_2}(j_{\sigma_{3,2}}(A)) \\
&= \big(A + j_{\sigma_{3,1}}(A)|(X_{p_1} \smallsetminus X_{p_3})\big)\big(j_{\sigma_{3,2}}(A) + j_{\sigma_{3,1}}(A)|(X_{p_1} \smallsetminus X_{p_3})\big) \\
&= (j_{\sigma_{3,1}}(A))^2 = i_{r,p_1}(j_{\sigma_{3,1}}(A))^2. \qquad \blacksquare
\end{aligned}$$

Lemma 5.17 *Suppose that p_1, p_2, p_3 are distinct elements in $\mathbb{P}$ that are pairwise in convenient position as witnessed by $\sigma_{j,i} \colon a_{p_i} \to a_{p_j}$ for $1 \le i < j \le 3$ such that $\Delta < a_{p_1} \smallsetminus \Delta < a_{p_2} \smallsetminus \Delta < a_{p_3} \smallsetminus \Delta$ and $n_\xi^{p_i} = n$ for some $n > 1$ and each $i \in \{1, 2, 3\}$ and that $v_1 = (v_0^1, \ldots, v_{n-1}^1), v_2 = (v_0^2, \ldots, v_{n-1}^2)$ are two orthogonal unit vectors of $\mathbb{C}^n$. Then there is $r \le p_1, p_2, p_3$ satisfying*

- $a_r = a_{p_1} \cup a_{p_2} \cup a_{p_3}$;
- $n_\xi^r = n_\xi^{p_i} = n$ *if $\xi \in a_{p_i}$ for $1 \le i \le 3$;*
- $\|[i_{r,p_m}(A), i_{r,p_1}(j_{\sigma_{m,1}}(A))]\| = 1/2$, *for $m = 2, 3$ and for every nonexpanding $A \in \mathcal{B}_{X_{p_m}}$ such that there is $\xi \in a_m \smallsetminus a_{p_1}$ with*

$$A\Big(\sum_{k<n} v_k^1 e_{\xi,k}\Big) = \sum_{k<n} v_k^1 e_{\xi,k} \quad and \quad A\Big(\sum_{k<n} v_k^2 e_{\xi,k}\Big) = 0.$$

The element r is called the amalgamation of p_1, p_2, p_3 of type 3 for vectors v_1 and v_2.

Proof First consider $s_2 \le p_1, p_2$ and $s_3 \le p_1, p_3$, which are the (v_1, v_2)-anticommuting amalgamations of p_1, p_2 and p_1, p_3 as in Lemma 5.14. It is clear that s_1 and s_2 are in convenient position as witnessed by $\mathrm{Id}_{a_{p_1}} \cup \sigma_{3,2} \colon a_{p_1} \cup a_{p_2} \to a_{p_1} \cup a_{p_3}$. Now let r be the disjoint amalgamation of s_1 and s_2 as in Lemma 5.11. The properties

of s_1 and s_2 from the (ν_1, ν_2)-anti-commuting amalgamations s_1 and s_2 pass to r by Definition 5.1(iv). ∎

5.6 Inductive Limits of Directed Families in $\mathbb{P}$

In this section we adopt the terminology where a directed set is a partial order $(X, \le)$ where for any two $x, y \in X$, there is $z \in X$ such that $z \le x, y$. In this section we will consider inductive limits $\mathcal{A}^{\mathbb{G}}$ of systems $(\mathcal{A}_{X_p}^p : p \in \mathbb{G})$ where $\mathbb{G} \subseteq \mathbb{P}$ is a directed subset of $\mathbb{P}$ with the order $\le\, =\, \le_{\mathbb{P}}$. Here, for $p \le q$ the embeddings $i_{pq} \colon \mathcal{A}_{X_q}^q \to \mathcal{A}_{X_p}^p$ are given by Definition 5.1(c); *i.e.*, they satisfy $i_{pq}(A_{\xi,m,n}^q) = A_{\xi,m,n}^p$ for $\xi \in a_q$ and $n, m \in [0, n_\xi^q)$. Formally, we define $\mathcal{A}^{\mathbb{G}}$ differently in order to work with its convenient representation in $\mathcal{B}_{\omega_1 \times \mathbb{N}}$, but then, in Lemma 5.22 we prove that the constructed algebra is the corresponding inductive limit.

Definition 5.18 We say that $\mathbb{G} \subseteq \mathbb{P}$ is covering if and only if

$$\omega_1 \times \mathbb{N} \subseteq \bigcup \{X_p : p \in \mathbb{G}\}.$$

Definition 5.19 Suppose that $\mathbb{G} \subseteq \mathbb{P}$ is directed and covering. Then $A_{\xi,n,m}^{\mathbb{G}} \in \mathcal{B}_{\omega_1 \times \mathbb{N}}$ is given by

$$\left\langle A_{\xi,n,m}^{\mathbb{G}}(e_{\eta,k}), e_{\eta,l} \right\rangle = \left\langle A_{\xi,n,m}^p(e_{\eta,k}), e_{\eta,l} \right\rangle$$

for any (all) $p \in \mathbb{G}$ such $(\eta, k), (\eta, l), (\xi, n), (\xi, m) \in X_p$.

Note that $A_{\xi,n,m}^{\mathbb{G}}$ are well defined if $\mathbb{G}$ is directed and covering. This is because given two $p, p' \in \mathbb{G}$ such that $(\eta, k), (\eta, l), (\xi, n), (\xi, m) \in X_p, X_{p'}$, there is $q \le p, p'$, which implies that $X_p, X_{p'} \subseteq X_q$, and so

$$\left\langle A_{\xi,n,m}^p(e_{\eta,k}), e_{\eta,l} \right\rangle = \left\langle A_{\xi,n,m}^q(e_{\eta,k}), e_{\eta,l} \right\rangle = \left\langle A_{\xi,n,m}^{p'}(e_{\eta,k}), e_{\eta,l} \right\rangle$$

by Definition 5.1(c)–(d). The following definition is parallel to Definition 5.2.

Definition 5.20 Suppose that $\mathbb{G} \subseteq \mathbb{P}$ is directed and covering. Then $\mathcal{A}^{\mathbb{G}}$ is the subalgebra of $\mathcal{B}_{\omega_1 \times \mathbb{N}}$ generated by the operators $A_{\xi,m,n}^{\mathbb{G}}$ for all $\xi \in \omega_1$ and $m, n \in \mathbb{N}$.

Let X be a subset of $\omega_1 \times \mathbb{N}$. We define $\mathcal{A}_X^{\mathbb{G}}$ to be the C^*-subalgebra of $\mathcal{A}^{\mathbb{G}}$ generated by $(A_{\xi,m,n}^{\mathbb{G}} : (\xi, n), (\xi, m) \in X)$. In particular, for every $\alpha < \omega_1$, by $\mathcal{A}_\alpha^{\mathbb{G}}$ we mean the C^*-subalgebra of $\mathcal{A}^{\mathbb{G}}$ generated by $\{A_{\xi,m,n}^{\mathbb{G}} : \xi < \alpha, m, n \in \mathbb{N}\}$.

