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TAIL BEHAVIOR OF RANDOMLY WEIGHTED SUMS

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Abstract

Let $\{X_t,\ t \geq 1\}$ be a sequence of identically distributed and pairwise asymptotically independent random variables with regularly varying tails, and let $\{\Theta_t,\ t \geq 1\}$ be a sequence of positive random variables independent of the sequence $\{X_t,\ t \geq 1\}$. We will discuss the tail probabilities and almost-sure convergence of $X_{(\infty)} = \sum_{t=1}^{\infty} \Theta_t X_t^+$ (where $X^+ = \max\{0, X\}$) and $\max_{1\leq k<\infty} \sum_{t=1}^{k} \Theta_t X_t$, and provide some sufficient conditions motivated by Denisov and Zwart (2007) as alternatives to the usual moment conditions. In particular, we illustrate how the conditions on the slowly varying function involved in the tail probability of X_1 help to control the tail behavior of the randomly weighted sums. Note that, the above results allow us to choose $X_1, X_2, \ldots$ as independent and identically distributed positive random variables. If X_1 has a regularly varying tail of index $-\alpha$, where $\alpha > 0$, and if $\{\Theta_t,\ t \geq 1\}$ is a positive sequence of random variables independent of $\{X_t\}$, then it is known—which can also be obtained from the sufficient conditions in this article—that, under some appropriate moment conditions on $\{\Theta_t,\ t \geq 1\}$, $X_{(\infty)} = \sum_{t=1}^{\infty} \Theta_t X_t$ converges with probability 1 and has a regularly varying tail of index $-\alpha$. Motivated by the converse problems in Jacobsen, Mikosch, Rosiński and Samorodnitsky (2009) we ask the question: if $X_{(\infty)}$ has a regularly varying tail then does X_1 have a regularly varying tail under some appropriate conditions? We obtain appropriate sufficient moment conditions, including the nonvanishing Mellin transform of $\sum_{t=1}^{\infty} \Theta_t$ along some vertical line in the complex plane, so that the above is true. We also show that the condition on the Mellin transform cannot be dropped.

Keywords: Regular variation; heavy tail; asymptotic independence; Breiman's theorem; product of random variables; subexponential

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1. Introduction

Let $\{X_t,\ t \geq 1\}$ be a sequence of identically distributed, pairwise asymptotically independent (see (2.1) below) random variables, and let $\{\Theta_t,\ t \geq 1\}$ be a sequence of positive random variables independent of the sequence $\{X_t,\ t \geq 1\}$. We will discuss the tail probabilities and almost-sure convergence of $X_{(\infty)} = \sum_{t=1}^{\infty} \Theta_t X_t^+$ (where $X^+ = \max\{0, X\}$) and $\max_{1\leq k<\infty} \sum_{t=1}^{k} \Theta_t X_t$, in particular, when the X_ts belong to the class of random variables with regularly varying tail and $\{\Theta_t,\ t \geq 1\}$ satisfies some moment conditions. We will say that a random variable X with distribution function F has a *regularly varying tail of index* $-\alpha$ if $\overline{F}(x) := 1 - F(x)$ is a regularly varying function of index $-\alpha$. That is, for any $t > 0$, as

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$x \to \infty$, $\overline{F}(tx) \sim t^{-\alpha}\overline{F}(x)$. Here and later, for two positive functions $a(x)$ and $b(x)$, we write $a(x) \sim b(x)$ as $x \to \infty$ if $\lim_{x\to\infty} a(x)/b(x) = 1$. For $\alpha > 0$, the convergence in the limit of the ratio of the tail probabilities is uniform in t on intervals of the form $[a, \infty)$ with $a > 0$. Note that we require the upper endpoint of the support of X to be ∞. Recently, there has been quite a few articles devoted to the asymptotic tail behavior of randomly weighted sums and their maxima. (See, for example, [2], [8], and [17]–[19].)

The question about the tail behavior of the infinite series $X_{(\infty)}$ with nonrandom Θ_t and independent and identically distributed (i.i.d.) X_t having regularly varying tails has been well studied in the literature, as it arises in the context of the linear processes, including ARMA and FARIMA processes. We refer the reader to [10] for a review of the results. The case in which the Θ_ts are random arises in various areas, especially in actuarial and economic applications, as well as stochastic recurrence equations. For various applications, see [8] and [19].

Resnick and Willekens [17] showed that if $\{X_t\}$ is a sequence of i.i.d. nonnegative random variables with a regularly varying tail of index $-\alpha$, where $\alpha > 0$ and $\{\Theta_t\}$ is another sequence of positive random variables independent of $\{X_t\}$, the series $X_{(\infty)}$ has a regularly varying tail under the following conditions, which we will call the RW conditions.

(RW1) If $0 < \alpha < 1$ then, for some $\varepsilon \in (0, \alpha)$, $\sum_{t=1}^{\infty} \mathrm{E}[\Theta_t^{\alpha+\varepsilon} + \Theta_t^{\alpha-\varepsilon}] < \infty$.

(RW2) If $1 \le \alpha < \infty$ then, for some $\varepsilon \in (0, \alpha)$, $\sum_{t=1}^{\infty} (\mathrm{E}[\Theta_t^{\alpha+\varepsilon} + \Theta_t^{\alpha-\varepsilon}])^{1/(\alpha+\varepsilon)} < \infty$.

In this case, we have $\mathrm{P}[X_{(\infty)} > x] \sim \sum_{t=1}^{\infty} \mathrm{E}[\Theta_t^{\alpha}]\,\mathrm{P}[X_1 > x]$ as $x \to \infty$.

Remark 1.1. Each of the RW conditions implies the other for the respective ranges of α. In particular, if $0 < \alpha < 1$, choose $\varepsilon' < \varepsilon$ such that $\alpha + \varepsilon' < 1$. Note that

$$\begin{aligned}
\sum_{t=1}^{\infty} \mathrm{E}[\Theta_t^{\alpha+\varepsilon'} + \Theta_t^{\alpha-\varepsilon'}] &\le 2\sum_{t=1}^{\infty} \mathrm{E}[\Theta_t^{\alpha+\varepsilon'}\mathbf{1}_{[\Theta_t \ge 1]} + \Theta_t^{\alpha-\varepsilon'}\mathbf{1}_{[\Theta_t < 1]}] \\
&\le 2\sum_{t=1}^{\infty} \mathrm{E}[\Theta_t^{\alpha+\varepsilon}\mathbf{1}_{[\Theta_t \ge 1]} + \Theta_t^{\alpha-\varepsilon}\mathbf{1}_{[\Theta_t < 1]}] \\
&\le 2\sum_{t=1}^{\infty} \mathrm{E}[\Theta_t^{\alpha+\varepsilon} + \Theta_t^{\alpha-\varepsilon}] \\
&< \infty.
\end{aligned}$$

Furthermore, since $\alpha + \varepsilon' < 1$, we also have $\sum_{t=1}^{\infty} (\mathrm{E}[\Theta_t^{\alpha+\varepsilon'} + \Theta_t^{\alpha-\varepsilon'}])^{1/(\alpha+\varepsilon')} < \infty$. On the other hand, if $\alpha \ge 1$ and $\varepsilon > 0$, then $\alpha + \varepsilon > 1$ and condition (RW2) implies that $\sum_{t=1}^{\infty} \mathrm{E}[\Theta_t^{\alpha+\varepsilon} + \Theta_t^{\alpha-\varepsilon}] < \infty$.

Zhang *et al.* [19] considered the tails of $\sum_{t=1}^{n} \Theta_t X_t$ and the tails of their maxima when the $\{X_t\}$ are pairwise asymptotically independent and have an extended regularly varying and negligible left tail and the $\{\Theta_t\}$ are positive random variables independent of $\{X_t\}$. The sufficient conditions for the tails to be regularly varying are almost similar.

While the tail behavior of $X_{(\infty)}$ requires only the αth moments of the Θ_ts, we require existence and summability of some extra moments in the RW conditions. Note that $\Theta_t^{\alpha+\varepsilon}$ acts as a dominator for $[\Theta_t \ge 1]$ and $\Theta_t^{\alpha-\varepsilon}$ acts as a dominator for $[\Theta_t \le 1]$. In some cases, the assumption of existence and summability of extra moments can become restrictive. For example, consider $\{\Theta_t\}$ such that $\sum_{t=1}^{\infty} \mathrm{E}[\Theta_t^{\alpha+\varepsilon}] = \infty$ for all $\varepsilon > 0$ but $\sum_{t=1}^{\infty} \mathrm{E}[\Theta_t^{\alpha}] < \infty$. (A particular choice of such $\{\Theta_t\}$ for $\alpha < 1$ is as follows: Θ_t takes the values $2^t/t^{2/\alpha}$ and 0

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with probability $2^{-t\alpha}$ and $1-2^{-t\alpha}$, respectively.) Also, let $\{X_t\}$ be i.i.d. Pareto with parameter $\alpha < 1$, independent of $\{\Theta_t\}$. Then it turns out, after some easy calculations, that $\sum_{t=1}^{\infty} \Theta_t X_t$ is finite almost surely and has a regularly varying tail of index $-\alpha$. This leads to the question of whether we can reduce the moment conditions on Θ_t to obtain the regular variation of the tail for $X_{(\infty)}$.

The situation becomes clearer when we consider a general term of the series. It involves the product $\Theta_t X_t$. Using Breiman's theorem (cf. [1] and [3]), the tail behavior of the product depends on the moments of Θ_t. Breiman's theorem states that, if X is a random variable with a regularly varying tail of index $-\alpha$ for some $\alpha > 0$ and it is independent of a positive random variable Θ satisfying $\mathrm{E}[\Theta^{\alpha+\varepsilon}] < \infty$ for some $\varepsilon > 0$, then

$$\lim_{x\to\infty} \mathrm{P}[\Theta X > x] \sim \mathrm{E}[\Theta^{\alpha}]\,\mathrm{P}[X > x]. \tag{1.1}$$

Note that, in this case, we work with a probability measure $\mathrm{P}[\Theta_t \in \cdot]$, unlike in the problem of the weighted sum, where a σ-finite measure $\sum_{t=1}^{\infty} \mathrm{P}[\Theta \in \cdot]$ is considered. In this case, we can consider the dominator as 1 on $[\Theta \le 1]$ instead of $\Theta^{\alpha-\varepsilon}$, since 1 is integrable with respect to a probability measure.

Denisov and Zwart [5] relaxed the existence of the $(\alpha+\varepsilon)$th moments in Breiman's theorem to $\mathrm{E}[\Theta^{\alpha}] < \infty$. They also made the further natural assumption that $\mathrm{P}[\Theta > x] = o(\mathrm{P}[X > x])$. However, to obtain (1.1), the weaker moment assumption needed to be compensated. They obtained some sufficient conditions for (1.1) to hold. We would like to find conditions similar to those obtained in [5], which will guarantee the regular variation of $X_{(\infty)}$.

