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Stokes flow of incompressible liquid through a conical diffuser with partial slip boundary condition

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For the first time, an analytical solution has been derived for Stokes flow through a conical diffuser under the condition of partial slip. Recurrent relations are obtained that allow determination of the velocity, pressure and stream function for a certain slip length λ. The solution is analysed in the first order of decomposition with respect to a small dimensionless parameter λ/r. It is shown that the sliding of the liquid over the surface of the cone leads to a vorticity of the flow. At zero slip length, we obtain the well-known solution to the problem of a diffuser with a no-slip boundary condition corresponding to strictly radial streamlines. To solve that problem, we use an alternative form of the general solution of the linearised, stationary, axisymmetric Navier–Stokes equations for an incompressible fluid in spherical coordinates. A previously published solution to this problem, dating back to the paper by Sampson (1891 *Phil. Trans. R. Soc. A*, vol. **182**, pp. 449–518), is given in terms of a stream function that leads to formulae that are difficult to apply in practice. By contrast, the new general solution is derived in the vector potential representation and is simpler to apply.

Key words: low-Reynolds-number flows, Navier-Stokes equations, microfluidics

1. Introduction

In recent decades, investigations of the interaction of a liquid flow with a solid surface with partial slip boundary conditions have been developing rapidly (Vinogradova 1995; Neto *et al.* 2005; Lauga, Brenner & Stone 2007; Rothstein 2010; Dubov *et al.* 2018). This is due to the revolutionary development of technologies for creating functionalised surfaces, including hydrophobic and superhydrophobic coatings. The velocity of an aqueous flow in contact with such a solid coating is non-zero (Vinogradova 1999; Boinovich & Emelyanenko 2008; Simpson, Hunter & Aytug 2015).

There are studies of the nature of this phenomenon at the atomic and molecular levels, as well as calculations of flow near surfaces of different shapes: flat, cylindrical and spherical or their combinations (Vinogradova 1999; Lauga & Stone 2003; Neto *et al.* 2005; Lauga *et al.* 2007). Small-sized surfaces and slow fluid flow are usually considered, corresponding to small Reynolds numbers. For example, the sedimentation of a small spherical particle has been theoretically and experimentally considered. The drag force acting on a solid sphere moving through a fluid under a partial slip condition is described by generalisation of the well-known Stokes law previously obtained for the non-slip condition (Boehnke *et al.* 1999; Lauga *et al.* 2007).

So-called Stokes equations are a linearised form of the stationary Navier–Stokes equations for incompressible liquids (Happel & Brenner 1983; Landau & Lifshitz 1987; Batchelor 2000):

$$\eta \overrightarrow{\nabla^2} \boldsymbol{V} = \boldsymbol{\nabla} p, \quad (\boldsymbol{\nabla} \cdot \boldsymbol{V}) = 0, \tag{1.1}$$

where $\boldsymbol{V}$ and η are the velocity and the dynamic viscosity of a liquid, respectively, and p is the pressure. The vector Laplacian is determined by (Moon & Spencer 1971)

$$\overrightarrow{\nabla^2} \boldsymbol{V} = \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \boldsymbol{V}) - \boldsymbol{\nabla} \times [\boldsymbol{\nabla} \times \boldsymbol{V}]. \tag{1.2}$$

Equations (1.1) and (1.2) in a spherical coordinate system (figure 1) for an axisymmetric problem can be rewritten as (Batchelor 2000)

$$\nabla^2 V_r - \frac{2V_r}{r^2} - \frac{2}{r^2 \sin\theta} \frac{\partial}{\partial \theta} (V_\theta \sin\theta) = \eta^{-1} \frac{\partial}{\partial r} p, \tag{1.3}$$

$$\nabla^2 V_\theta - \frac{V_\theta}{r^2 \sin^2\theta} + \frac{2}{r^2} \frac{\partial}{\partial \theta} V_r = \eta^{-1} \frac{1}{r} \frac{\partial}{\partial \theta} p, \tag{1.4}$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r^2 V_r\right) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} (V_\theta \sin\theta) = 0, \tag{1.5}$$

where the non-zero components of velocity $V_r(r, \theta)$, $V_\theta(r, \theta)$ and pressure p are independent of the azimuthal angle φ.

A cone is a geometric figure that traditionally attracts attention because of its importance in many applications and the relative simplicity of the mathematical formulation of the boundary problem.