Lemma 5.21 *Suppose that $\mathbb{G} \subset \mathbb{P}$ is directed and covering and $p \in \mathbb{G}$. There is a $*$-embedding $i_{\mathbb{G},p} \colon \mathcal{A}_{X_p}^p \to \mathcal{A}_{X_p}^{\mathbb{G}}$ such that*

(i) $i_{\mathbb{G},p}(A_{\xi,m,n}^p) = A_{\xi,m,n}^{\mathbb{G}},$
(ii) $i_{\mathbb{G},p}(A_{\xi,m,n}^p)|X_p = A_{\xi,m,n}^p$
for every ξ, n, m such that $(\xi, n), (\xi, m) \in X$.

Proof By Definitions 5.1 and 5.19, a map sending $A^p_{\xi,m,n}$ to $A^{\mathbb{G}}_{\xi,m,n}$ extends to a $*$-homomorphism of $\mathcal{A}^p_{X_p}$ into $\mathcal{A}^{\mathbb{G}}_{X_p}$. Its kernel must be null as the kernels of $i_{q,p}$ for $q \le p$ are null.

To prove the second part of the lemma, use the first part and Definition 5.19. ∎

Lemma 5.22 *Suppose that $\mathbb{G} \subseteq \mathbb{P}$ is directed and covering. There is a $*$-isomorphism j of $\mathcal{A}^{\mathbb{G}}$ and the inductive limit $\lim_{p \in \mathbb{G}} \mathcal{A}^p_{X_p}$ of the system $(\mathcal{A}^p_{X_p} : p \in \mathbb{G})$ with maps $(i_{p,q} : p \le q)$ such that*

$$j(A^{\mathbb{G}}_{\xi,n,m}) = \lim_{p \in \mathbb{G}} A^p_{\xi,n,m}$$

for each $\xi \in \omega_1$ and $m, n \in \mathbb{N}$.

Proof As in [39, Ex. 1. Chapter 6] it is enough to prove that for every $p, q \in \mathbb{G}$ satisfying $p \le q$, the diagram

$$
\begin{array}{ccc}
\mathcal{A}^q_{X_q} & \xrightarrow{\;i_{\mathbb{G},q}\;} & A^{\mathbb{G}}_{X_q} \\
{\scriptstyle i_{p,q}}\downarrow & & \downarrow{\scriptstyle \subseteq} \\
\mathcal{A}^p_{X_p} & \xrightarrow{\;i_{\mathbb{G},p}\;} & A^{\mathbb{G}}_{X_p}
\end{array}
$$

commutes. This follows from the fact that by Definition 5.19, we have

$$i_{\mathbb{G},p}\big(i_{p,q}(A^q_{\xi,n,m}) \big) = \mathcal{A}^{\mathbb{G}}_{\xi,n,m} = i_{\mathbb{G},q}(A^q_{\xi,n,m})$$

for ξ, m, n such that $(\xi, m), (\xi, n) \in X_q$. But these elements generate $\mathcal{A}^q_{X_q}$. ∎

Definition 5.23 A family $\mathbb{G} \subseteq \mathbb{P}$ is called $\mathbb{F}$-*rich* if and only if $\mathbb{G}$ is directed, covering and $\mathbb{F}_{X,\alpha} \cap \mathbb{G} \ne \varnothing$ for every finite $X \subseteq \omega_1 \times \mathbb{N}$ and $\alpha \in X$, where $\mathbb{F}_{X,\alpha}$ is defined in Lemma 5.7.

Lemma 5.24 *Let $\mathbb{G} \subseteq \mathbb{P}$ be an $\mathbb{F}$-rich family. Then for every $\alpha < \omega_1$, the following hold:*

(i) *$\mathcal{A}^{\mathbb{G}}_\alpha$ is an ideal of $\mathcal{A}^{\mathbb{G}}$ equal to $\{A \in \mathcal{A}^{\mathbb{G}} : A|[\alpha, \omega_1) = 0\}$;*

(ii) *there is a $*$-isomorphism $j_\alpha : \mathcal{A}^{\mathbb{G}}/\mathcal{A}^{\mathbb{G}}_\alpha \to \mathcal{A}^{\mathbb{G}}|[\alpha, \omega_1)$;*

(iii) *the representation $\pi_\alpha : \mathcal{A}^{\mathbb{G}}|[\alpha, \omega_1) \to \mathcal{A}^{\mathbb{G}}|\{\alpha\}$ given by $\pi_\alpha(A) = A|\{\alpha\}$ is faithful.*

Proof As $\ell_2(\{\xi\} \times \mathbb{N})$ are $\mathcal{A}^{\mathbb{G}}$-invariant, it is clear that sending $A \in \mathcal{A}^{\mathbb{G}}$ to $A|[\alpha, \omega_1)$ is a $*$-homomorphism. So for (i) and (ii) we are left with proving that its kernel is equal to $\mathcal{A}^{\mathbb{G}}_\alpha$.

First, note that the kernel contains every generator $A^{\mathbb{G}}_{\xi,n,m}$ for $\xi < \alpha$ and $m, n \in \mathbb{N}$ of $\mathcal{A}^{\mathbb{G}}_\alpha$ and so includes $\mathcal{A}^{\mathbb{G}}_\alpha$. This is true by Definition 5.1(iv).

For the other inclusion, let $A \in \mathcal{A}^{\mathbb{G}}$ satisfy $A|[\alpha, \omega_1) = 0$. Since $\mathcal{A}^{\mathbb{G}}$ is the inductive limit of $\mathcal{A}^p_{X_p}$s for $p \in \mathbb{G}$ by Lemma 5.22, for every $\varepsilon > 0$ there is $p \in \mathbb{G}$ and $B \in \mathcal{A}^p_{X_p}$ such that $\|i_{\mathbb{G},p}(B) - A\| < \varepsilon$, and so $\|i_{\mathbb{G},p}(B)|[\alpha, \omega_1)\| < \varepsilon$. By Lemma 5.3, $B|\alpha \in \mathcal{A}^p_{X_p \cap (\alpha \times \mathbb{N})} \subseteq \mathcal{A}^p_{X_p}$ and $B|[\alpha, \omega_1) \in \mathcal{A}^p_{X_p}$, so we can apply $i_{\mathbb{G},p}$ to them. By Lemma 5.4

and Definition 5.19 we have that

$$\left\| i_{\mathbb{G},p}\big(B|[\alpha,\omega_1) \big) \right\| = \left\| i_{\mathbb{G},p}(B) \mid [\alpha,\omega_1) \right\|.$$

So we have

$$\left\| A - i_{\mathbb{G},p}(B|\alpha) \right\| = \left\| A - i_{\mathbb{G},p}(B) + i_{\mathbb{G},p}\big(B|[\alpha,\omega_1) \big) \right\|$$
$$\leq \|A - i_{\mathbb{G},p}(B)\| + \|i_{\mathbb{G},p}(B)|[\alpha,\omega_1)\| \leq 2\varepsilon.$$

But $i_{\mathbb{G},p}(B|\alpha) \in \mathcal{A}_\alpha^{\mathbb{G}}$, since $B|\alpha \in \mathcal{A}_{X_p \cap (\alpha \times \mathbb{N})}^p$. As $\varepsilon > 0$ was arbitrary and $i_{\mathbb{G},p}(B|\alpha) \in \mathcal{A}_\alpha^{\mathbb{G}}$, we conclude that $A \in \mathcal{A}_\alpha^{\mathbb{G}}$, completing the proof of (i) and (ii).