In the above discussion, we considered the effect of the tail of X_1 in determining the tail of $X_{(\infty)}$. However, the converse problem is also equally interesting. More specifically, let $\{X_t\}$ be a sequence of identically distributed, asymptotically independent, positive random variables, independent of another sequence of positive random variables $\{\Theta_t\}$. As before, we define $X_{(\infty)} = \sum_{t=1}^{\infty} \Theta_t X_t$, and assume that $X_{(\infty)}$ converges with probability 1 and has a regularly varying tail of index $-\alpha$ with $\alpha > 0$. It is relevant to obtain sufficient conditions which will ensure that X_1 has a regularly varying tail of index $-\alpha$ as well.

The converse of Breiman's theorem (1.1) has recently been considered in the literature. Suppose that X and Y are positive random variables with $\mathrm{E}[Y^{\alpha+\varepsilon}] < \infty$, and that the product XY has a regularly varying tail of index $-\alpha$, $\alpha > 0$. It was shown by Jacobsen *et al.* [9] that X has a regularly varying tail of the same index and, hence, (1.1) holds. They have also obtained results for the weighted series, when the weights $\{\Theta_t\}$ are nonrandom. We extend their result for the product to the case of a randomly weighted series under appropriate conditions.

In Section 2 we first describe the various classes of heavy-tailed distributions and describe the conditions imposed in [5]. We study the tail behavior of finite randomly weighted sums. In Section 3 we describe the tail behavior of the series of randomly weighted sums. In Section 4 we consider the converse problem described above. We prove the converse result under the RW conditions and an additional assumption of a nonvanishing Mellin transform. We also demonstrate the necessity of the additional assumption.

2. Notation and preliminary results

We first introduce a few classes of random variables, which will be required for the rest of the discussion. A random variable X with distribution function F is called *long tailed* if, for any fixed $y \in \mathbb{R}$, as $x \to \infty$, we have $\overline{F}(x-y) \sim \overline{F}(x)$. The class of long-tailed distributions is denoted by $\mathcal{L}$. Observe that, for $F \in \mathcal{L}$, we require $\overline{F}(x) > 0$ for all $x > 0$. The class $\mathcal{L}$ is related to the class of distributions with regularly varying tails by the fact that $F \in \mathcal{L}$

if and only if $\overline{F}(\log(\cdot))$ is slowly varying, that is, regularly varying of index 0. Equivalently, the random variable X has a distribution function in the class $\mathcal{L}$ if and only if $\exp(X)$ has a regularly varying tail of index 0.

A nonnegative function f, which does not vanish for all large x, is in the class of *subexponential densities* (denoted by $\mathcal{S}_d$), if it satisfies the property

$$\lim_{x\to\infty}\int_0^x \frac{f(x-y)}{f(x)} f(y)\,\mathrm{d}y = 2\int_0^\infty f(u)\,\mathrm{d}u < \infty.$$

(Note that, following [12], we do not require the subexponential density class function f to satisfy $\int_0^\infty f(x)\,\mathrm{d}x = 1$. However, some recent authors, e.g. Foss *et al.* [7], required the integral to be 1.) If, for some nonnegative random variable X, the tail of the distribution function $\overline{F}(x) = \mathrm{P}[X > x]$ is in the class of subexponential densities, we say that X is in $\mathcal{S}^*$. Again, if $X \in \mathcal{S}^*$, we need $\overline{F}(x) > 0$ for all $x > 0$. Also, for a nonnegative random variable X with distribution function F in the class $\mathcal{S}^*$, we have $\int_0^\infty \overline{F}(x)\,\mathrm{d}x = \mathrm{E}[X] < \infty$.

A distribution function F belongs to the class $\mathcal{S}(\gamma)$ with $\gamma \geq 0$ if, for all real u,

$$\lim_{x\to\infty}\frac{\overline{F}(x-u)}{\overline{F}(x)} = \mathrm{e}^{\gamma u} \quad\text{and}\quad \lim_{x\to\infty}\frac{\overline{F*F}(x)}{\overline{F}(x)} = 2\int_0^\infty \mathrm{e}^{\gamma y}F(\mathrm{d}y) < \infty.$$

The class $\mathcal{S} := \mathcal{S}(0)$ is called the class of *subexponential distribution functions*. See [6], [11], and [12] for properties of these classes.

We call two random variables X_1 and X_2 *asymptotically independent* if

$$\lim_{x\to\infty}\frac{\mathrm{P}[X_1 > x, X_2 > x]}{\mathrm{P}[X_t > x]} = 0 \quad \text{for } t = 1, 2. \tag{2.1}$$

See [13], [14], or [16, Chapter 6.5] for discussions on asymptotic independence. Note that we require $\overline{F}_t(x) > 0$ for all $x > 0$ and $t = 1, 2$. Observe that if X_1 and X_2 are independent, then they are also asymptotically independent. Thus, the results under the pairwise asymptotic independence condition continue to hold in the independent setup.

A random variable X is said to have *negligible left tail with respect to the right tail* if

$$\lim_{x\to\infty}\frac{\mathrm{P}[X < -x]}{\mathrm{P}[X > x]} = 0. \tag{2.2}$$

Note that we require $\mathrm{P}[X > x] > 0$ for all $x > 0$.

The random variables with regularly varying tails will play a central role in this article. Note that, if X has a regularly varying tail of index $-\alpha$ then $x^\alpha\,\mathrm{P}[X > x]$ is a slowly varying function, that is, a regularly varying function with index 0. By Karamata's representation, a slowly varying function L can be one of the following four representations (cf. [5, Lemma 2.1]):

1. $L(x) = c(x)$,
2. $L(x) = c(x)/\mathrm{P}[V > \log x]$,
3. $L(x) = c(x)\,\mathrm{P}[U > \log x]$,
4. $L(x) = c(x)\,\mathrm{P}[U > \log x]/\mathrm{P}[V > \log x]$.

In the above representations, $c(x)$ is a function converging to $c \in (0, \infty)$, and U and V are two long-tailed random variables with hazard rates converging to 0. (The hazard rate of a distribution function F with density f is defined as $f/\overline{F}$.) We will refer to a slowly varying function L as of type 1, type 2, type 3, or type 4, according to the above representations.

Denisov and Zwart [5] introduced the following sufficient conditions on the slowly varying part L of the regularly varying tail of index $-\alpha$ of a random variable X with distribution function $\overline{F}(x) = x^{-\alpha}L(x)$ for Breiman's theorem (1.1) to hold.

(DZ1) Assume that $\lim_{x\to\infty}\sup_{y\in[1,x]} L(y)/L(x) =: D_1 < \infty$.

(DZ2) Assume that L is of type 3 or type 4 and $L(\mathrm{e}^x) \in \mathcal{S}_d$.

(DZ3) Assume that L is of type 3 or type 4, $U \in \mathcal{S}^*$, and $\mathrm{P}[\Theta > x] = o(x^{-\alpha}\,\mathrm{P}[U > \log x])$.

(DZ4) When $\mathrm{E}[U] = \infty$ or, equivalently, $\mathrm{E}[X^\alpha] = \infty$, define $m(x) = \int_0^x v^\alpha F(\mathrm{d}v) \to \infty$. Assume that $\limsup_{x\to\infty}\sup_{\sqrt{x}\le y\le x} L(y)/L(x) =: D_2 < \infty$ and $\mathrm{P}[\Theta > x] = o(\mathrm{P}[X > x]/m(x))$.

We will refer to these conditions as the DZ conditions. For further discussions on the DZ conditions, we refer the reader to [5]. Denisov and Zwart proved the following lemma.

Lemma 2.1. (Section 2 of [5].) *Let X be a nonnegative random variable with a regularly varying tail of index $-\alpha$, $\alpha \ge 0$, and let Θ be a positive random variable independent of X such that $\mathrm{E}[\Theta^\alpha] < \infty$ and $\mathrm{P}[\Theta > x] = o(\mathrm{P}[X > x])$. If X and Θ satisfy any one of the DZ conditions, then (1.1) holds.*

The next result shows that asymptotic independence is preserved under multiplication, when the DZ conditions are assumed.

Lemma 2.2. *Let X_1 and X_2 be two positive, asymptotically independent, identically distributed random variables with regularly varying tails of index $-\alpha$, where $\alpha > 0$. Let Θ_1 and Θ_2 be two other positive random variables independent of the pair (X_1, X_2) satisfying $\mathrm{E}[\Theta_t^\alpha] < \infty$, $t = 1, 2$. Also, suppose that $\mathrm{P}[\Theta_t > x] = o(\mathrm{P}[X_1 > x])$ for $t = 1, 2$, and that the pairs (Θ_1, X_1) and (Θ_2, X_2) satisfy any one of the DZ conditions. Then Θ_1X_1 and Θ_2X_2 are asymptotically independent.*

Proof. Here and later, G will denote the joint distribution function of (Θ_1, Θ_2) and G_t will denote the marginal distribution functions of Θ_t. We have

$$\begin{aligned}\frac{\mathrm{P}[\Theta_1X_1 > x,\ \Theta_2X_2 > x]}{\mathrm{P}[X_1 > x]} &= \biggl(\iint_{u\le v} + \iint_{u>v}\biggr)\frac{\mathrm{P}[X_1 > x/u,\ X_2 > x/v]}{\mathrm{P}[X_1 > x]}G(\mathrm{d}u,\ \mathrm{d}v)\\ &\le \int_0^\infty \frac{\mathrm{P}[X_1 > x/v,\ X_2 > x/v]}{\mathrm{P}[X_1 > x/v]}\frac{\mathrm{P}[X_1 > x/v]}{\mathrm{P}[X_1 > x]}(G_1 + G_2)(\mathrm{d}v).\end{aligned}$$

The integrand converges to 0. Also, the first factor of the integrand is bounded by 1 and, hence, the integrand is bounded by the second factor, which converges to v^α. Furthermore, using Lemma 2.1, we have

$$\begin{aligned}\int_0^\infty \frac{\mathrm{P}[X_1 > x/v]}{\mathrm{P}[X_1 > x]}(G_1 + G_2)(\mathrm{d}v) &= \frac{\mathrm{P}[\Theta_1X_1 > x] + \mathrm{P}[\Theta_2X_1 > x]}{\mathrm{P}[X_1 > x]}\\ &\to \mathrm{E}[\Theta_1^\alpha] + \mathrm{E}[\Theta_2^\alpha]\\ &= \int_0^\infty v^\alpha(G_1 + G_2)(\mathrm{d}v).\end{aligned}$$

Then the result follows using Pratt's lemma; cf. [15].

The next lemma shows that if the left tail of X is negligible when compared to the right tail then the same holds for the product.

Lemma 2.3. *Let X have a regularly varying tail of index $-\alpha$, $\alpha > 0$, satisfying (2.2), and let Θ be independent of X, satisfying* $\mathrm{E}[\Theta^\alpha] < \infty$ *and* $\mathrm{P}[\Theta > x] = o(\mathrm{P}[X > x])$. *Also, suppose that the pair (Θ, X) satisfies one of the DZ conditions. Then, for any $u > 0$,*

$$\lim_{x\to\infty} \frac{\mathrm{P}[\Theta X < -ux]}{\mathrm{P}[\Theta X > x]} = 0.$$

The proof is similar to that of Lemma 2.2, except for the fact that the first factor in the integrand is bounded, as, using (2.2), $\mathrm{P}[X < -x]/\,\mathrm{P}[X > x]$ is bounded. We omit the proof.