The Navier condition is a mathematical formulation of the boundary conditions of slip-with-friction, in which the tangential component of the fluid flow velocity on a solid surface is proportional to the rate of strain or, equivalently, to the viscous stress tensor component that corresponds to this rate developing (Vinogradova 1995; Neto *et al.* 2005; Lauga *et al.* 2007; Rothstein 2010; Dubov *et al.* 2018). So, for the surface of a conical diffuser in a spherical coordinate system (figure 1), the Navier condition takes the form

$$\sigma_{\theta r} (r, \theta_0) = \frac{\eta}{\lambda} V_r (r, \theta_0), \tag{1.6}$$

where λ is the slip length (a constant, characterising the properties of the solid surface and the liquid) and $\sigma_{\theta r}$ is the component of the viscous stress tensor corresponding to the shear strain rate (Landau & Lifshitz 1987; Batchelor 2000):

$$\sigma_{\theta r}(r, \theta) = \eta \left(\frac{1}{r} \frac{\partial V_r}{\partial \theta} + \frac{\partial V_\theta}{\partial r} - \frac{V_\theta}{r} \right). \tag{1.7}$$

The second boundary condition manifests the impermeability of the conical surface:

$$V_\theta (r, \theta_0) = 0. \tag{1.8}$$

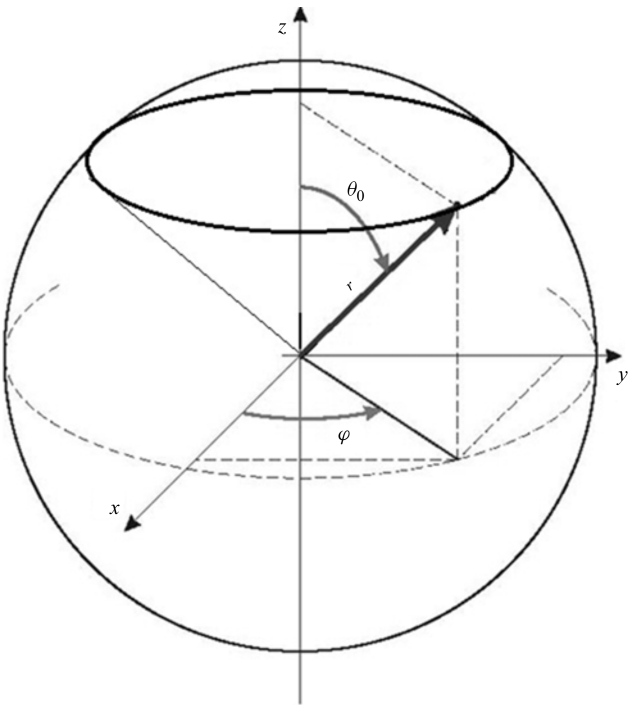

Figure 1. A spherical coordinate system (r, θ, φ) and conical diffuser with polar angle θ_0.

Taking into account (1.7) and (1.8), equation (1.6) can be rewritten as

$$\lambda \left(\frac{1}{r} \frac{\partial V_r(r, \theta)}{\partial \theta} + \frac{\partial V_\theta(r, \theta)}{\partial r} \right)_{\theta=\theta_0} = V_r\,(r, \theta_0)\,. \tag{1.9}$$

For more than a century, the solution of the Stokes equations (1.3)–(1.5) with no-slip conditions (1.8) and (1.9), where $\lambda = 0$, has been known (Harrison 1920; Slezkin 1955). However, for conditions of partial slip ($\lambda \neq 0$) the solution to this problem has not yet been published. One of the reasons for this situation may be as follows.

The general solution of axisymmetric Stokes equations in a spherical coordinate system, performed in the formalism of the stream function, is presented in the monograph by Happel & Brenner (1983), summarising earlier works in which this solution was obtained (Sampson 1891; Savic 1953; Haberman & Sayre 1958). This solution has a quite complex formulation. It is difficult to use for practical problems in which the typical situation is that the boundary conditions are imposed directly on the velocity components, and not on the stream function. The disadvantage of such a solution, in our opinion, is that it is based on obtaining a stream function, which is not an optimal choice of the generating function, although it is very important for visualising two-dimensional flows.