To prove (iii) first note that since $\ell_2(\{\alpha\} \times \omega_1)$ is $\mathcal{A}^{\mathbb{G}}$-invariant, it is clear that π_α is a representation of $\mathcal{A}^{\mathbb{G}}|[\alpha,\omega_1)$. Now suppose that $A \in \mathcal{A}_{X_q}^{\mathbb{G}}$ for $q \in \mathbb{G}$. By Lemma 5.21, there is $B \in \mathcal{A}_{X_q}^q$ such that $i_{\mathbb{G},q}(B) = A$. Since $\mathbb{G}$ is assumed to be $\mathbb{F}$-rich, by Lemmas 5.7 and 5.8, there is $p \in \mathbb{F}_{X_q,\alpha}$ such that $p \leq q$. By Lemma 5.4 and Definition 5.19, we have $\|A|[\alpha,\omega_1)\| = \|i_{p,q}(B)|[\alpha,\omega_1)\|$. By the fact that $p \in \mathbb{F}_{X_q,\alpha}$, we have that

$$\|A|\{\alpha\}\| \geq \|i_{p,q}(B)|\{\alpha\}\| \geq \|i_{p,q}(B)|[\alpha,\omega_1)\| = \|A|[\alpha,\omega_1)\|.$$

This shows that π_α is an isometry when restricted to $\bigcup_{q \in \mathbb{G}} \mathcal{A}_{X_q}^{\mathbb{G}}|[\alpha,\omega_1)$, which is dense in $\mathcal{A}^{\mathbb{G}}|[\alpha,\omega_1)$ by Lemma 5.22, and so the representation is faithful. ∎

Proposition 5.25 *Suppose that $\mathbb{G} \subseteq \mathbb{P}$ is an $\mathbb{F}$-rich family. Then $\mathcal{A}^{\mathbb{G}}$ is a scattered thin-tall fully noncommutative C^*-algebra such that*

(i) $\mathcal{I}_\alpha^{At}(\mathcal{A}^{\mathbb{G}}) = \mathcal{A}_\alpha^{\mathbb{G}}$;

(ii) *there is a $*$-isomorphism $j_\alpha : \mathcal{A}^{\mathbb{G}}/\mathcal{I}_\alpha^{At}(\mathcal{A}^{\mathbb{G}}) \to \mathcal{A}^{\mathbb{G}}|[\alpha,\omega_1)$ satisfying*

$$j_\alpha([A]_{\mathcal{I}_\alpha^{At}(\mathcal{A})}) = A|[\alpha,\omega_1);$$

(iii) *the collection $\{[A_{\alpha,m,n}]_{\mathcal{I}_\alpha^{At}(\mathcal{A})} : n, m \in \mathbb{N}\}$ satisfies the matrix units relations and generates the essential ideal $At(\mathcal{A}^{\mathbb{G}}/\mathcal{I}_\alpha^{At}(\mathcal{A}^{\mathbb{G}}))$.*

Proof By [18, Theorem 1.4] it is enough to prove (i)–(iii) to conclude that $\mathcal{A}$ is a scattered thin-tall fully noncommutative C^*-algebra.

The proof of (i)–(iii) is by induction on $\alpha < \omega_1$. For $\alpha = 0$ we have that $\mathcal{I}_\alpha = \{0\}$ and so (i) and (ii) are trivial. Also, $A_{0,n,m}^{\mathbb{G}} = 1_{0,n,m}$ by Definition 5.1, so these elements satisfy the matrix unit relations. Moreover, they generate the algebra of all compact operators on $\ell_2(\{0\} \times \mathbb{N})$, which is an essential ideal in $\mathcal{B}_{\{0\} \times \mathbb{N}}$. Since π_0 from Lemma 5.24 is faithfull, the collection $\{A_{0,m,n} : n, m \in \mathbb{N}\}$ generates an essential ideal isomorphic to an algebra of all compact operators on a Hilbert space, so by [18, Theorem 1.2(4)] this ideal is $\mathcal{I}^{At}(\mathcal{A}^{\mathbb{G}})$ as required.

Now suppose we are done for $\beta < \alpha < \omega_1$.

(i) If α is a limit ordinal, then by [18, 1.4] and the inductive hypothesis we have

$$I_\alpha^{At}(\mathcal{A}) = \overline{\bigcup_{\beta < \alpha} I_\beta^{At}(\mathcal{A})} = \overline{\bigcup_{\beta < \alpha} \mathcal{A}_\beta} = \mathcal{A}_\alpha.$$

If $\alpha = \beta + 1$, then (iii) of the inductive hypothesis implies (i).

(ii) follows from Lemma 5.24.

(iii) is proved as in the case $\alpha = 0$. ∎

6 An Operator Algebra Along a Construction Scheme

In this section, we adopt the terminology and the notation of Section 5. We will use the constructions scheme of [54] described in Section 2.2 to build appropriate $\mathbb{F}$-rich families $\mathbb{G}$ in the partial order $\mathbb{P}$ of approximations whose inductive limit $\mathcal{A}^{\mathbb{G}}$ will have interesting properties described in the introduction. To prove the main theorem of this section we need one more general lemma.

Lemma 6.1 *Suppose that $\mathcal{A}$ is an AF C*-algebra and $\{\mathcal{A}_D : D \in \mathcal{D}\}$ is a directed family of finite-dimensional subalgebras with dense union. Let $P \in \mathcal{A}$ be a projection. Then for every $0 < \varepsilon < 1$ there is $D \in \mathcal{D}$ and a projection $Q \in \mathcal{A}_D$ such that $\|Q - P\| < \varepsilon$.*

Proof Let $D \in \mathcal{D}$ be such that there is $A \in A_D$ satisfying $\|A - P\| < \varepsilon/6$. By considering $(A + A^*)/2$ instead of A we may assume that A is self-adjoint and $\|A - P\| < \varepsilon/6$. As $\mathcal{A}_D$ is finite dimensional, it is $*$-isomorphic to the direct sum of full matrix algebras. Let π be the isomorphism. The matrix $\pi(A)$ is self-adjoint, so it can be diagonalized. As $\|A - P\| < \varepsilon/6$, we have that $\|A^2 - A\| < \varepsilon/2$ and so the distance of each entry on the diagonal of the diagonalized $\pi(A)$ from 0 or 1 cannot be bigger than $\varepsilon/2$, so there is a projection $Q \in \mathcal{A}_D$ such that $\|\pi(Q) - \pi(A)\| < \varepsilon/2$, and hence $\|Q - A\| < \varepsilon/2$ and $\|Q - P\| < \varepsilon$, as required. ∎