The following result from [4] considers the case of the tail of the sum of finitely many random variables.

Lemma 2.4. ([4, Lemma 2.1].) *Suppose that $Y_1, Y_2, \dots, Y_k$ are nonnegative, pairwise asymptotically independent (but need not be identically distributed) random variables with regularly varying tails of common index $-\alpha$, where $\alpha > 0$. If, for $t = 1, 2, \dots, k$, $\mathrm{P}[Y_t > x]/\,\mathrm{P}[Y_1 > x] \to c_t$ then*

$$\frac{\mathrm{P}[\sum_{t=1}^k Y_t > x]}{\mathrm{P}[Y_1 > x]} \to c_1 + c_2 + \cdots + c_k.$$

We have the following corollary by applying Lemma 2.4 with $Y_t = \Theta_t X_t^+$ and the modified Breiman theorem in Lemma 2.1 under the DZ conditions.

Corollary 2.1. *Let $\{X_t\}$ be a sequence of pairwise asymptotically independent, identically distributed random variables with common regularly varying tail of index $-\alpha$, where $\alpha > 0$, which is independent of another sequence of positive random variables $\{\Theta_t\}$ satisfying* $\mathrm{E}[\Theta_t^\alpha] < \infty$ *for all t. Also, assume that, for all t,* $\mathrm{P}[\Theta_t > x] = o(\mathrm{P}[X_1 > x])$ *and that the pairs (Θ_t, X_t) satisfy one of the DZ conditions. Then*

$$\mathrm{P}\Bigg[\sum_{t=1}^k \Theta_t X_t^+ > x\Bigg] \sim \mathrm{P}[X_1 > x] \sum_{t=1}^k \mathrm{E}[\Theta_t^\alpha].$$

Using Lemmas 2.1–2.4 and Corollary 2.1 and arguing as in Theorem 3.1(a) of [19], we have the following result. (Note that the proof of Theorem 3.1(a) of [19] required only the results obtained in Lemmas 2.1–2.4 and Corollary 2.1.)

Proposition 2.1. *Let $\{X_t\}$ be a sequence of pairwise asymptotically independent, identically distributed random variables with common regularly varying tail of index $-\alpha$, $\alpha > 0$, satisfying (2.2), which is independent of another sequence of positive random variables $\{\Theta_t\}$. Assume further that, for all t,* $\mathrm{P}[\Theta_t > x] = o(\mathrm{P}[X_1 > x])$ *and* $\mathrm{E}[\Theta_t^\alpha] < \infty$. *Also, assume that the pairs (Θ_t, X_t) satisfy one of the DZ conditions. Then*

$$\mathrm{P}\Bigg[\max_{1\le k\le n} \sum_{t=1}^k \Theta_t X_t > x\Bigg] \sim \mathrm{P}\Bigg[\sum_{t=1}^n \Theta_t X_t^+ > x\Bigg] \sim \mathrm{P}[X_1 > x] \sum_{t=1}^n \mathrm{E}[\Theta_t^\alpha].$$

3. The tail of the weighted sum under the DZ conditions

In Proposition 2.1, we saw that the conditions on the slowly varying function help to reduce the moment conditions on $\{\Theta_t\}$ for the finite sum. However, we need some additional hypotheses to handle the infinite series. To study the almost-sure convergence of $X_{(\infty)} = \sum_{t=1}^\infty \Theta_t X_t^+$, observe that the partial sums $S_n = \sum_{t=1}^n \Theta_t X_t^+$ increase to $X_{(\infty)}$. We will show in the

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following results that $\mathrm{P}[X_{(\infty)} > x] \sim \mathrm{P}[X_1 > x] \sum_{t=1}^{\infty} \mathrm{E}[\Theta_t^\alpha]$ under suitable conditions. Thus, if $\sum_{t=1}^{\infty} \mathrm{E}[\Theta_t^\alpha] < \infty$ then $\lim_{x\to\infty} \mathrm{P}[X_{(\infty)} > x] = 0$ and $X_{(\infty)}$ is finite almost surely.

To obtain the required tail behavior, we will assume the following conditions, which weaken the moment requirements of $\{\Theta_t\}$ assumed in conditions (RW1) and (RW2) given in [17].

(RW1′) For $0 < \alpha < 1$, $\sum_{t=1}^{\infty} \mathrm{E}[\Theta_t^\alpha] < \infty$.

(RW2′) For $1 \le \alpha < \infty$ and some $\varepsilon > 0$, $\sum_{t=1}^{\infty} (\mathrm{E}[\Theta_t^\alpha])^{1/(\alpha+\varepsilon)} < \infty$.

We will call these conditions the modified RW moment conditions.

Remark 3.1. Hult and Samorodnitsky [8] provided a set of sufficient conditions as an alternative to conditions (RW1) and (RW2). For the case $0 < \alpha < 1$, the condition provided in [8] coincides with (RW1). On the other hand, for $\alpha \ge 1$, Condition (3.7) of [8] is strictly weaker than (RW2). However, for all α, Conditions (3.7)–(3.9) of [8] required, for some $\varepsilon > 0$, the existence of $\mathrm{E}[\Theta_t^{\alpha+\varepsilon}]$ for all t and, hence, is not weaker than the modified RW moment conditions introduced in this paper. Furthermore, for the case $\alpha = 1$ or $\alpha \ge 2$, Conditions (3.8) and (3.9) of [8] involve moments of a random series rather than a series involving moments of a random variable as in the RW conditions and, thus, the conditions in [8] may be more difficult to verify in practice. So we extend only the RW conditions in this paper and leave the extensions of the conditions in [8] for a future work.

Remark 3.2. As observed in Remark 1.1, for $\alpha \ge 1$ and $\varepsilon > 0$, the finiteness of the sum in (RW2′) implies that $\sum_{t=1}^{\infty} (\mathrm{E}[\Theta_t^\alpha]) < \infty$. Thus, to check the almost-sure finiteness of $X_{(\infty)}$, it is enough to check the tail asymptotics condition:

$$\mathrm{P}[X_{(\infty)} > x] \sim \mathrm{P}[X_1 > x] \sum_{t=1}^{\infty} \mathrm{E}[\Theta_t^\alpha].$$

We will prove it under the above model together with the assumption that $\mathrm{P}[\Theta_t > x] = o(\mathrm{P}[X_1 > x])$ and one of the DZ conditions. We need to assume an extra summability condition for uniform convergence, when condition (DZ2), (DZ3), or (DZ4) holds.

Furthermore, note that $\Theta_1 X_1 \le \max_{1\le n<\infty} \sum_{t=1}^{n} \Theta_t X_t \le X_{(\infty)}$ and, hence, the almost-sure finiteness of $X_{(\infty)}$ guarantees that $\max_{1\le n<\infty} \sum_{t=1}^{n} \Theta_t X_t$ is finite almost surely.

Theorem 3.1. *Suppose that $\{X_t\}$ is a sequence of pairwise asymptotically independent, identically distributed random variables with a regularly varying tail of index $-\alpha$, $\alpha > 0$, satisfying (2.2), which is independent of another sequence of positive random variables $\{\Theta_t\}$. Also, assume that $\mathrm{P}[\Theta_t > x] = o(\mathrm{P}[X_1 > x])$, and that the pairs (Θ_t, X_t) satisfy one of the four DZ conditions and, depending on the value of α, one of the modified RW moment conditions. If the pairs (Θ_t, X_t) satisfy condition (DZ2), (DZ3), or (DZ4), define*

$$C_t = \begin{cases} \sup_x \dfrac{\mathrm{P}[\Theta_t > x]}{\mathrm{P}[X_1 > x]} & \textit{when (DZ2) holds,} \\ \sup_x \dfrac{\mathrm{P}[\Theta_t > x]}{x^{-\alpha}\,\mathrm{P}[U > \log x]} & \textit{when (DZ3) holds,} \\ \sup_x \dfrac{\mathrm{P}[\Theta_t > x]}{\mathrm{P}[X_1 > x]} m(x) & \textit{when (DZ4) holds,} \end{cases} \tag{3.1}$$

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and assume further that

$$\sum_{t=1}^{\infty} C_t < \infty \quad \textit{when } \alpha < 1, \tag{3.2}$$

$$\sum_{t=1}^{\infty} C_t^{1/(\alpha+\varepsilon)} < \infty \quad \textit{when } \alpha \geq 1. \tag{3.3}$$

Then

$$\mathrm{P}\left[\max_{1\leq n<\infty} \sum_{t=1}^{n} \Theta_t X_t > x\right] \sim \mathrm{P}[X_{(\infty)} > x] \sim \mathrm{P}[X_1 > x] \sum_{t=1}^{\infty} \mathrm{E}[\Theta_t^{\alpha}]$$

and $X_{(\infty)}$ is almost surely finite.

Proof. For any $m \geq 1$, we have, by Proposition 2.1,

$$\mathrm{P}\left[\max_{1\leq n<\infty} \sum_{t=1}^{n} \Theta_t X_t > x\right] \geq \mathrm{P}\left[\max_{1\leq n\leq m} \sum_{t=1}^{n} \Theta_t X_t > x\right] \sim \mathrm{P}[X_1 > x] \sum_{t=1}^{m} \mathrm{E}[\Theta_t^{\alpha}],$$

leading to

$$\liminf_{x\to\infty} \frac{\mathrm{P}[\max_{1\leq n<\infty} \sum_{t=1}^{n} \Theta_t X_t > x]}{\mathrm{P}[X_1 > x]} \geq \sum_{t=1}^{\infty} \mathrm{E}[\Theta_t^{\alpha}].$$

Similarly, comparing with the partial sums and using Proposition 2.1, we also obtain

$$\liminf_{x\to\infty} \frac{\mathrm{P}[X_{(\infty)} > x]}{\mathrm{P}[X_1 > x]} \geq \sum_{t=1}^{\infty} \mathrm{E}[\Theta_t^{\alpha}].$$

For the other inequality, observe that, for any natural number m, $0 < \delta < 1$, and $x \geq 0$,

$$\mathrm{P}\left[\max_{1\leq n<\infty} \sum_{t=1}^{n} \Theta_t X_t > x\right] \leq \mathrm{P}\left[\max_{1\leq n\leq m} \sum_{t=1}^{n} \Theta_t X_t > (1-\delta)x\right] + \mathrm{P}\left[\sum_{t=m+1}^{\infty} \Theta_t X_t^+ > \delta x\right].$$

For the first term, by Proposition 2.1 and the regular variation of the tail of X_1, we have

$$\lim_{x\to\infty} \frac{\mathrm{P}[\max_{1\leq n\leq m} \sum_{t=1}^{n} \Theta_t X_t > (1-\delta)x]}{\mathrm{P}[X_1 > x]} = (1-\delta)^{-\alpha} \sum_{t=1}^{m} \mathrm{E}[\Theta_t^{\alpha}] \leq (1-\delta)^{-\alpha} \sum_{t=1}^{\infty} \mathrm{E}[\Theta_t^{\alpha}].$$

Also, for $X_{(\infty)}$, we have

$$\mathrm{P}[X_{(\infty)} > x] \leq \mathrm{P}\left[\sum_{t=1}^{m} \Theta_t X_t^+ > (1-\delta)x\right] + \mathrm{P}\left[\sum_{t=m+1}^{\infty} \Theta_t X_t^+ > \delta x\right],$$

and a similar result holds for the first term.