The general solution of axisymmetric Stokes equations in a spherical coordinate system can be obtained by a mathematically equivalent, but alternative, method in the representation of a vector potential, rather than a stream function. This makes it possible to take full advantage of the well-developed apparatus of both the Legendre polynomials and the associated Legendre polynomials, through which the relations for the velocity and pressure components are expressed. The general solution we have obtained is shown in table 1, and its derivation is given in supplementary material I available at https://doi.org/10.1017/jfm.2025.10874. The solution is divided into internal and external problems for

Internal problem	External problem
$V_r(r,\theta) = \sum_{l=1}^{\infty} l(l+1)\left\{\frac{\alpha_l r^{l+1}}{4l+6} + \gamma_l r^{l-1}\right\} P_l(\cos\theta)$	$V_r(r,\theta) = -\sum_{l=1}^{\infty} l(l+1)\left\{\frac{\beta_l r^{-l}}{4l-2} + \delta_l r^{-l-2}\right\} P_l(\cos\theta) + \delta_0 r^{-2}$
$V_\theta(r,\theta) = \sum_{l=1}^{\infty}\left\{\alpha_l \frac{l+3}{4l+6} r^{l+1} + \gamma_l (l+1) r^{l-1}\right\} P_l^1(\cos\theta)$	$V_\theta(r,\theta) = \sum_{l=1}^{\infty}\left\{\beta_l \frac{l-2}{4l-2} r^{-l} + l\delta_l r^{-l-2}\right\} P_l^1(\cos\theta)$
$p(r,\theta) = \frac{\eta}{R}\sum_{l=0}^{\infty}(l+1)\alpha_l\, r^l P_l(\cos\theta)$	$p(r,\theta) = -\eta\sum_{l=1}^{\infty} l\beta_l\, r^{-l-1} P_l(\cos\theta)$
$\psi(r,\theta) = -\sin\theta\sum_{l=1}^{\infty}\left\{\frac{\alpha_l r^{l+3}}{4l+6} + \gamma_l r^{l+1}\right\} P_l^1(\cos\theta)$	$\psi(r,\theta) = \sin\theta\sum_{l=1}^{\infty}\left\{\frac{\beta_l r^{2-l}}{4l-2} + \delta_l r^{-l}\right\} P_l^1(\cos\theta) - d_0\cos\theta$
$\phi(r,\ \theta) = -\sum_{l=1}^{\infty}\alpha_l r^l P_l^1(\cos\theta)$	$\phi(r,\ \theta) = \sum_{l=1}^{\infty}\beta_l r^{-l-1} P_l^1(\cos\theta)$

Table 1. Solutions of internal and external axisymmetric problems in a spherical coordinate system obtained in the representation of a vector potential. The radial and polar components of the incompressible fluid velocity, pressure, stream function and vorticity are shown in the rows from top to bottom; $P_l(\cos\theta)$ is the Legendre polynomial, $P_l^1(\cos\theta)$ is the associated Legendre function of the first order. This table is a copy of table S2 from supplementary material I, which presents the derivation of these formulae.

methodological purposes. However, it must be borne in mind that in the general case, the solution is the sum of the corresponding formulae for internal and external problems (left- and right-hand columns of table 1).

Investigation of the conical diffuser problem allows us to validate the general solution under consideration in the case of no slip ($\lambda = 0$) and to obtain the new result for the boundary condition of partial slip ($\lambda \neq 0$).

2. Flow in a conical diffuser with a polar angle $0 < \theta_0 < \pi$

Let us consider a conical diffuser, the working surface of which, interacting with the flow, is limited by a polar angle with permissible values in the range $0 < \theta_0 < \pi$ (figure 1). The flow rate of liquid passing through the diffuser is determined by

$$Q = 2\pi r^2 \int_0^{\theta_0} V_r(r,\theta)\sin\theta\, \mathrm{d}\theta. \tag{2.1}$$

The flow rate must be constant in any section of the cone: $Q = \text{const}$. Then

$$\int_0^{\theta_0} V_r(r,\theta)\sin\theta\, \mathrm{d}\theta \sim \frac{1}{r^2}. \tag{2.2}$$

Therefore, the velocity component $V_r(r,\theta)$ tends to zero at $r\to\infty$:

$$V_r(\infty,\theta) = 0. \tag{2.3}$$

According to table 1, to satisfy this condition, it is necessary to use the general solution of the external problem (right-hand column of table 1). In this case, the following conditions will obviously also be automatically met:

$$V_\theta(\infty,\theta) = 0, \tag{2.4}$$

$$p(\infty,\theta) = 0. \tag{2.5}$$

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Conditions (1.8) and (1.9) are met on the work surface of the diffuser. Substituting $V_\theta(r, \theta_0)$ from table 1 into condition (1.8), we obtain

$$\sum_{l=1}^{\infty} \left\{ \beta_l \frac{l-2}{4l-2} r^{-l} + l\delta_l r^{-l-2} \right\} P_l^1(\cos\theta_0) = 0. \tag{2.6}$$