Theorem 6.2 *Suppose that there exists a construction scheme $\mathcal{F}$ with allowed parameters $(r_k)_{k \in \mathbb{N}}$ and $(n_k)_{k \in \mathbb{N}}$, where $n_k = 3$ for each $k \in \mathbb{N} \setminus \{0\}$ and a partition $(P_m)_{m \in \mathbb{N}}$ of $\mathbb{N}$ into infinite sets such that for every $m \in \mathbb{N}$ and every uncountable Δ-system T of finite subsets of ω_1 there exist $F \in \mathcal{F}$ of arbitrarily large rank in P_m that fully captures a subsystem of T.*

Then there is an $\mathbb{F}$-rich family $\mathbb{G}$ of elements of $\mathbb{P}$ such that the scattered thin-tall fully noncommutative C*-algebra $\mathcal{A}^{\mathbb{G}}$ has the following properties:

(i) *There is a nondecreasing unbounded sequence $(l_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ and a directed family of finite dimensional algebras $\{A_X^{\mathbb{G}} : X = F \times [0, l_k), F \in \mathcal{F}_k, k \in \mathbb{N}\}$ whose union $\mathcal{B}$ is dense in $\mathcal{A}$, such that if $(P_\xi : \xi < \omega_1) \subseteq \mathcal{B}$ is a family of projections which generate a nonseparable subalgebra of $\mathcal{A}^{\mathbb{G}}$, then for every $\varepsilon > 0$,*

    (a) *there are $\xi_1 < \xi_2 < \xi_3 < \omega_1$ such that $\|P_{\xi_1} - P_{\xi_2} P_{\xi_3}\| < \varepsilon$,*

    (b) *there are $\xi_1 < \xi_2 < \omega_1$ such that $\|[P_{\xi_1}, P_{\xi_2}]\| < \varepsilon$,*

    (c) *there are $\xi_1 < \xi_2 < \omega_1$ such that $\|[P_{\xi_1}, P_{\xi_2}]\| > 1/2 - \varepsilon$.*

(ii) *$\mathcal{A}^{\mathbb{G}}$ has no uncountable irredundant subset.*

(iii) *$\mathcal{A}^{\mathbb{G}}$ has no nonseparable abelian subalgebra.*

Proof Fix an enumeration $((v^m, w^m) : m \geq 3)$, with possible repetitions, of all pairs of orthonormal complex vectors with finitely many coordinates, all of them rational, such that $v^m, w^m \in \mathbb{C}^m$ for each $m \geq 3$ (we abuse notation and identify $\mathbb{C}^{m'}$ with a subset of $\mathbb{C}^m$ for $m' \leq m$).

We construct the sequence $(l_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ and $\mathbb{G} = \{p_F : F \in \mathcal{F}\} \subseteq \mathbb{P}$ by induction with respect to $k \in \mathbb{N}$ such that $F \in \mathcal{F}_k$. Moreover, for each $k \in \mathbb{N}$ we require that

whenever $F, F' \in \mathcal{F}_k$ are such that $F \smallsetminus F' < F' \smallsetminus F$ (*cf.* Definition 2.10(ii)), then

(6.1) p_F and $p_{F'}$ are in convenient position as witnessed by $\phi_{F',F}$.

(6.2) $a_{p_F} = F$,

(6.3) $n_\xi^{p_F} = l_k$ for all $\xi \in F$ and $F \in \mathcal{F}_k$ and $k \in \mathbb{N}$.

For $k = 0$, we have that $\mathcal{F}_1 = [\omega_1]^1$ by Definition 2.10(i), so we define p_F for $F = \{\xi\}$ to be the element of $\mathbb{P}$ such that

- $a_{p_F} = \{\xi\}$;
- $n_{p_F}^\xi = l_0 = 1$;
- $A_{\xi,0,0}^{p_F} = 1_{\xi,0,0}$.

Suppose that we have constructed p_F for all $F \in F_{k'}$ for $k' \leq k$ satisfying (6.1)–(6.3). Now we need to define the p_F for $F \in \mathcal{F}_{k+1}$. Since $n_{k+1} = 3$, each $F \in \mathcal{F}_{k+1}$ is the union of the maximal elements G_1, G_2, G_3 of $\mathcal{F}|F$ which form an increasing Δ-system by Definition 2.10(iii). If $k \in P_1$, then we define p_F as the amalgamation of $p_{G_1}, p_{G_2}, p_{G_3}$ of type 1 from Lemma 5.15. If $k \in P_2$, then we define p_F as the amalgamation of $p_{G_1}, p_{G_2}, p_{G_3}$ of type 2 from Lemma 5.16. If $k \in P_m$ for $m \geq 3$, and $l_k < m$, then we define p_F as the amalgamation of $p_{G_1}, p_{G_2}, p_{G_3}$ of type 1 from Lemma 5.15. If $k \in P_m$ for $m \geq 3$, and $l_k \geq m$, then we define p_F as the amalgamation of $p_{G_1}, p_{G_2}, p_{G_3}$ of type 3 for vectors (v^m, w^m) from Lemma 5.17.

Observe that amalgamation of type 1 increases l_k, so $l_k \to \infty$ when $k \to \infty$.

First, let us note that our inductive hypothesis (6.1)–(6.3) is preserved when we pass from $k \in \mathbb{N}$ to $k+1$. Let $F, F' \in \mathcal{F}_{k+1}$ be such that $F \smallsetminus F' < F' \smallsetminus F$. By Definition 2.10(ii), there is an order preserving bijection $\phi_{F',F} \colon F \to F'$ and $F \cap F' < F \smallsetminus F' < F' \smallsetminus F$. In particular Definition 2.10(ii) implies that the maximal elements of $\mathcal{F}|F$ are sent by $\phi_{F',F}$ onto the maximal elements of $\mathcal{F}|F'$, on the other hand, Definition 2.10(iii) implies that these maximal elements form the canonical decomposition consisting of elements in $\mathcal{F}_k$, which in fact are used in the construction of p_F or $p_{F'}$. Now to verify Definition 5.9 in order to check (6.1), we note that the amalgamations described in Lemmas 5.15, 5.16, and 5.17 consist of constructions of operators that depend only on the place of the involved objects in F, so Definition 5.9 and (6.1) are satisfied for p_F and $p_{F'}$. Equations (6.2) and (6.3) follow from the descriptions of the amalgamations from Lemmas 5.15, 5.16, 5.17. We have $l_{k+1} = l_k$ if $k \in \mathbb{N} \smallsetminus P_1$ and $l_{k+1} > l_k$ if $k \in P_1$, and Definition 2.10 guarantees that the amalgamations that follow Lemma 5.15 can be done in "the same way" up to the bijection $\phi_{F',F}$, and so obtaining $n_{p_F}^\xi = l_{k+1} = n_{p_{F'}}^{\xi'}$ for any $F, F' \in \mathcal{F}_{k+1}$ and $\xi \in F$ and $\xi' \in F'$. This completes the construction of $\mathbb{G} = \{p_F \colon F \in \mathcal{F}\}$ and completely determines the C*-algebra $\mathcal{A}^\mathbb{G}$ as in Definition 5.20.

