On some properties of the quadrilateral.

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§ 1. In a triangle ABC (fig. 37), BE is made equal to CF; to find the locus of the middle point of EF.

Take K the middle point of BC and P the middle point of EF, then PK is the locus required. For if E′ and F′ are the middle points of BE and CF, the middle point of E′F′ will lie in PK (namely, at the middle point of PK); and again if BE′ and CF′ are bisected in E″ and F″, the middle point of E″F″ will lie in PK (namely, at the middle point of KR); and so on. At any stage we may double the parts cut off from BA and CA instead of bisecting them. Hence the locus required is such that any part of it, however small, contains an infinite number of collinear points; and hence the locus is a straight line.

§ 2. The proof of the above paragraph obviously holds good if BE and CF, instead of being taken equal, are taken in a constant ratio.

Hence the proposition may be stated as a property of the quad rilateral, as follows:—

If the points P and Q divide the sides AB and DC of a quadrilateral ABCD in the same (variable) ratio, the locus of the middle point of PQ is a straight line.

§ 3. GENERALISATION.

If L and M (fig. 38) divide the sides AB and DC of a quadrilateral ABCD in the same (variable) ratio $l{:}m$, then the locus of a point P that divides LM in a given ratio is a straight line.

For this is true, by last paragraph, if we take the ratio LP:LM to be 1:2; and hence also if we take it to be 1:4 or 1:8 or 3:8; or, in general, if we take it to be $m{:}2^n$; and hence it must be true generally.

Note.—In this way the whole plane may be divided into a number of quadrilaterals whose sides are proportional (but which are not similar).

§ 4. SECOND METHOD OF PROOF.

Lemma.—Let ABC, DEF (fig. 39) be two straight lines and DA,

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EB, FC, be perpendicular to AC; and let further A′B′C′ (fig. 40) be a straight line and D′A′, E′B′, F′C′ be perpendiculars to A′C′, equal respectively to DA, EB, FC; then if AB:BC = A′B′:B′C′, the points D′, E′, F′, are collinear; and conversely.

The proof of this lemma comes at once on drawing through D and D′ lines parallel respectively to AC and A′C′.

[This lemma is obviously connected with a property of the simlpest kind of homogeneous strain.]

Now take (fig. 41) BF:FD = CG:GE. Bisect BC and DE in P and Q, and join PQ. Draw perpendiculars BH, CH′, etc., to PQ. Then, obviously, BH = CH′, DL = EL′; and HK:KL = H′K′:K′L′; and hence FK = GK′ and FR = RG.

The more general theorem of (§ 3) may also be proved in this way·

§ 5. Third Method of Proof.

As neither of the preceding proofs is exactly Euclidian in character, it may be as well to add the following proof.

Let ABC (fig. 42) be any triangle.

Make AC′ = AC; AB′ = AB.

Then BB′ is parallel to CC′, and P,Q,R,S, the middle points of BC, BC′, B′C, B′C′, are collinear.

Now BQ = CR = $\frac{1}{2}$ (BA + AC).

If we make BD = CE, we have still DQ = ER = $\frac{1}{2}$ (DA + AE); and hence RQ passes through P′, the middle point of DE.

The more general theorem, in which BD:CE is a constant ratio, may be proved in much the same way.

§ 6. By means of the proposition of the preceding paragraphs, a simple proof may be given of the following well-known theorem *:—

The middle points of the diagonals of a complete quadrilateral are concurrent. (Fig. 43).

Make BHDK a parallelogram.

Then AH/HE = (AH/HB)(HB/HE)
= (BK/KF)(KC/KB)
= KC/KF.

Hence the middle points of HK, AC and EF are collinear.

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* *Numerous proofs of this theorem have been given, one of the simplest being that contained in Taylor's* Ancient and Modern Geometry of Conics, § 107. *For some account of the history of the theorem, see the last paper in this volume of the* Proceedings, *by Dr J. S Mackay.*

§ 7. The following proof, by means of co-ordinates, of the general theorem of (§ 3) is so simple, that it may be worth while giving it here.

Put (fig. 38) $\qquad AL:LB = DM:MC = \lambda:\mu$;

$$AR:RD = LP:PM = BS:SC = p:q.$$

Let the co-ordinates of R, P, Q, be (ξ_1, η_1), (ξ_2, η_2), (ξ_3, η_3); the co-ordinates of A be (x_1, y_1), etc.

Then $\xi_1 = (qx_1 + px_4)/(p+q)$.

$$\xi_2 = \{p(\lambda x_3 + \mu x_4)/(\lambda + \mu) + q(\lambda x_2 + \mu x_1)/(\lambda + \mu)\}/(v + q)$$
$$= \{p(\lambda x_3 + \mu x_4) + q(\lambda x_2 + \mu x_1)\}/(\lambda + \mu)(p + q).$$
$$\xi_3 = (px_3 + qx_2)/(p + q).$$

Now we may easily show that if we put

$$P = \{p(x_4 - x_3) + q(x_1 - x_2)\}/(\lambda + \mu)(p + q),$$
$$Q = \{p(y_4 - y_3) + q(y_1 - y_2)\}/(\lambda + \mu)(p + q),$$

then $\qquad \xi_2 - \xi_3 = \mu P$; $\xi_3 - \xi_1 = -(\lambda + \mu)P$; $\xi_1 - \xi_2 = \lambda P$.

Hence $\qquad \eta_1(\xi_2 - \xi_3) + \eta_2(\xi_3 - \xi_1) + \eta_3(\xi_1 - \xi_2)$

$$= \eta_1 \mu P - \eta_2(\lambda + \mu)P + \eta_3 \lambda P$$
$$= P\{\lambda(\eta_3 - \eta_2) + \mu(\eta_1 - \eta_2)\}$$
$$= P\{\lambda(-\mu Q) + \mu(\lambda Q)\}$$
$$= PQ(-\lambda\mu + \lambda\mu) = 0.$$

Hence R, P and S are collinear.

An Apparatus of Professor Tait's was exhibited which gives the same curve as a glissette, either of a hyperbola or an ellipse.

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R. E. ALLARDICE, Esq., M.A., Vice-President, in the Chair.

On the Moduluses of Elasticity of an Elastic Solid according to Boscovich's Theory.

By Sir WILLIAM THOMSON.

The substance of this paper will be found in the *Proceedings of the Royal Society of Edinburgh*, Vol. xvi., pp. 693–724; and Thomson's *Mathematical and Physical Papers*, Vol. iii., Art. xcvii., pp. 395–498.