Then, as X_1 is a random variable with a regularly varying tail, to complete the proof, it is enough to show that

$$\lim_{m\to\infty} \limsup_{x\to\infty} \frac{\mathrm{P}[\sum_{t=m+1}^{\infty} \Theta_t X_t^+ > x]}{\mathrm{P}[X_1 > x]} = 0. \tag{3.4}$$

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Now,

$$
\begin{aligned}
\mathrm{P}\bigg[\sum_{t=m+1}^{\infty} \Theta_t {X_t}^{+} > x\bigg]
&\le \mathrm{P}\bigg[\bigvee_{t=m+1}^{\infty} \Theta_t X_t^{+} > x\bigg] + \mathrm{P}\bigg[\sum_{t=m+1}^{\infty} \Theta_t X_t^{+} > x,\ \bigvee_{t=m+1}^{\infty} \Theta_t X_t^{+} \le x\bigg]\\
&\le \sum_{t=m+1}^{\infty} \mathrm{P}[\Theta_t X_t > x] + \mathrm{P}\bigg[\sum_{t=m+1}^{\infty} \Theta_t X_t^{+} \mathbf{1}_{[\Theta_t X_t^{+} \le x]} > x\bigg]. \qquad (3.5)
\end{aligned}
$$

We bound the final term of (3.5) separately in the cases $\alpha < 1$ and $\alpha \ge 1$. In the rest of the proof, for $\alpha \ge 1$, we will choose $\varepsilon > 0$, so that condition (RW2$'$) holds. We first consider the case $\alpha < 1$. By Markov's inequality, the final term of (3.5) gets bounded above by

$$
\begin{aligned}
\sum_{t=m+1}^{\infty} \frac{1}{x} \mathrm{E}[\Theta_t X_t^{+} \mathbf{1}_{[\Theta_t X_t^{+} \le x]}] &= \sum_{t=m+1}^{\infty} \int_0^{\infty} \frac{1}{x/v} \mathrm{E}[X_t^{+} \mathbf{1}_{[X_t^{+} \le x/v]}] G_t(\mathrm{d}v)\\
&= \sum_{t=m+1}^{\infty} \int_0^{\infty} \frac{\mathrm{E}[X_t^{+} \mathbf{1}_{[X_t^{+} \le x/v]}]}{x/v\, \mathrm{P}[X_t^{+} > x/v]} \mathrm{P}\bigg[X_t > \frac{x}{v}\bigg] G_t(\mathrm{d}v). \qquad (3.6)
\end{aligned}
$$

Now, using Karamata's theorem (cf. [16, Theorem 2.1]), we have

$$
\lim_{x\to\infty} \frac{\mathrm{E}[X_t^{+} \mathbf{1}_{[X_t^{+} \le x]}]}{x\, \mathrm{P}[X_t^{+} > x]} = \frac{\alpha}{1-\alpha}
$$

and, for $x < 1$, we have

$$
\frac{\mathrm{E}[X_t^{+} \mathbf{1}_{[X_t^{+} \le x]}]}{x\, \mathrm{P}[X_t^{+} > x]} \le \frac{1}{\mathrm{P}[X_t^{+} > 1]}.
$$

Thus, $\mathrm{E}[X_t^{+} \mathbf{1}_{[X_t^{+} \le x]}]/(x\, \mathrm{P}[X_t^{+} > x])$ is bounded on $(0, \infty)$. So the final term of (3.5) becomes bounded by a multiple of $\sum_{t=m+1}^{\infty} \mathrm{P}[\Theta_t X_t > x]$.

When $\alpha \ge 1$, using Markov's inequality on the final term of (3.5), we obtain a bound for it as

$$
\frac{1}{x^{\alpha+\varepsilon}} \mathrm{E}\bigg[\bigg(\sum_{t=m+1}^{\infty} \Theta_t X_t^{+} \mathbf{1}_{[\Theta_t X_t^{+} \le x]}\bigg)^{\alpha+\varepsilon}\bigg],
$$

and then using Minkowski's inequality, this gets further bounded by

$$
\begin{aligned}
&\bigg\{\sum_{t=m+1}^{\infty} \bigg(\mathrm{E}\bigg[\frac{1}{x^{\alpha+\varepsilon}} (\Theta_t X_t^{+})^{\alpha+\varepsilon} \mathbf{1}_{[\Theta_t X_t^{+} \le x]}\bigg]\bigg)^{1/(\alpha+\varepsilon)}\bigg\}^{\alpha+\varepsilon}\\
&\quad = \bigg\{\sum_{t=m+1}^{\infty} \bigg(\int_0^{\infty} \bigg(\frac{x}{v}\bigg)^{-(\alpha+\varepsilon)} \mathrm{E}[(X_t^{+})^{\alpha+\varepsilon} \mathbf{1}_{[X_t^{+} \le x/v]}] G_t(\mathrm{d}v)\bigg)^{1/(\alpha+\varepsilon)}\bigg\}^{\alpha+\varepsilon}\\
&\quad = \bigg\{\sum_{t=m+1}^{\infty} \bigg(\int_0^{\infty} \frac{\mathrm{E}[(X_t^{+})^{\alpha+\varepsilon} \mathbf{1}_{[X_t^{+} \le x/v]}]}{(x/v)^{\alpha+\varepsilon}\, \mathrm{P}[X_t^{+} > x/v]} \mathrm{P}\bigg[X_t > \frac{x}{v}\bigg] G_t(\mathrm{d}v)\bigg)^{1/(\alpha+\varepsilon)}\bigg\}^{\alpha+\varepsilon}. \qquad (3.7)
\end{aligned}
$$

Then, again using Karamata's theorem, the first factor of the integrand converges to α/ε and, arguing as in the case $\alpha < 1$, is bounded. Thus, the final term of (3.5) is bounded by a multiple of $[\sum_{t=m+1}^{\infty} (\mathrm{P}[\Theta_t X_t > x])^{1/(\alpha+\varepsilon)}]^{\alpha+\varepsilon}$.

Combining the two cases for α, we obtain, for some $L_1 > 0$,

$$\frac{\mathrm{P}[\sum_{t=m+1}^{\infty} \Theta_t X_t^+ > x]}{\mathrm{P}[X_1 > x]} \le \begin{cases} L_1 \displaystyle\sum_{t=m+1}^{\infty} \frac{\mathrm{P}[\Theta_t X_t > x]}{\mathrm{P}[X_1 > x]} & \text{when } \alpha < 1, \\ \displaystyle\sum_{t=m+1}^{\infty} \frac{\mathrm{P}[\Theta_t X_t > x]}{\mathrm{P}[X_1 > x]} & \\ \quad + L_1 \left[\displaystyle\sum_{t=m+1}^{\infty} \left(\frac{\mathrm{P}[\Theta_t X_t > x]}{\mathrm{P}[X_1 > x]} \right)^{1/(\alpha+\varepsilon)} \right]^{\alpha+\varepsilon} & \text{when } \alpha \ge 1. \end{cases}$$

To prove (3.4), we must show that

$$\frac{\mathrm{P}[\Theta_t X_t > x]}{\mathrm{P}[X_1 > x]} \le B_t \tag{3.8}$$

for all large values of x, where

$$\sum_{t=1}^{\infty} B_t < \infty \quad \text{for } \alpha < 1, \qquad \sum_{t=1}^{\infty} B_t^{1/(\alpha+\varepsilon)} < \infty \quad \text{for } \alpha \ge 1. \tag{3.9}$$

As mentioned in Remark 3.2, for $\alpha \ge 1$ and $\varepsilon > 0$, $\sum_{t=1}^{\infty} B_t^{1/\alpha+\varepsilon} < \infty$ will also imply that $\sum_{t=1}^{\infty} B_t < \infty$. Thus, for both the cases $\alpha < 1$ and $\alpha \ge 1$, the sums involved will be bounded by the tail sum of a convergent series and, hence, (3.4) will hold.

First observe that

$$\frac{\mathrm{P}[\Theta_t X_t > x]}{\mathrm{P}[X_1 > x]} = \int_0^{\infty} \frac{\mathrm{P}[X_1 > x/v]}{\mathrm{P}[X_1 > x]} G_t(\mathrm{d}v). \tag{3.10}$$

We break the range of integration into three intervals, $(0, 1]$, $(1, x]$, and (x, ∞), where we choose a suitably large x greater than 1.

Since $\overline{F}$ is regularly varying of index $-\alpha$ with $\alpha > 0$, $\mathrm{P}[X_1 > x/v]/\mathrm{P}[X_1 > x]$ converges uniformly to v^{α} for $v \in (0, 1)$ or, equivalently, $1/v \in (1, \infty)$. Hence, the integral in (3.10) over the first interval can be bounded for all large enough x as

$$\int_0^1 \frac{\mathrm{P}[X_1 > x/v]}{\mathrm{P}[X_1 > x]} G_t(\mathrm{d}v) \le 2\,\mathrm{E}[\Theta_t^{\alpha}]. \tag{3.11}$$

For the integral in (3.10) over the third interval, we have, for all large enough x, by (3.1) (for conditions (DZ2), (DZ3), and (DZ4) only),

$$\begin{aligned} &\int_x^{\infty} \frac{\mathrm{P}[X_1 > x/v]}{\mathrm{P}[X_1 > x]} G_t(\mathrm{d}v) \\ &\quad \le \frac{\mathrm{P}[\Theta_t > x]}{\mathrm{P}[X_1 > x]} \\ &\quad \le \begin{cases} \dfrac{\mathrm{E}[\Theta_t^{\alpha}]}{L(x)} \le 2D_1 \dfrac{\mathrm{E}[\Theta_t^{\alpha}]}{L(1)} & \text{by Markov's inequality, when (DZ1) holds,} \\ C_t & \text{when (DZ2) holds,} \\ \dfrac{\mathrm{P}[\Theta_t > x]}{c(x)x^{-\alpha}\,\mathrm{P}[U > \log x]} \le \dfrac{2}{c} C_t & \text{when (DZ3) holds,} \\ C_t & \text{as } m(x) \to \infty, \text{ when (DZ4) holds.} \end{cases} \end{aligned} \tag{3.12}$$

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Note that, when condition (DZ3) holds and L is of type 4, we can ignore the factor $\mathrm{P}[V > \log x]$, as it is bounded by 1.