Now note that $\mathbb{G}$ is $\mathbb{F}$-rich as in Definition 5.23. First note that $p_{F'} \leq p_F$ whenever $F' \subseteq F$ and $F, F' \in \mathcal{F}$. This can be proved by induction on $k \in \mathbb{N}$ such that $F \in \mathcal{F}_k$. Note that it is true if F' is a maximal element of $F|\mathcal{F}$, because then F' is in the canonical decomposition of F by Definition 2.10(iii) and we use $p_{F'}$ in the construction of p_F, obtaining $p_{F'} \leq p_F$ by the Lemmas 5.15–5.17. Now we proceed with the inductive argument; given $F' \subsetneq F$, either F' is below a maximal element G of $\mathcal{F}|F$ or it is one of the maximal elements. The latter case is proved above, and the former follows from the inductive assumption for the pair F', G and from the transitivity of the order in $\mathbb{P}$.

To prove the directedness of $\mathbb{G}$ take $F, F' \in \mathcal{F}$ and use the cofinality of $\mathcal{F}$ in $[\omega_1]^{<\omega}$ (Definition 2.10) to find $F'' \in \mathcal{F}$ such that $F \cup F' \subseteq F''$. By the above arguments, we have $p_F, p_{F'} \leq p_{F''}$.

Now let $X = a \times [0, l) \in [\omega_1 \times \mathbb{N}]^{<\omega}$ and $\alpha \in \omega_1$ and aim at proving further parts of the $\mathbb{F}$-richness. Consider the Δ-system

$$ T = \left\{ a \cup \{\alpha, \xi\} : \max(a \cup \{\alpha\}) < \xi < \omega_1 \right\} $$

of finite subsets of ω_1. By the hypothesis there is $k \in P_1$ with $l_k \geq l$ and $F \in \mathcal{F}$ such that F fully captures a subsystem of T. In particular, $F = G_1 \cup G_2 \cup G_3$ for some $G_1, G_2, G_3 \in \mathcal{F}_k$ and $X \subseteq X_{p_{G_1}}$ and $\alpha \in a_{p_{G_1}}$. By the construction, we do the amalgamation of type 1 as in Lemma 5.15 while constructing p_F, and so p_F is in $\mathbb{F}_{X_{p_{G_1}}, \alpha}$, but this implies that it is in $\mathbb{F}_{X, \alpha}$, as required for $\mathbb{F}$-richness in Definition 5.23.

Proposition 5.25 implies that $\mathcal{A}^{\mathbb{G}}$, as in Definition 5.20, is a thin-tall fully noncommutative scattered C*-algebra.

To prove (i), the directed family of finite dimensional subalgebras of $\mathcal{A}^{\mathbb{G}}$ is $\{\mathcal{A}_{X_p}^{\mathbb{G}} : p \in \mathbb{G}\}$ as in Definition 5.20. By Lemma 5.21, the algebras $\mathcal{A}_{X_p}^{\mathbb{G}}$ are $*$-isomorphic to the algebras $\mathcal{A}_{X_p}^p$ and they are finite dimensional, since they are equal to $\mathcal{B}_{X_p}$ by Lemma 5.3. Let $\mathcal{B} = \bigcup \{\mathcal{A}_{X_p}^{\mathbb{G}} : p \in \mathbb{G}\}$.

Suppose that $\{P_\xi : \xi < \omega_1\} \subseteq \mathcal{B}$ is a collection of projections that generate a nonseparable subalgebra of $\mathcal{A}^{\mathbb{G}}$. So, there must be distinct $\alpha_\xi \in \omega_1$ such that $P_\xi|(\{\alpha_\xi\} \times \mathbb{N}) \neq 0$. Since $\mathcal{B}_{\{\alpha_\xi\} \times \mathbb{N}}$ is invariant for $\mathcal{A}^{\mathbb{G}}$, it follows that $P_\xi|(\{\alpha_\xi\} \times \mathbb{N})$ is a non-zero projection. Moreover, it is not the unit of $\mathcal{B}_{\{\alpha_\xi\} \times \mathbb{N}}$, because such a unit would produce a unit of $\mathcal{A}^{\mathbb{G}}/\mathcal{I}_{\alpha_\xi}^{At}(\mathcal{A}^{\mathbb{G}})$ by Lemma 5.24 and Theorem 5.25, which is impossible because $\mathcal{A}^{\mathbb{G}}$ is the union of proper ideals $\mathcal{I}_\alpha^{At}(\mathcal{A}^{\mathbb{G}})$ for $\alpha < \omega_1$.

Let $F_\xi \in \mathcal{F}$ be such that $\alpha_\xi \in F_\xi$, $P_\xi \in \mathcal{A}_{X_{p_{F_\xi}}}^{\mathbb{G}} = \mathcal{A}_{F_\xi \times [0, l_\xi)}^{\mathbb{G}}$ for each $\xi \in \omega_1$, where $l_\xi = l_k$ for $F_\xi \in \mathcal{F}_k$ and $P_\xi|(\{\alpha_\xi\} \times [0, l_\xi))$ is a nonzero projection that is not the unit of $\mathcal{B}_{\{\alpha_\xi\} \times [0, l_\xi)}$. This can be obtained from the cofinality of $\mathcal{F}$ and the fact that $l_k \to \infty$ when $k \to \infty$.

Let $Q_\xi \in \mathcal{A}_{F_\xi \times [0, l_\xi)}^{p_{F_\xi}}$ be such that $i_{\mathbb{G}, p_{F_\xi}}(Q_\xi) = P_\xi$. Note that by Lemma 5.21 the Q_ξs are projections and $Q_\xi|(\{\alpha_\xi\} \times [0, l_\xi))$ is a nonzero projection that is not the unit of $\mathcal{B}_{\{\alpha_\xi\} \times [0, l_\xi)}$ for each $\xi < \omega_1$.

By passing to an uncountable subset, we can assume that $T = \{F_\xi : \xi < \omega_1\}$ forms an increasing Δ-system of elements of $\mathcal{F}_{k'}$ for a fixed $k' \in \mathbb{N}$ and that

$$ \left| \left\langle Q_\xi(e_{\eta, l}), e_{\eta, l'} \right\rangle - \left\langle Q_{\xi'}(e_{\phi_{F_{\xi'}, F_\xi}(\eta), l}), e_{\phi_{F_{\xi'}, F_\xi}(\eta), l'} \right\rangle \right| < \varepsilon/2 l_{k'} $$

for every $(\eta, l), (\eta, l') \in F_\xi \times [0, l_{k'})$ and every $\xi < \xi' < \omega_1$. This guarantees that

$$ (6.4) \qquad \| j_{\phi_{F_\xi, F_{\xi'}}}(Q_\xi) - Q_{\xi'} \| < \varepsilon/2 $$