Similarly, substituting $V_r(r, \theta)$, one can find the left-hand side of (1.9):

$$\left(\frac{1}{r}\frac{\partial V_r(r,\theta)}{\partial\theta} + \frac{\partial V_\theta(r,\theta)}{\partial r} \right)_{\theta=\theta_0} = -\sum_{l=1}^{\infty} l \left\{ \frac{\beta_l}{2} r^{-l-1} + \delta_l(2l+3) r^{-l-3} \right\} P_l^1(\cos\theta_0), \tag{2.7}$$

where the transformations take into account the following expression (Arfken, Weber & Harris 2012):

$$\frac{\partial P_l}{\partial\theta} = -\sin\theta \frac{\partial P_l}{\partial\cos\theta} = P_l^1. \tag{2.8}$$

Substituting (2.7) and $V_r(r, \theta)$ from table 1 in (1.9), one can obtain

$$\begin{aligned} &\lambda \sum_{l=1}^{\infty} l \left\{ \frac{\beta_l}{2} r^{-l-1} + \delta_l(2l+3) r^{-l-3} \right\} P_l^1(\cos\theta_0) \\ &\quad = \sum_{l=1}^{\infty} l(l+1) \left\{ \frac{\beta_l r^{-l}}{4l-2} + d_l r^{-l-2} \right\} P_l(\cos\theta_0) - d_0 r^{-2}. \end{aligned} \tag{2.9}$$

There are two terms under summation in (2.6). The first of them can be represented as

$$\sum_{l=1}^{\infty} \beta_l \frac{l-2}{4l-2} r^{-l} P_l^1(\cos\theta_0) = -\frac{\beta_1}{2r} P_1^1(\cos\theta_0) + \sum_{l=3}^{\infty} \beta_l \frac{l-2}{4l-2} r^{-l} P_l^1(\cos\theta_0). \tag{2.10}$$

Shifting the count of summation in l by two units downwards in the right-hand part of (2.10) and making a redefinition, we get

$$\sum_{l=1}^{\infty} \beta_l \frac{l-2}{4l-2} r^{-l} P_l^1(\cos\theta_0) = -\frac{\beta_1}{2r} P_1^1(\cos\theta_0) + \sum_{l=1}^{\infty} \beta_{l+2} \frac{l r^{-l-2}}{4l+6} P_{l+2}^1(\cos\theta_0). \tag{2.11}$$

Substituting (2.11) in (2.6), we have

$$-\frac{\beta_1}{2r} P_1^1(\cos\theta_0) + \sum_{l=1}^{\infty} l \left\{ \frac{\beta_{l+2}}{4l+6} P_{l+2}^1(\cos\theta_0) + \delta_l P_l^1(\cos\theta_0) \right\} r^{-l-2} = 0. \tag{2.12}$$

Therefore,

$$\beta_1 P_1^1(\cos\theta_0) = 0, \quad \frac{\beta_{l+2}}{4l+6} P_{l+2}^1(\cos\theta_0) + \delta_l P_l^1(\cos\theta_0) = 0, \quad l = 1, 2, 3, \ldots \tag{2.13}$$

Similarly, let us convert the sum on the right-hand side of the condition (2.9):

$$\begin{aligned} &\sum_{l=1}^{\infty} l(l+1) \left\{ \frac{\beta_l r^{-l}}{4l-2} + \delta_l r^{-l-2} \right\} P_l(\cos\theta_0) = \left\{ \frac{\beta_1}{r} + 2\delta_1 r^{-3} \right\} P_1(\cos\theta_0) \\ &\quad + \sum_{l=1}^{\infty} (l+1)(l+2) \left\{ \frac{\beta_{l+1} r^{-l-1}}{4l+2} + \delta_{l+1} r^{-l-3} \right\} P_{l+1}(\cos\theta_0). \end{aligned} \tag{2.14}$$