Finally, we consider the integral in (3.10) over the second interval separately for each of the DZ conditions. We begin with condition (DZ1). In this case we have, for all large enough x,

$$\begin{aligned}\int_1^x \frac{\mathrm{P}[X_1 > x/v]}{\mathrm{P}[X_1 > x]} G_t(\mathrm{d}v) &\le \int_1^x v^\alpha \frac{L(x/v)}{L(x)} G_t(\mathrm{d}v)\\ &\le \sup_{y\in[1,x]} \frac{L(y)}{L(x)} \mathrm{E}[\Theta_t^\alpha]\\ &\le 2D_1 \mathrm{E}[\Theta_t^\alpha]. \end{aligned}\tag{3.13}$$

Next we consider condition (DZ2). Integrating by parts, we have

$$\int_1^x \frac{\mathrm{P}[X_1 > x/v]}{\mathrm{P}[X_1 > x]} G_t(\mathrm{d}v) \le \mathrm{P}[\Theta_t > 1] + \int_1^x \frac{\mathrm{P}[\Theta_t > v]}{\mathrm{P}[X_1 > x]} \mathrm{d}_v \mathrm{P}\biggl[X_1 > \frac{x}{v}\biggr].$$

Note that in the integral on the right-hand side, x is kept constant and, as $\mathrm{P}[X_1 > x/v]$ is a nondecreasing function in v, we interpret the integral as the Riemann–Stieltjes integral with respect to the variable v. Similar notation will also be used in the sequel.

Using Markov's inequality and (3.1) respectively in each of the terms, we have

$$\int_1^x \frac{\mathrm{P}[X_1 > x/v]}{\mathrm{P}[X_1 > x]} G_t(\mathrm{d}v) \le \mathrm{E}[\Theta_t^\alpha] + C_t \int_1^x \frac{\mathrm{P}[X_1 > v]}{\mathrm{P}[X_1 > x]} \mathrm{d}_v \mathrm{P}\biggl[X_1 > \frac{x}{v}\biggr].$$

Substituting $u = \log v$, the second term becomes, for all large x,

$$\begin{aligned} C_t \int_0^{\log x} \frac{\mathrm{P}[\log X_1 > u]}{\mathrm{P}[\log X_1 > \log x]} \mathrm{d}_u \mathrm{P}[\log X_1 > \log x - u] &\le 2C_t \mathrm{E}[\exp(\alpha(\log X_1)^+)]\\ &\le 2C_t \mathrm{E}[X_1^\alpha], \end{aligned}$$

where the inequalities follow, using the fact that $L(\mathrm{e}^x) \in \mathcal{S}_d$ implies that $(\log X_1)^+ \in \mathcal{S}(\alpha)$; cf. [12]. Thus,

$$\int_1^x \frac{\mathrm{P}[X_1 > x/v]}{\mathrm{P}[X_1 > x]} G_t(\mathrm{d}v) \le \mathrm{E}[\Theta_t^\alpha] + 2C_t \mathrm{E}[X_1^\alpha]. \tag{3.14}$$

Next we consider condition (DZ3). In this case, we have

$$\begin{aligned}\int_1^x \frac{\mathrm{P}[X_1 > x/v]}{\mathrm{P}[X_1 > x]} G_t(\mathrm{d}v) &= \int_1^x \frac{L(x/v)}{L(x)} v^\alpha G_t(\mathrm{d}v)\\ &\le \sup_{v\in[1,x]} \frac{c(x/v)}{c(x)} \int_1^x \frac{\mathrm{P}[U > \log x - \log v]}{\mathrm{P}[U > \log x]} v^\alpha G_t(\mathrm{d}v). \end{aligned}$$

If L is of type 4, the ratio $L(x/v)/L(x)$ has an extra factor $\mathrm{P}[V > \log x]/\mathrm{P}[V > \log x - \log v]$, which is bounded by 1. Thus, the above estimate works if L is either of type 3 or of type 4. Since $c(x) \to c \in (0, \infty)$, we have $\sup_{v\in[N,x)} c(x/v)/c(x) =: L_2 < \infty$. Integrating by parts, the integral becomes

$$\begin{aligned} &\int_1^x \frac{\mathrm{P}[U > \log x - \log v]}{\mathrm{P}[U > \log x]} v^\alpha G_t(\mathrm{d}v)\\ &\quad\le \mathrm{P}[\Theta_t > 1] + \int_1^x \frac{\mathrm{P}[U > \log x - \log v]\,\mathrm{P}[\Theta_t > v]}{\mathrm{P}[U > \log x]} \alpha v^{\alpha-1}\, \mathrm{d}v\\ &\qquad + \int_1^x \frac{\mathrm{P}[\Theta_t > v] v^\alpha}{\mathrm{P}[U > \log x]} \mathrm{d}_v \mathrm{P}[U > \log x - \log v]. \end{aligned}$$

The first term is bounded by $\mathrm{E}[\Theta_1^\alpha]$ by Markov's inequality. By (3.1), the second term gets bounded by, for all large enough x,

$$\alpha C_t \int_1^x \frac{\mathrm{P}[U > \log x - \log v]\,\mathrm{P}[U > \log v]}{\mathrm{P}[U > \log x]}\,\mathrm{d}(\log v) \le 2\alpha C_t\,\mathrm{E}[U],$$

as U belongs to $\mathcal{S}^*$. Again, by (3.1), the third term gets bounded by, for all large enough x,

$$C_t \int_1^x \frac{\mathrm{P}[U > \log v]\,\mathrm{d}_v \mathrm{P}[U > \log x - \log v]}{\mathrm{P}[U > \log x]} \le 4C_t,$$

because U belongs to $\mathcal{S}^*$ and, hence, is subexponential; cf. [11]. Combining the bounds for the three terms, we obtain

$$\int_1^x \frac{\mathrm{P}[X_1 > x/v]}{\mathrm{P}[X_1 > x]} G_t(\mathrm{d}v) \le L_2\{\mathrm{E}[\Theta_t^\alpha] + 2(\alpha\,\mathrm{E}[U] + 2)C_t\}. \tag{3.15}$$

Finally, we consider condition (DZ4). In this case we split the interval $(1, x]$ into two subintervals, $(1, \sqrt{x}]$ and $(\sqrt{x}, x]$, and bound the integrals on each of the subintervals separately. We begin with the integral on the subinterval $(1, \sqrt{x}]$:

$$\int_1^{\sqrt{x}} \frac{L(x/v)}{L(x)} v^\alpha G_t(\mathrm{d}v) \le \sup_{v \in (1,\sqrt{x}]} \frac{L(x/v)}{L(x)} \int_1^{\sqrt{x}} v^\alpha G_t(\mathrm{d}v) \le D_2\,\mathrm{E}[\Theta_t^\alpha].$$

For the integral over $(\sqrt{x}, x]$, we integrate by parts to obtain

$$\int_{\sqrt{x}}^x \frac{L(x/v)}{L(x)} v^\alpha G_t(\mathrm{d}v) \le \mathrm{P}[\Theta_t > \sqrt{x}]x^{\alpha/2}\frac{L(\sqrt{x})}{L(x)} + \int_{\sqrt{x}}^x \frac{\mathrm{P}[\Theta_t > v]}{L(x)} \mathrm{d}_v\left(v^\alpha L\left(\frac{x}{v}\right)\right).$$

By Markov's inequality, the first term is bounded by $D_2\,\mathrm{E}[\Theta_t^\alpha]$. The second term becomes, using (3.1),

$$\begin{aligned}
\int_{\sqrt{x}}^x \frac{\mathrm{P}[\Theta_t > v]}{L(x)} x^\alpha\,\mathrm{d}_v\left(\mathrm{P}\left[X_1 \le \frac{x}{v}\right]\right) &\le C_t \int_{\sqrt{x}}^x \frac{\mathrm{P}[X_1 > v]}{L(x)m(v)} x^\alpha\,\mathrm{d}_v\left(\mathrm{P}\left[X_1 \le \frac{x}{v}\right]\right)\\
&\le \frac{C_t}{m(\sqrt{x})} \int_{\sqrt{x}}^x \frac{L(v)}{L(x)}\left(\frac{x}{v}\right)^\alpha \mathrm{d}_v\left(\mathrm{P}\left[X_1 \le \frac{x}{v}\right]\right)\\
&\le \frac{D_2 C_t}{m(\sqrt{x})} \int_1^{\sqrt{x}} y^\alpha\,\mathrm{d}_y(\mathrm{P}[X_1 \le y])\\
&\le D_2 C_t.
\end{aligned}$$

Combining the bounds for the integrals over each subinterval, we obtain

$$\int_1^x \frac{\mathrm{P}[X_1 > x/v]}{\mathrm{P}[X_1 > x]} G_t(\mathrm{d}v) \le D_2(2\,\mathrm{E}[\Theta_t^\alpha] + C_t). \tag{3.16}$$

Combining all the bounds in (3.11)–(3.16), for some constant B, we can choose the bound in (3.8) as

$$B_t = \begin{cases} B\,\mathrm{E}[\Theta_t^\alpha] & \text{when condition (DZ1) holds,}\\ B(\mathrm{E}[\Theta_t^\alpha] + C_t) & \text{when condition (DZ2), (DZ3), or (DZ4) holds.}\end{cases}$$

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Then, for $\alpha < 1$, the summability condition (3.9) follows from condition (RW1′) alone under condition (DZ1) and from condition (RW1′) together with (3.2) under condition (DZ2), (DZ3), or (DZ4). For $\alpha \geq 1$, under condition (DZ1), the summability condition (3.9) follows from condition (RW2′). Finally, to check the summability condition (3.9) for $\alpha \geq 1$, under condition (DZ2), (DZ3), or (DZ4), observe that, as $\alpha \geq 1$ and $\varepsilon > 0$, we have

$$(\mathrm{E}[\Theta_t^\alpha] + C_t)^{1/(\alpha+\varepsilon)} \leq (\mathrm{E}[\Theta_t^\alpha])^{1/(\alpha+\varepsilon)} + C_t^{1/(\alpha+\varepsilon)},$$

and we obtain the desired condition from condition (RW2′), together with (3.3).

4. The tails of the summands from the tail of the sum

In this section we address the converse problem of studying the tail behavior of X_1 based on the tail behavior of $X_{(\infty)}$. For the converse problem, we restrict ourselves to the setup where the sequence $\{X_t\}$ is positive and pairwise asymptotically independent, and the other sequence $\{\Theta_t\}$ is positive and independent of the sequence $\{X_t\}$, such that $X_{(\infty)}$ is finite with probability 1 and has a regularly varying tail of index $-\alpha$. Depending on the value of α, we assume the usual RW moment conditions, (RW1) or (RW2), for the sequence $\{\Theta_t\}$, instead of the modified RW moment conditions. Then, under the further assumption of a nonvanishing Mellin transform along the vertical line of the complex plane with real part α, we will show that X_1 also has a regularly varying tail of index $-\alpha$.

We extend the notion of the product of two independent positive random variables to the product convolution of two measures on $(0, \infty)$, which we allow to be σ-finite. For two σ-finite measures ν and ρ on $(0, \infty)$, we define the product convolution as

$$\nu \circledast \rho(B) = \int_0^\infty \nu(x^{-1}B)\rho(\mathrm{d}x)$$

for any Borel subset B of $(0, \infty)$. In particular, for two independent nonnegative random variables Θ and X distributed as F and G, respectively, we will have $\mathrm{P}[\Theta X \in B] = F \circledast G(B)$. However, we will be more interested in using the notation when the measures involved are infinite but σ-finite measures, rather than probability measures. We will need the following result from [9].