for every $\xi < \xi' < \omega_1$. Now let us prove item (a) of (i). By the hypothesis on $\mathcal{F}$, there is $k \in P_2$ bigger than k' and $F \in \mathcal{F}_{k+1}$, which fully captures T; *i.e.*, the canonical decomposition of F is $\{G_1, G_2, G_3\}$, and there are $\xi_1 < \xi_2 < \xi_3 < \omega_1$ such that $F_{\xi_i} \subseteq G_i$ and $\phi_{G_j, G_i}[F_{\xi_i}] = F_{\xi_j}$ for all $1 \leq i, j \leq 3$. As ϕ_{G_j, G_i} are order preserving, they must

agree with $\phi_{F_{\xi_j},F_{\xi_i}}$ on F_{ξ_i}, so (6.4) implies that

$$\|j_{\phi_{G_3,G_i}}(Q_{\xi_3}) - Q_{\xi_i}\| < \varepsilon/2$$

holds for $i = 1, 2$. Since we use amalgamation of type 2 at the construction of p_F for $k \in P_2$, by Lemma 5.16, we have

$$i_{p_F,p_{G_3}}(Q_{\xi_3})i_{p_F,p_{G_2}}(j_{\phi_{G_3,G_2}}(Q_{\xi_3})) = i_{p_F,p_{G_1}}(j_{\phi_{G_3,G_1}}(Q_{\xi_3}))^2,$$

and so

$$\|i_{p_F,p_{G_3}}(Q_{\xi_3})i_{p_F,p_{G_2}}(Q_{\xi_2}) - i_{p_F,p_{G_1}}(Q_{\xi_1})^2\| < \varepsilon,$$

and hence $\|P_{\xi_3}P_{\xi_2} - P_{\xi_1}\| < \varepsilon$, since

$$i_{\mathbb{G},p_F} \circ i_{p_F,p_{G_{\xi_i}}}(Q_{\xi_i}) = i_{\mathbb{G},p_{G_{\xi_i}}}(Q_{\xi_i}) = i_{\mathbb{G},p_{G_{\xi_i}}}(i_{p_{G_{\xi_i}},p_{F_{\xi_i}}}(Q_{\xi_i}))$$
$$= i_{\mathbb{G},p_{F_{\xi_i}}}(Q_{\xi_i}) = P_{\xi_i}$$

by Definition 5.19 and Lemma 5.21. This completes the proof of (a) of (i). Item (b) follows from (a) for $\varepsilon/2$ and by taking adjoints.

Now let us prove item (c) of (i). For $\xi < \omega_1$ let $Q'_\xi \in \mathcal{B}_{F_\xi \times [0,l_{k'})}$ be such projections that $\|Q_\xi - Q'_\xi\| < \varepsilon/8$, and there is an orthonormal basis in $\mathcal{B}_{\{\alpha_\xi\} \times [0,l_{k'})}$ of eigenvectors for Q'_ξ consisting only of vectors with all rational coordinates with respect to our canonical basis $(e_{\alpha_\xi,l} : 0 \le l < l_{k'})$. Note that by Lemma 5.3, we have that $Q'_\xi \in \mathcal{A}_{X_{p_{F_\xi}}}^{p_{F_\xi}}$. Since $Q_\xi|(\{\alpha_\xi\} \times [0,l_{k'}))$ is a nonzero projection that is not the unit of $\mathcal{B}_{\{\alpha_\xi\} \times [0,l_{k'})}$ for each $\xi < \omega_1$, Q'_ξ can be assumed to have the same rank as Q_ξ and so there are orthogonal unit vectors $v^\xi, w^\xi \in \mathbb{C}^{l_{k'}}$ with all rational coordinates such that

$$Q'_\xi\Big(\sum_{l<l_{k'}} v^\xi_l e_{\alpha_\xi,l}\Big) = \sum_{l<l_{k'}} v^\xi_l e_{\alpha_\xi,l}, Q'_\xi\Big(\sum_{l<l_{k'}} w^\xi_l e_{\alpha_\xi,l}\Big) = 0.$$

As there are only countably many such vectors, we may assume that all of them are equal to a pair (v, w), and moreover that

$$(6.5) \qquad \|j_{\phi_{F_\xi,F_{\xi'}}}(Q'_\xi) - Q'_{\xi'}\| < \varepsilon/4$$

for every $\xi < \xi' < \omega_1$.

By the hypothesis on $\mathcal{F}$ there is $k \in P_m$ bigger than k' such that $v_m = v = (v_1, \ldots, v_{l_{k'}})$ and $w_m = w = (w_1, \ldots, w_{l_{k'}})$ and there is $F \in \mathcal{F}_{k+1}$ which fully captures T, i.e., the canonical decomposition of F is $\{G_1, G_2, G_3\}$ and there are $\xi_1 < \xi_2 < \xi_3 < \omega_1$ such that $F_{\xi_i} \subseteq G_i$ and $\phi_{G_j,G_i}[F_{\xi_i}] = F_{\xi_j}$ for all $1 \le i, j \le 3$. Note that α_{ξ_i}s are not in the root of $\{G_1, G_2, G_3\}$ as they are not in the root of the F_ξs. As ϕ_{G_j,G_i} are order preserving, they must agree with $\phi_{F_{\xi_j},F_{\xi_i}}$ on F_{ξ_i}, so (6.5) implies that

$$\|j_{\phi_{G_1,G_i}}(Q'_{\xi_1}) - Q'_{\xi_i}\| < \varepsilon/4$$

holds for $i = 2, 3$. Since we use amalgamation of type 3 at the construction of p_F for $k \in P_m$ by Lemma 5.17, we have

$$\|[i_{p_F,p_{G_1}}(Q'_{\xi_1}), i_{p_F,p_{G_2}}(j_{\phi_{G_1,G_2}}(Q'_{\xi_1}))]\| = 1/2$$

and so

$$\|[i_{p_F,p_{G_1}}(Q'_{\xi_1}), i_{p_F,p_{G_2}}(Q'_{\xi_2})]\| \ge 1/2 - \varepsilon/2,$$

and hence

$$\left\| \left[i_{p_F, p_{G_1}}(Q_{\xi_1}), i_{p_F, p_{G_2}}(Q_{\xi_2}) \right] \right\| \geq 1/2 - \varepsilon,$$

as $\|Q_\xi - Q'_{\xi'}\| < \varepsilon/8$ for each $\xi < \xi' < \omega_1$; and finally

$$\left\| [P_{\xi_1}, P_{\xi_2}] \right\| \geq 1/2 - \varepsilon,$$

since

$$i_{\mathbb{G}, p_F} \circ i_{p_F, p_{G_{\xi_i}}}(Q_{\xi_i}) = i_{\mathbb{G}, p_{G_{\xi_i}}}(Q_{\xi_i}) = i_{\mathbb{G}, p_{G_{\xi_i}}}\left(i_{p_{G_{\xi_i}}, p_{F_{\xi_i}}}(Q_{\xi_i}) \right)$$
$$= i_{\mathbb{G}, p_{F_{\xi_i}}}(Q_{\xi_i}) = P_{\xi_i},$$

by Definition 5.19 and Lemma 5.21. This completes the proof of (c) of (i).

The proof of (ii) will be based on (i)(a) and Lemma 6.1. Suppose that $\mathcal{A}^{\mathbb{G}}$ contains an uncountable irredundant set $\{Q_\xi : \xi < \omega_1\}$. By Lemma 3.3, we can assume that all Q_ξs are projections. For each ξ, let $\mathcal{A}_{\omega_1 \setminus \{\xi\}}$ be the C*-subalgebra of $\mathcal{A}^{\mathbb{G}}$ generated by the set $\{Q_\eta : \eta \in \omega_1 \setminus \{\xi\}\}$. By passing to an uncountable subset, we can assume that there is $\varepsilon > 0$ such that for each $\xi < \omega_1$ we have $\|A - Q_\xi\| \geq \varepsilon$ for each $A \in \mathcal{A}_{\omega_1 \setminus \{\xi\}}$. Let $P_\xi \in \mathcal{B}$ be a projection satisfying $\|P_\xi - Q_\xi\| < \varepsilon/4$, which is obtained using Lemma 6.1. By (i)(a) there are $\xi_1 < \xi_2 < \xi_3 < \omega_1$ such that $\|P_{\xi_1} - P_{\xi_2}P_{\xi_3}\| < \varepsilon/4$. This implies that $\|Q_{\xi_1} - Q_{\xi_2}Q_{\xi_3}\| < \varepsilon$, which contradicts the defining property of ε and completes the proof of (ii).

The proof of (iii) will be based on (i)(c) and Lemma 6.1. Suppose that $\mathcal{A}^{\mathbb{G}}$ contains a nonseparable abelian subalgebra. As subalgebras of scattered algebras are scattered and scattered locally compact spaces are totally disconnected, it follows that $\mathcal{A}^{\mathbb{G}}$ contains an uncountable Boolean algebra of (commuting) projections $\{Q_\xi : \xi < \omega_1\}$. In particular $\|Q_\xi - Q_{\xi'}\| = 1$ for all $\xi < \xi' < \omega_1$.

Let $P_\xi \in \mathcal{B}$ for $\xi < \omega_1$ be projections satisfying $\|P_\xi - Q_\xi\| < \frac{1}{10}$ for each $\xi < \omega_1$, which is obtained using Lemma 6.1. In particular, $\|P_\xi - P_{\xi'}\| \geq \frac{8}{10}$ for all $\xi < \xi' < \omega_1$ and so they generate a nonseparable C*-algebra.

We have $\|P_{\xi_1}P_{\xi_2} - Q_{\xi_1}Q_{\xi_2}\| < \frac{1}{5}$ and $\|P_{\xi_2}P_{\xi_1} - Q_{\xi_2}Q_{\xi_1}\| < \frac{1}{5}$ for each $\xi_1 < \xi_2 < \omega_1$, so $[P_{\xi_1}, P_{\xi_2}] < \frac{2}{5}$ for each $\xi_1 < \xi_2 < \omega_1$. But by (i)(c), there are $\xi_1 < \xi_2 < \omega_1$ such that $\|[P_{\xi_1}, P_{\xi_2}]\| \geq \frac{2}{5}$, a contradiction. ∎

References

[1] U. Abraham, M. Rubin, and S. Shelah, *On the consistency of some partition theorems for continuous colorings, and the structure of $\aleph_1$-dense real order types*. Ann. Pure Appl. Logic 29(1985), no. 2, 123–206.   https://doi.org/10.1016/0168-0072(84)90024-1

[2] C. Akemann, *Left ideal structure of C*-algebras*. J. Functional Analysis 6(1970), 305–317. https://doi.org/10.1016/0022-1236(70)90063-7

[3] C. Akemann and N. Weaver, *Consistency of a counterexample to Naimark's problem*. Proc. Natl. Acad. Sci. USA 101(2004), no. 20, 7522–7525.   https://doi.org/10.1073/pnas.0401489101

[4] C. Akemann and N. Weaver, $\mathcal{B}(H)$ *has a pure state that is not multiplicative on any masa*. Proc. Natl. Acad. Sci. USA 105(2008), no. 14, 5313–5314.   https://doi.org/10.1073/pnas.0801176105

[5] W. Arveson, *An invitation to C*-algebras*, Graduate Texts in Mathematics, 39, Springer-Verlag, New York-Heidelberg, 1976.

[6] M. Bell, J. Ginsburg, and S. Todorcevic, *Countable spread of $exp\,Y$ and λY*. Topology Appl. 14(1982), no. 1, 1–12.   https://doi.org/10.1016/0166-8641(82)90043-8

[7] T. Bice and P. Koszmider, *A note on the Akemann-Doner and Farah-Wofsey constructions*. Proc. Amer. Math. Soc. 145(2017), no. 2, 681–687.   https://doi.org/10.1090/proc/13242

[8] C. Brech and P. Koszmider, *Thin-very tall compact scattered spaces which are hereditarily separable*. Trans. Amer. Math. Soc. 363(2011), no. 1, 501–519. https://doi.org/10.1090/S0002-9947-2010-05149-9

[9] C. Brech and P. Koszmider, *On biorthogonal systems whose functionals are finitely supported*. Fund. Math. 213(2011), no. 1, 43–66.   https://doi.org/10.4064/fm213-1-3

[10] G. Carotenuto, *An introduction to OCA*. Notes on lectures by Matteo Viale. 2014. http://www.logicatorino.altervista.org/matteo_viale/OCA.pdf.

[11] K. Davidson, *C*-algebras by example*, Fields Institute Monographs, 6, American Mathematical Society, Providence, RI, 1996.   https://doi.org/10.1090/fim/006

[12] M. Dzamonja and I. Juhasz, *CH, a problem of Rolewicz and bidiscrete systems*. Topol. Appl. 158(2011), 2458–2494.   https://doi.org/10.1016/j.topol.2011.08.005

[13] P. Enflo and H. Rosenthal, *Some results concerning $L_p(\mu)$-spaces*. J. Functional Analysis 14(1973), 325–348.   https://doi.org/10.1016/0022-1236(73)90050-5

[14] R. Engelking, *General topology. Translated from the Polish by the author*, Second ed., Sigma Series in Pure Mathematics, 6, Heldermann Verlag, Berlin, 1989.

[15] I. Farah, *Analytic quotients: theory of liftings for quotients over analytic ideals on the integers*. Mem. Amer. Math. Soc. 148(2000), no. 702.   https://doi.org/10.1090/memo/0702

[16] I. Farah and I. Hirshberg, *Simple nuclear C*-algebras not isomorphic to their opposites*. Proc. Natl. Acad. Sci. USA 114(2017), no. 24, 6244–6249.   https://doi.org/10.1073/pnas.1619936114

[17] I. Farah and T. Katsura, *Nonseparable UHF algebras I: Dixmier's problem*. Adv. Math. 225(2010), no. 3, 1399–1430.   https://doi.org/10.1016/j.aim.2010.04.006

[18] S. Ghasemi and P. Koszmider, *Noncommutative Cantor-Bendixson derivatives and scattered C*-algebras*. Topology Appl. 240(2018), 183–209.   https://doi.org/10.1016/j.topol.2018.03.008

[19] S. Ghasemi and P. Koszmider, *A non-stable C*-algebra with an elementary essential composition series*.   arxiv:1712.02090

[20] P. Hajek, V. Montesinos Santalucia, J. Vanderwerff, and V. Zizler, *Biorthogonal systems in Banach spaces*, CMS Books in Mathematics/Ouvrages de Mathematiques de la SMC, 26, Springer, New York, 2008.