Theorem 4.1. (Theorem 2.3 of [9].) *Suppose that the nonzero σ-finite measure ρ on $(0, \infty)$ satisfies, for some $\alpha > 0$, $\varepsilon \in (0, \alpha)$, and all $\beta \in \mathbb{R}$,*

$$\int_0^\infty (y^{\alpha-\varepsilon} \vee y^{\alpha+\varepsilon})\rho(\mathrm{d}y) < \infty \tag{4.1}$$

and

$$\int_0^\infty y^{\alpha+\mathrm{i}\beta}\rho(\mathrm{d}y) \neq 0. \tag{4.2}$$

Suppose that, for the nonzero σ-finite measure ν on $(0, \infty)$, the product convolution measure $\nu \circledast \rho$ has a regularly varying tail of index $-\alpha$ and

$$\lim_{b\to 0} \limsup_{x\to\infty} \frac{\int_0^b \rho(x/y, \infty)\nu(\mathrm{d}y)}{(\nu \circledast \rho)(x, \infty)} = 0. \tag{4.3}$$

Then the measure ν has a regularly varying tail of index $-\alpha$ as well and

$$\lim_{x\to\infty} \frac{\nu \circledast \rho(x, \infty)}{\nu(x, \infty)} = \int_0^\infty y^\alpha \rho(\mathrm{d}y).$$

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Conversely, if (4.1) holds but (4.2) fails for the measure ρ, then there exists a σ-finite measure ν without a regularly varying tail, such that $\nu \circledast \rho$ has a regularly varying tail of index $-\alpha$ and (4.3) holds.

Remark 4.1. Jacobsen *et al.* [9] gave an explicit construction of the σ-finite measure ν in Theorem 4.1 above. In fact, if (4.2) fails for $\beta = \beta_0$ then, for any real numbers a and b satisfying $0 < a^2 + b^2 \leq 1$, we can define $g(x) = 1 + a\cos(\beta_0 \log x) + b\sin(\beta_0 \log x)$ and $\mathrm{d}\nu = g\,\mathrm{d}\nu_\alpha$ will qualify for the measure in the converse part, where ν_α is the σ-finite measure given by $\nu_\alpha(x,\infty) = x^{-\alpha}$ for any $x > 0$.

It is easy to check that $0 \leq g(x) \leq 2$ for all $x > 0$ and, hence,

$$\nu(x,\infty) \leq 2x^{-\alpha}. \tag{4.4}$$

Also, it is known from Theorem 2.1 of [9] that

$$\nu \circledast \rho = \|\rho\|_\alpha \nu_\alpha, \tag{4.5}$$

where $\|\rho\|_\alpha = \int_0^\infty y^\alpha \rho(\mathrm{d}y) < \infty$, by (4.1).

We are now ready to state the main result of this section.

Theorem 4.2. *Let $\{X_t,\ t \geq 1\}$ be a sequence of identically distributed, pairwise asymptotically independent positive random variables, and let $\{\Theta_t,\ t \geq 1\}$ be a sequence of positive random variables independent of $\{X_t\}$, such that $X_{(\infty)} = \sum_{t=1}^\infty \Theta_t X_t$ is finite with probability 1 and has a regularly varying tail of index $-\alpha$, $\alpha > 0$. Let $\{\Theta_t,\ t \geq 1\}$ satisfy the appropriate RW condition, (RW1) or (RW2), depending on the value of α. If we further have, for all $\beta \in \mathbb{R}$,*

$$\sum_{t=1}^\infty \mathrm{E}[\Theta_t^{\alpha+\mathrm{i}\beta}] \neq 0 \tag{4.6}$$

then X_1 has a regularly varying tail of index $-\alpha$ and, as $x \to \infty$,

$$\mathrm{P}[X_{(\infty)} > x] \sim \mathrm{P}[X_1 > x] \sum_{t=1}^\infty \mathrm{E}[\Theta_t^\alpha] \quad \textit{as } x \to \infty.$$

We will prove Theorem 4.2 in several steps. We collect the preliminary steps, which will also be useful for a converse to Theorem 4.2, into three separate lemmas. The first lemma controls the tail of the sum $X_{(\infty)}$.

Lemma 4.1. *Let $\{X_t\}$ be a sequence of identically distributed positive random variables, and let $\{\Theta_t\}$ be a sequence of positive random variables independent of $\{X_t\}$. Suppose that the tail of X_1 is dominated by a bounded regularly varying function R of index $-\alpha$, $\alpha > 0$, that is, for all $x > 0$,*

$$\mathrm{P}[X_1 > x] \leq R(x). \tag{4.7}$$

Also, assume that $\{\Theta_t\}$ satisfies the appropriate RW condition depending on the value of α. Then

$$\lim_{m\to\infty} \limsup_{x\to\infty} \frac{\mathrm{P}[\sum_{t=m+1}^\infty \Theta_t X_t > x]}{R(x)} = 0$$

and

$$\lim_{m\to\infty} \limsup_{x\to\infty} \sum_{t=m+1}^\infty \frac{\mathrm{P}[\Theta_t X_t > x]}{R(x)} = 0.$$

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Proof. From (3.5) we have

$$\mathrm{P}\Bigg[\sum_{t=m+1}^{\infty} \Theta_t X_t > x\Bigg] \le \sum_{t=m+1}^{\infty} \mathrm{P}[\Theta_t X_t > x] + \mathrm{P}\Bigg[\sum_{t=m+1}^{\infty} \Theta_t X_t \mathbf{1}_{[\Theta_t X_t \le x]} > x\Bigg]. \tag{4.8}$$

Using (4.7), the summands in the first term on the right-hand side of (4.8) can be bounded as

$$\mathrm{P}[\Theta_t X_t > x] = \int_0^\infty \mathrm{P}\bigg[X_t > \frac{x}{u}\bigg] G_t(\mathrm{d}u) \le \int_0^\infty R\bigg(\frac{x}{u}\bigg) G_t(\mathrm{d}u). \tag{4.9}$$

Before analyzing the second term on the right-hand side of (4.8), observe that, for $\gamma > \alpha$, we have, using Fubini's theorem, (4.7), and Karamata's theorem successively,

$$\begin{aligned} \mathrm{E}[X_t^\gamma \mathbf{1}_{[X_t \le x]}] &\le \gamma \int_0^x u^{\gamma-1} \mathrm{P}[X_t > u]\,\mathrm{d}u \\ &\le \gamma \int_0^x u^{\gamma-1} R(u)\,\mathrm{d}u \\ &\sim \frac{\gamma}{\gamma-\alpha} x^\gamma R(x) \quad \text{as } x \to \infty. \end{aligned}$$

Thus, there exists a constant $M \equiv M(\gamma)$ such that, for all $x > 0$,

$$x^{-\gamma}\,\mathrm{E}[X_t^\gamma \mathbf{1}_{[X_t \le x]}] \le M R(x). \tag{4.10}$$

We bound the second term on the right-hand side of (4.8), using (4.10), separately for the cases $\alpha < 1$ and $\alpha \ge 1$. For $\alpha < 1$, we use (3.6) and (4.10) with $\gamma = 1$ to obtain

$$\mathrm{P}\Bigg[\sum_{t=m+1}^{\infty} \Theta_t X_t \mathbf{1}_{[\Theta_t X_t \le x]} > x\Bigg] \le M(1) \sum_{t=m+1}^{\infty} \int_0^\infty R\bigg(\frac{x}{u}\bigg) G_t(\mathrm{d}u). \tag{4.11}$$

For $\alpha \ge 1$, we use (3.7) and (4.10) with $\gamma = \alpha + \varepsilon$ to obtain

$$\mathrm{P}\Bigg[\sum_{t=m+1}^{\infty} \Theta_t X_t \mathbf{1}_{[\Theta_t X_t \le x]} > x\Bigg] \le M(\alpha+\varepsilon)\Bigg[\sum_{t=m+1}^{\infty} \bigg(\int_0^\infty R\bigg(\frac{x}{u}\bigg) G_t(\mathrm{d}u)\bigg)^{1/(\alpha+\varepsilon)}\Bigg]^{\alpha+\varepsilon}. \tag{4.12}$$

Combining (4.9), (4.11), and (4.12) with the bound in (4.8), the proof will be complete if we show that

$$\begin{aligned} &\lim_{m\to\infty} \limsup_{x\to\infty} \sum_{t=m+1}^{\infty} \int_0^\infty \frac{R(x/u)}{R(x)} G_t(\mathrm{d}u) = 0 \quad \text{for } \alpha < 1 \\ \text{and} \quad &\lim_{m\to\infty} \limsup_{x\to\infty} \sum_{t=m+1}^{\infty} \bigg(\int_0^\infty \frac{R(x/u)}{R(x)} G_t(\mathrm{d}u)\bigg)^{1/(\alpha+\varepsilon)} = 0 \quad \text{for } \alpha \ge 1. \end{aligned} \tag{4.13}$$

Note that, for $\alpha \ge 1$, as in Remark 1.1, the second limit above gives the first limit as well.

We bound the integrand using a variant of Potter's bound (for details, see [17, Lemma 2.2]). Let $\varepsilon > 0$ be as in the RW conditions. Then there exists an x_0 and a constant $M > 0$ such that, for $x > x_0$, we have

$$\frac{R(x/u)}{R(x)} \le \begin{cases} M u^{\alpha-\varepsilon} & \text{if } u < 1, \\ M u^{\alpha+\varepsilon} & \text{if } 1 \le u \le x/x_0. \end{cases} \tag{4.14}$$

We split the range of integration in (4.13) into three intervals, namely $(0, 1]$, $(1, x/x_0]$, and $(x/x_0, \infty)$. For $x > x_0$, we bound the integrand over the first two intervals using (4.14) and, hence, the integrals get bounded by multiples of $\mathrm{E}[\Theta_t^{\alpha-\varepsilon}]$ and $\mathrm{E}[\Theta_t^{\alpha+\varepsilon}]$, respectively. As R is bounded, by Markov's inequality, the third integral gets bounded by a multiple of $x_0^{\alpha+\varepsilon}\,\mathrm{E}[\Theta_t^{\alpha+\varepsilon}]/\{x^{\alpha+\varepsilon}R(x)\}$. Putting all the bounds together, we have

$$\int_0^\infty \frac{R(x/u)}{R(x)} G_t(\mathrm{d}u) \le M\biggl(\mathrm{E}[\Theta_t^{\alpha-\varepsilon}] + \mathrm{E}[\Theta_t^{\alpha+\varepsilon}] + \frac{x_0^{\alpha+\varepsilon}\,\mathrm{E}[\Theta_t^{\alpha+\varepsilon}]}{x^{\alpha+\varepsilon}R(x)}\biggr).$$

Then, (4.13) holds for $\alpha < 1$ using condition (RW1) and the fact that R is regularly varying of index $-\alpha$. For $\alpha \ge 1$, we need to further observe that, as $\alpha + \varepsilon > 1$, we have

$$\begin{aligned}
\biggl(\int_0^\infty \frac{R(x/u)}{R(x)} G_t(\mathrm{d}u)\biggr)^{1/(\alpha+\varepsilon)}
&\le M^{1/(\alpha+\varepsilon)}\biggl[(\mathrm{E}[\Theta_t^{\alpha-\varepsilon}] + \mathrm{E}[\Theta_t^{\alpha+\varepsilon}]) + \frac{x_0^{\alpha+\varepsilon}\,\mathrm{E}[\Theta_t^{\alpha+\varepsilon}]}{x^{\alpha+\varepsilon}R(x)}\biggr]^{1/(\alpha+\varepsilon)}\\
&\le M^{1/(\alpha+\varepsilon)}(\mathrm{E}[\Theta_t^{\alpha-\varepsilon}] + \mathrm{E}[\Theta_t^{\alpha+\varepsilon}])^{1/(\alpha+\varepsilon)} + \frac{x_0(\mathrm{E}[\Theta_t^{\alpha+\varepsilon}])^{1/(\alpha+\varepsilon)}}{xR(x)^{1/(\alpha+\varepsilon)}}
\end{aligned}$$

and (4.13) holds using condition (RW2) and the fact that R is regularly varying of index $-\alpha$.