[21] R. Hodel, *Cardinal functions. I*. In: *Handbook of set-theoretic topology*, North-Holland, Amsterdam, 1984, pp. 1–61.

[22] K. Hofmann and K.-H. Neeb, *Epimorphisms of C*-algebras are surjective*. Arch. Math. (Basel) 65(1995), no. 2, 134–137.   https://doi.org/10.1007/BF01270691

[23] L. Heindorf, *A note on irredundant sets*. Algebra Universalis 26(1989), no. 2, 216–221. https://doi.org/10.1007/BF01236868

[24] C. Hida, *Two cardinal inequalities about bidiscrete systems*. Topology Appl. 212(2016), 71–80. https://doi.org/10.1016/j.topol.2016.09.006

[25] T. Jech, *Set theory*, The third millennium ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003.

[26] H. Jensen, *Scattered C*-algebras*. Math. Scand. 41(1977), no. 2, 308–314. https://doi.org/10.7146/math.scand.a-11723

[27] H. Jensen, *Scattered C*-algebras. II*. Math. Scand. 43(1979), no. 2, 308–310. https://doi.org/10.7146/math.scand.a-11782

[28] S. Koppelberg, *Handbook of Boolean algebras. Vol. 1*. North-Holland Publishing Co., Amsterdam, 1989.

[29] P. Koszmider, *On a problem of Rolewicz about Banach spaces that admit support sets*. J. Funct. Anal. 257(2009), no. 9, 2723–2741.

[30] P. Koszmider, *Some topological invariants and biorthogonal systems in Banach spaces*. Extracta Math. 26(2011), 271–294.

[31] P. Koszmider, *On the problem of compact totally disconnected reflection of nonmetrizability*. Topology Appl. 213(2016), 154–166.   https://doi.org/10.1016/j.topol.2016.08.017

[32] P. Koszmider, *On constructions with 2-cardinals*. Arch. Math. Logic 56(2017), no. 7–8, 849–876. https://doi.org/10.1007/s00153-017-0544-9

[33] K. Kunen, *An introduction to independence proofs*, Studies in Logic and the Foundations of Mathematics, 102, North-Holland Publishing Co., Amsterdam-New York, 1980.

[34] M. Kusuda, *C*-algebras in which every C*-subalgebra is AF*. Q. J. Math. 63(2012), no. 3, 675–680. https://doi.org/10.1093/qmath/har014

[35] H. X. Lin, *The structure of quasimultipliers of C*-algebras*. Trans. Amer. Math. Soc. 315(1989), no. 1, 147–172. https://doi.org/10.2307/2001377

[36] F. Lopez and S. Todorcevic, *Trees and gaps from a construction scheme*. Proc. Amer. Math. Soc. 145(2017), no. 2, 871–879. https://doi.org/10.1090/proc/13431

[37] F. Lopez, *Banach spaces from a construction scheme*. J. Math. Anal. Appl. 446(2017), no. 1, 426–435. https://doi.org/10.1016/j.jmaa.2016.08.068

[38] A. Mostowski and A. Tarski, *Booleshe Ringe mit ordneter basis*. Fund. Math. 32 69–86.

[39] G. J. Murphy, *C*-algebras and operator theory*. Academic Press, Inc., Boston, MA, 1990.

[40] S. Negrepontis, *Banach spaces and topology*. In: *Handbook of set-theoretic topology*, North-Holland, Amsterdam, 1984, pp. 1045–1142.

[41] T. Ogasawara, *Finite-dimensionality of certain Banach algebras*. J. Sci. Hiroshima Univ. Ser. A 17(1954), 359–364.

[42] C. Olsen and W. Zame, *Some C*-alegebras with a single generator*. Trans. Amer. Math. Soc. 215(1976), 205–217. https://doi.org/10.2307/1999722

[43] A. J. Ostaszewski, *On countably compact, perfectly normal spaces*. J. London Math. Soc. (2) 14(1976), no. 3, 505–516. https://doi.org/10.1112/jlms/s2-14.3.505

[44] A. Pełczyński and Z. Semadeni, *Spaces of continuous functions. III. Spaces $C(\Omega)$ for ω without perfect subsets*. Studia Math. 18(1959), 211–222. https://doi.org/10.4064/sm-18-2-211-222

[45] S. Popa, *Orthogonal pairs of *-subalgebras in finite von Neumann algebras*. J. Operator Theory 9(1983), no. 2, 253–268.

[46] M. Rubin, *A Boolean algebra with few subalgebras, interval Boolean algebras and retractiveness*. Trans. Amer. Math. Soc. 278(1983), no. 1, 65–89. https://doi.org/10.2307/1999302

[47] J. G. Stampfli, *The norm of a derivation*. Pacific J. Math. 33(1970), 737–747.

[48] H. Thiel, *The generator rank for C*-algebras*. arxiv:1210.6608

[49] H. Thiel and W. Winter, *The generator problem for $\mathcal{Z}$-stable C*-algebras*. Trans. Amer. Math. Soc. 366(2014), no. 5, 2327–2343. https://doi.org/10.1090/S0002-9947-2014-06013-3

[50] J. Tomiyama, *A characterization of C*-algebras whose conjugate spaces are separable*. Tohoku Math. J. 15(1963), 96–102. https://doi.org/10.2748/tmj/1178243872

[51] S. Todorcevic, *Partition problems in topology*, Contemporary Mathematics, 84, American Mathematical Society, Providence, RI, 1989. https://doi.org/10.1090/conm/084

[52] S. Todorcevic, *Irredundant sets in Boolean algebras*. Trans. Am. Math. Soc. 339(1993), 35–44. https://doi.org/10.2307/2154207

[53] S. Todorcevic, *Biorthogonal systems and quotient spaces via Baire category methods*. Math. Ann. 335(2006), 687–715. https://doi.org/10.1007/s00208-006-0762-7

[54] S. Todorcevic, *A construction scheme for non-separable structures*. Adv. Math. 313(2017), 564–589. https://doi.org/10.1016/j.aim.2017.04.015

[55] D. Velleman, *ω-morasses, and a weak form of Martin's axiom provable in ZFC*. Trans. Amer. Math. Soc. 285(1984), 617–627. https://doi.org/10.2307/1999454

[56] P. Wojtaszczyk, *On linear properties of separable conjugate spaces of C^*-algebras*. Studia Math. 52(1974), 143–147. https://doi.org/10.4064/sm-52-2-143-147

Departamento de Matemática, Instituto de Matemática e Estatística, Universidade de São Paulo, Caixa Postal 66281, 05314-970, São Paulo, Brazil
e-mail : suguio@ime.usp.br

Institute of Mathematics of the Polish Academy of Sciences, ul. Śniadeckich 8, 00-656 Warszawa, Poland
e-mail : piotr.koszmider@impan.pl