In the next lemma we consider the joint distribution of $(\Theta_1 X_1, \Theta_2 X_2)$ and show that they are 'somewhat' asymptotically independent, if (X_1, X_2) are asymptotically independent.

Lemma 4.2. *Let (X_1, X_2) and (Θ_1, Θ_2) be two independent random vectors such that each coordinate of either vector is positive. We assume that X_1 and X_2 have the same distribution with their common tail dominated by a regularly varying function R of index $-\alpha$ with $\alpha > 0$, as in (4.7). We also assume that R stays bounded away from 0 on any bounded interval. We further assume that both Θ_1 and Θ_2 have $\alpha + \varepsilon$ finite moments. Then*

$$\lim_{x\to\infty} \frac{\mathrm{P}[\Theta_1 X_1 > x,\ \Theta_2 X_2 > x]}{R(x)} = 0.$$

Proof. By asymptotic independence and (4.7), we have

$$\mathrm{P}[X_1 > x,\ X_2 > x] = o(R(x)). \tag{4.15}$$

Furthermore, since R is bounded away from 0 on any bounded interval, $\mathrm{P}[X_1 > x,\ X_2 > x]$ is bounded by a multiple of $R(x)$. Then

$$\begin{aligned}
\frac{\mathrm{P}[\Theta_1 X_1 > x,\ \Theta_2 X_2 > x]}{R(x)} &= \int_0^\infty \int_0^\infty \frac{\mathrm{P}[X_1 > x/u, X_2 > x/v]}{R(x)} G(\mathrm{d}u,\ \mathrm{d}v)\\
&= \biggl(\iint_{u>v} + \iint_{u\le v}\biggr)\frac{\mathrm{P}[X_1 > x/u,\ X_2 > x/v]}{R(x)} G(\mathrm{d}u,\ \mathrm{d}v)\\
&\le \int_0^\infty \frac{\mathrm{P}[X_1 > x/u,\ X_2 > x/u]}{R(x)} (G_1 + G_2)(\mathrm{d}u)\\
&\le \int_0^\infty \frac{\mathrm{P}[X_1 > x/u, X_2 > x/u]}{R(x)}\,\mathbf{1}_{[0,x/x_0]}(u)(G_1 + G_2)(\mathrm{d}u)\\
&\quad + \frac{x_0^{\alpha+\varepsilon}(\mathrm{E}[\Theta_1^{\alpha+\varepsilon}] + \mathrm{E}[\Theta_1^{\alpha+\varepsilon}])}{x^{\alpha+\varepsilon}R(x)} \quad \text{for any } x_0 > 0.
\end{aligned}$$

The integrand in the first term goes to 0, using (4.15) and the regular variation of R. Furthermore, choose x_0 as in Potter's bound (4.14). Then, the integrand in the first term is bounded by a multiple of $1 + u^{\alpha+\varepsilon}$, which is integrable with respect to $G_1 + G_2$. So, by the dominated convergence theorem, the first term goes to 0. For this choice of x_0, the second term also goes to 0, as R is regularly varying of index $-\alpha$.

In the next lemma we compare $\sum_{t=1}^{m} \mathrm{P}[\Theta_t X_t > x]$ and $\mathrm{P}[\sum_{t=1}^{m} \Theta_t X_t > x]$.

Lemma 4.3. *Let $\{X_t\}$ and $\{\Theta_t\}$ be two sequences of positive random variables. Then we have, for any $\frac{1}{2} < \delta < 1$ and $m \geq 2$,*

$$\mathrm{P}\biggl[\sum_{t=1}^{m} \Theta_t X_t > x\biggr] \geq \sum_{t=1}^{m} \mathrm{P}[\Theta_t X_t > x] - \sum\sum_{1\leq s\neq t\leq m} \mathrm{P}[\Theta_s X_s > x,\ \Theta_t X_t > x] \tag{4.16}$$

and

$$\mathrm{P}\biggl[\sum_{t=1}^{m} \Theta_t X_t > x\biggr] \leq \sum_{t=1}^{m} \mathrm{P}[\Theta_t X_t > x] + \sum\sum_{1\leq s\neq t\leq m} \mathrm{P}\biggl[\Theta_s X_s > \frac{1-\delta}{m-1}x,\ \Theta_t X_t > \frac{1-\delta}{m-1}x\biggr]. \tag{4.17}$$

Proof. Inequality (4.16) follows from the fact that

$$\biggl[\sum_{t=1}^{m} \Theta_t X_t > x\biggr] \subseteq \bigcup_{t=1}^{m} [\Theta_t X_t > x]$$

and Bonferroni's inequality.

For inequality (4.17), observe that

$$\mathrm{P}\biggl[\sum_{t=1}^{m} \Theta_t X_t > x\biggr] \leq \sum_{t=1}^{m} \mathrm{P}[\Theta_t X_t > \delta x] + \mathrm{P}\biggl[\sum_{t=1}^{m} \Theta_t X_t > x,\ \bigvee_{t=1}^{m} \Theta_t X_t \leq \delta x\biggr].$$

Next we estimate the second term as

$$\begin{aligned}
\mathrm{P}\biggl[\sum_{t=1}^{m} \Theta_t X_t > x,\ \bigvee_{t=1}^{m} \Theta_t X_t \leq \delta x\biggr] &= \mathrm{P}\biggl[\sum_{t=1}^{m} \Theta_t X_t > x,\ \bigvee_{t=1}^{m} \Theta_t X_t \leq \delta x,\ \bigvee_{t=1}^{m} \Theta_t X_t > \frac{x}{m}\biggr] \\
&\leq \sum_{s=1}^{m} \mathrm{P}\biggl[\sum_{t=1}^{m} \Theta_t X_t > x,\ \bigvee_{t=1}^{m} \Theta_t X_t \leq \delta x,\ \Theta_s X_s > \frac{x}{m}\biggr] \\
&\leq \sum_{s=1}^{m} \mathrm{P}\biggl[\sum_{t=1}^{m} \Theta_t X_t > x,\ \Theta_s X_s \leq \delta x,\ \Theta_s X_s > \frac{x}{m}\biggr] \\
&\leq \sum_{s=1}^{m} \mathrm{P}\biggl[\sum_{\substack{t=1\\ t\neq s}}^{m} \Theta_t X_t > (1-\delta)x,\ \Theta_s X_s > \frac{x}{m}\biggr] \\
&\leq \sum\sum_{1\leq s\neq t\leq m} \mathrm{P}\biggl[\Theta_t X_t > \frac{1-\delta}{m-1}x,\ \Theta_s X_s > \frac{x}{m}\biggr] \\
&\leq \sum\sum_{1\leq s\neq t\leq m} \mathrm{P}\biggl[\Theta_t X_t > \frac{1-\delta}{m-1}x,\ \Theta_s X_s > \frac{1-\delta}{m-1}x\biggr],
\end{aligned}$$

since $\delta > \frac{1}{2}$ and $m \geq 2$ imply that $(1-\delta)/(m-1) < 1/m$.

With the above three lemmas, we are now ready to show the tail equivalence of the distribution of $X_{(\infty)}$ and $\sum_{t=1}^{\infty} \mathrm{P}[\Theta_t X_t \in \cdot]$.

Proposition 4.1. *Let $\{X_t,\ t \geq 1\}$ be a sequence of identically distributed, pairwise asymptotically independent positive random variables, and let $\{\Theta_t,\ t \geq 1\}$ be a sequence of positive random variables independent of $\{X_t\}$ such that $X_{(\infty)} = \sum_{t=1}^{\infty} \Theta_t X_t$ is finite with probability 1 and has a regularly varying tail of index $-\alpha$, $\alpha > 0$. Let $\{\Theta_t,\ t \geq 1\}$ satisfy the appropriate RW condition, (RW1) or (RW2), depending on the value of α. Then, as $x \to \infty$,*

$$\sum_{t=1}^{\infty} \mathrm{P}[\Theta_t X_t > x] \sim \mathrm{P}[X_{(\infty)} > x].$$

Proof. We first show that the tail of X_1 can be dominated by a multiple of the tail of $X_{(\infty)}$, so that Lemmas 4.1 and 4.2 apply. Note that the tail of $X_{(\infty)}$ is bounded and stays bounded away from 0 on any bounded interval. As Θ_1 is a positive random variable, choose $\eta > 0$ such that $\mathrm{P}[\Theta_1 > \eta] > 0$. Then, for all $x > 0$,

$$\mathrm{P}[X_{(\infty)} > \eta x] \geq \mathrm{P}[\Theta_1 X_1 > \eta x,\ \Theta_1 > \eta] \geq \mathrm{P}[X_1 > x]\,\mathrm{P}[\Theta_1 > \eta].$$

Furthermore, using the regular variation of the tail of $X_{(\infty)}$, X_1 satisfies (4.7) with R a multiple of $\mathrm{P}[X_{(\infty)} > \cdot]$. Thus, from Lemmas 4.1 and 4.2, we have

$$\lim_{m\to\infty} \limsup_{x\to\infty} \frac{\mathrm{P}[\sum_{t=m+1}^{\infty} \Theta_t X_t > x]}{\mathrm{P}[X_{(\infty)} > x]} = 0, \tag{4.18}$$

$$\lim_{m\to\infty} \limsup_{x\to\infty} \sum_{t=m+1}^{\infty} \frac{\mathrm{P}[\Theta_t X_t > x]}{\mathrm{P}[X_{(\infty)} > x]} = 0, \tag{4.19}$$

and, for any $s \neq t$,

$$\lim_{x\to\infty} \frac{\mathrm{P}[\Theta_s X_s > x,\ \Theta_t X_t > x]}{\mathrm{P}[X_{(\infty)} > x]} = 0. \tag{4.20}$$

Choose any $\delta > 0$. Then

$$\mathrm{P}[X_{(\infty)} > (1+\delta)x] \leq \mathrm{P}\bigg[\sum_{t=1}^{m} \Theta_t X_t > x\bigg] + \mathrm{P}\bigg[\sum_{t=m+1}^{\infty} \Theta_t X_t > \delta x\bigg],$$

and from (4.18) and the regular variation of the tail of $X_{(\infty)}$, we have

$$\lim_{m\to\infty} \liminf_{x\to\infty} \frac{\mathrm{P}[\sum_{t=1}^{m} \Theta_t X_t > x]}{\mathrm{P}[X_{(\infty)} > x]} \geq 1.$$

Furthermore, using the trivial bound $\mathrm{P}[\sum_{t=1}^{m} \Theta_t X_t > x] \leq \mathrm{P}[X_{(\infty)} > x]$, we have

$$1 \leq \lim_{m\to\infty} \liminf_{x\to\infty} \frac{\mathrm{P}[\sum_{t=1}^{m} \Theta_t X_t > x]}{\mathrm{P}[X_{(\infty)} > x]} \leq \lim_{m\to\infty} \limsup_{x\to\infty} \frac{\mathrm{P}[\sum_{t=1}^{m} \Theta_t X_t > x]}{\mathrm{P}[X_{(\infty)} > x]} \leq 1. \tag{4.21}$$

We next replace $\mathrm{P}[\sum_{t=1}^{m} \Theta_t X_t > x]$ in the numerator by $\sum_{t=1}^{m} \mathrm{P}[\Theta_t X_t > x]$. We obtain the upper bound first. From (4.16), (4.20), and (4.21), we obtain

$$\limsup_{x\to\infty} \frac{\sum_{t=1}^{m} \mathrm{P}[\Theta_t X_t > x]}{\mathrm{P}[X_{(\infty)} > x]} \leq 1,$$

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and letting $m \to \infty$, we obtain the upper bound. The lower bound follows by a similar argument, but using (4.17) and the regular variation of the tail of $X_{(\infty)}$ instead of (4.16). Putting the bounds together, we obtain

$$1 \le \lim_{m\to\infty} \liminf_{x\to\infty} \frac{\sum_{t=1}^{m} \mathrm{P}[\Theta_t X_t > x]}{\mathrm{P}[X_{(\infty)} > x]} \le \lim_{m\to\infty} \limsup_{x\to\infty} \frac{\sum_{t=1}^{m} \mathrm{P}[\Theta_t X_t > x]}{\mathrm{P}[X_{(\infty)} > x]} \le 1. \tag{4.22}$$

Then the result follows combining (4.19) and (4.22).

We are now ready to prove Theorem 4.2.

Proof of Theorem 4.2. Let ν be the law of X_1, and define the measure

$$\rho(\cdot) = \sum_{t=1}^{\infty} \mathrm{P}[\Theta_t \in \cdot].$$

As noted in Remark 1.1, under the RW conditions, for all values of α, we have $\sum_{t=1}^{\infty} \mathrm{E}[\Theta_t^{\alpha+\varepsilon}] < \infty$. Thus, ρ is a σ-finite measure. Also, by Proposition 4.1 we have $\nu \circledast \rho(x,\infty) = \sum_{t=1}^{\infty} \mathrm{P}[\Theta_t X_t > x] \sim \mathrm{P}[X_{(\infty)} > x]$. Hence, $\nu \circledast \rho$ has a regularly varying tail of index $-\alpha$. As ν is a probability measure, by Remark 2.4 of [9], (4.3) holds. The RW condition implies (4.1). Finally, (4.2) holds, since, for all $\beta \in \mathbb{R}$, we have, from (4.6), $\int_0^\infty y^{\alpha+\mathrm{i}\beta}\rho(\mathrm{d}y) = \sum_{t=1}^{\infty} \mathrm{E}[\Theta_t^{\alpha+\mathrm{i}\beta}] \ne 0$. Hence, by Theorem 4.1, X_1 has a regularly varying tail of index $-\alpha$.

As in Theorem 4.1, (4.6) is necessary for Theorem 4.2 and we give its converse below.

Theorem 4.3. *Let $\{\Theta_t,\ t \ge 1\}$ be a sequence of positive random variables satisfying condition (RW1) or (RW2) for some $\alpha > 0$, but $\sum_{t=1}^{\infty} \mathrm{E}[\Theta_t^{\alpha+\mathrm{i}\beta_0}] = 0$ for some $\beta_0 \in \mathbb{R}$. Then there exists a sequence of i.i.d. positive random variables $\{X_t\}$ such that X_1 does not have a regularly varying tail, but $X_{(\infty)}$ is finite almost surely and has a regularly varying tail of index $-\alpha$.*

The proof depends on an analogue of Proposition 4.1.

Proposition 4.2. *Let $\{X_t,\ t \ge 1\}$ be a sequence of identically distributed, pairwise asymptotically independent positive random variables, and let $\{\Theta_t,\ t \ge 1\}$ be a sequence of positive random variables satisfying condition (RW1) or (RW2) for some $\alpha > 0$ and independent of $\{X_t\}$. If $\sum_{t=1}^{\infty} \mathrm{P}[\Theta_t X_t > x]$ is regularly varying of index $-\alpha$ then, as $x \to \infty$,*

$$\sum_{t=1}^{\infty} \mathrm{P}[\Theta_t X_t > x] \sim \mathrm{P}[X_{(\infty)} > x]$$

and $X_{(\infty)}$ is finite with probability 1.

Proof. We will define $R(x) = \sum_{t=1}^{\infty} \mathrm{P}[\Theta_t X_t > x]$. As Θ_1 is a positive random variable, choose $\eta > 0$ such that $\mathrm{P}[\Theta_1 > \eta] > 0$. Then, for all $x > 0$, we have $R(x) \ge \mathrm{P}[\Theta_1 X_1 > \eta x, \Theta_1 > \eta] \ge \mathrm{P}[X_1 > x]\,\mathrm{P}[\Theta_1 > \eta]$, and using the regular variation of R, the tail of X_1 is dominated by a constant multiple of R. Also, note that R is bounded and stays bounded away from 0 on any bounded interval. Then, from Lemmas 4.1 and 4.2, we have

$$\lim_{m\to\infty} \limsup_{x\to\infty} \frac{\mathrm{P}[\sum_{t=m+1}^{\infty} \Theta_t X_t > x]}{R(x)} = 0, \tag{4.23}$$

$$\lim_{m\to\infty} \limsup_{x\to\infty} \sum_{t=m+1}^{\infty} \frac{\mathrm{P}[\Theta_t X_t > x]}{R(x)} = 0, \tag{4.24}$$

and, for any $s \neq t$,

$$\lim_{x\to\infty} \frac{\mathrm{P}[\Theta_s X_s > x,\ \Theta_t X_t > x]}{R(x)} = 0. \tag{4.25}$$

Using (4.24), we have

$$1 \le \lim_{m\to\infty} \liminf_{x\to\infty} \frac{\sum_{t=1}^m \mathrm{P}[\Theta_t X_t > x]}{R(x)} \le \lim_{m\to\infty} \limsup_{x\to\infty} \frac{\sum_{t=1}^m \mathrm{P}[\Theta_t X_t > x]}{R(x)} \le 1.$$

As in the proof of Proposition 4.1, using (4.16), (4.17), and (4.25), the above inequalities reduce to

$$1 \le \lim_{m\to\infty} \liminf_{x\to\infty} \frac{\mathrm{P}[\sum_{t=1}^m \Theta_t X_t > x]}{R(x)} \le \lim_{m\to\infty} \limsup_{x\to\infty} \frac{\mathrm{P}[\sum_{t=1}^m \Theta_t X_t > x]}{R(x)} \le 1,$$

and the tail equivalence follows using (4.23) and the regular variation of R. Since $R(x) \to 0$, the tail equivalence also shows the almost-sure finiteness of $X_{(\infty)}$.

Next, we prove Theorem 4.3 using the converse part of Theorem 4.1.

Proof of Theorem 4.3. Define the measure $\rho(\cdot) = \sum_{t=1}^\infty \mathrm{P}[\Theta_t \in \cdot]$. By the RW moment condition, the measure ρ is σ-finite. Furthermore, we have, $\int_0^\infty y^{\alpha+\mathrm{i}\beta_0}\rho(\mathrm{d}y) = 0$. Now, by the converse part of Theorem 4.1, there exists a σ-finite measure ν, whose tail is not regularly varying, but $\nu \circledast \rho$ has a regularly varying tail. Next, define a probability measure μ using the σ-finite measure ν as in Theorem 3.1 of [9]. Choose $b > 1$ such that $\nu(b,\infty) \le 1$, and define a probability measure on $(0,\infty)$ by

$$\mu(B) = \nu(B \cap (b,\infty)) + (1 - \nu(b,\infty))\,\mathbf{1}_B(1), \quad \text{where } B \text{ is a Borel subset of } (0,\infty).$$

First observe that

$$\mu(y,\infty) = \begin{cases} \nu(y,\infty) & \text{for } y > b, \\ \nu(b,\infty) & \text{for } 1 < y \le b, \\ 1 & \text{for } y \le 1. \end{cases}$$

Thus, μ does not have a regularly varying tail and

$$\begin{aligned} \mu \circledast \rho(x,\infty) &= \int_0^\infty \mu\left(\frac{x}{u},\infty\right)\rho(\mathrm{d}u) \\ &= \int_0^{x/b} \nu\left(\frac{x}{u},\infty\right)\rho(\mathrm{d}u) + \nu(b,\infty)\rho\left[\frac{x}{b},x\right) + \rho[x,\infty) \\ &= \nu \circledast \rho(x,\infty) - 2x^{-\alpha}\int_{x/b}^\infty u^\alpha \rho(\mathrm{d}u) + \nu(b,\infty)\rho\left[\frac{x}{b},x\right) + \rho[x,\infty). \end{aligned}$$

Now, using the bounds from (4.4) and (4.5), the second term is bounded for $x > b$ by

$$2\frac{\nu \circledast \rho(x,\infty)}{\|\rho\|_\alpha}\int_{x/b}^\infty u^{\alpha+\varepsilon}\rho(\mathrm{d}u) = o(\nu \circledast \rho(x,\infty))$$

as $x \to \infty$, since $\int_0^\infty u^{\alpha+\varepsilon}\rho(\mathrm{d}u) < \infty$ by the RW conditions. The sum of the last two terms can be bounded by

$$\frac{1 + \nu(b,\infty)b^{\alpha+\varepsilon}}{x^{\alpha+\varepsilon}}\int_0^\infty u^{\alpha+\varepsilon}\rho(\mathrm{d}u) = o(\nu \circledast \rho(x,\infty)) \quad \text{as } x \to \infty,$$

since $\nu \circledast \rho(x, \infty)$ is regularly varying of index $-\alpha$. Thus, $\mu \circledast \rho(x, \infty) \sim \nu \circledast \rho(x, \infty)$ as $x \to \infty$ and, hence, is regularly varying of index $-\alpha$.

Let X_t be an i.i.d. sequence with common law μ. Then, X_1 does not have a regularly varying tail. Furthermore, by Proposition 4.2, $X_{(\infty)}$ is finite with probability 1 and $\mathrm{P}[X_{(\infty)} > x] \sim \mu \circledast \rho(x, \infty)$ is regularly varying of index $-\alpha$.

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