Substituting (2.14) in (2.9), one can obtain

$$
\begin{aligned}
&\lambda \sum_{l=1}^{\infty} l \left\{ \frac{\beta_l}{2} r^{-l-1} + (2l+3)\delta_l r^{-l-3} \right\} P_l^1\left(\cos\theta_0\right) \\
&\quad - \sum_{l=1}^{\infty} (l+1)(l+2) \left\{ \frac{\beta_{l+1}}{4l+2} r^{-l-1} + \delta_{l+1} r^{-l-3} \right\} P_{l+1}\left(\cos\theta_0\right) \\
&\quad = \left\{ \frac{\beta_1}{r} + 2\delta_1 r^{-3} \right\} P_1\left(\cos\theta_0\right) - \delta_0 r^{-2}.
\end{aligned} \tag{2.15}
$$

Rearranging and combining the terms by equal powers of $1/r$, we have

$$
\begin{aligned}
&\sum_{l=1}^{\infty} \left\{ \frac{\lambda}{2} \beta_l l P_l^1\left(\cos\theta_0\right) - \frac{(l+1)(l+2)}{4l+2} \beta_{l+1} P_{l+1}\left(\cos\theta_0\right) \right\} r^{-l-1} \\
&\quad + \sum_{l=1}^{\infty} \left\{ \delta_l (2l+3) l \lambda P_l^1\left(\cos\theta_0\right) - (l+1)(l+2)\delta_{l+1} P_{l+1}\left(\cos\theta_0\right) \right\} r^{-l-3} \\
&\quad = \left\{ \frac{\beta_1}{r} + 2\delta_1 r^{-3} \right\} P_1\left(\cos\theta_0\right) - \delta_0 r^{-2}.
\end{aligned} \tag{2.16}
$$

Let us select the first two terms in the first sum in (2.16), bring the rest to the system of counting powers of $1/r$ that corresponds to the second sum and shift the counting along l:

$$
\begin{aligned}
&\sum_{l=1}^{2} \left\{ \frac{\lambda}{2} \beta_l l P_l^1\left(\cos\theta_0\right) - \frac{(l+1)(l+2)}{4l+2} \beta_{l+1} P_{l+1}\left(\cos\theta_0\right) \right\} r^{-l-1} \\
&\quad + \sum_{l=1}^{\infty} \left\{ \frac{l+2}{2} \lambda \beta_{l+2} P_{l+2}^1\left(\cos\theta_0\right) - \frac{(l+3)(l+4)}{4l+10} \beta_{l+3} P_{l+3}\left(\cos\theta_0\right) \right\} r^{-l-3} \\
&\quad + \sum_{l=1}^{\infty} \left\{ (2l+3) l \lambda \delta_l P_l^1\left(\cos\theta_0\right) - (l+1)(l+2)\delta_{l+1} P_{l+1}\left(\cos\theta_0\right) \right\} r^{-l-3} \\
&\quad = \left\{ \frac{\beta_1}{r} + 2\delta_1 r^{-3} \right\} P_1\left(\cos\theta_0\right) - \delta_0 r^{-2}.
\end{aligned} \tag{2.17}
$$

Coefficients of equal powers of $1/r$ can be combined:

$$
\begin{aligned}
&\sum_{l=1}^{\infty} \left\{ \frac{l+2}{2} \lambda \beta_{l+2} P_{l+2}^1\left(\cos\theta_0\right) + (2l+3) l \lambda \delta_l P_l^1\left(\cos\theta_0\right) \right. \\
&\quad \left. - \frac{(l+3)(l+4)}{4l+10} \beta_{l+3} P_{l+3}\left(\cos\theta_0\right) - (l+1)(l+2)\delta_{l+1} P_{l+1}\left(\cos\theta_0\right) \right\} r^{-l-3} \\
&\quad = \frac{\beta_1}{r} P_1\left(\cos\theta_0\right) + \left\{ \beta_2 P_2\left(\cos\theta_0\right) - \delta_0 - \frac{\lambda}{2} \beta_1 P_1^1\left(\cos\theta_0\right) \right\} r^{-2} \\
&\quad + \left\{ 2\delta_1 P_1\left(\cos\theta_0\right) - \lambda \beta_2 P_2^1\left(\cos\theta_0\right) + \frac{6\beta_3}{5} P_3\left(\cos\theta_0\right) \right\} r^{-3}.
\end{aligned} \tag{2.18}
$$

To satisfy (2.18), it is necessary to require the following conditions:

$$\beta_1 P_1(\cos\theta_0) = 0, \tag{2.19}$$

$$\beta_2 P_2(\cos\theta_0) - \delta_0 - \frac{\lambda}{2}\beta_1 P_1^1(\cos\theta_0) = 0, \tag{2.20}$$

$$2\delta_1 P_1(\cos\theta_0) - \lambda\beta_2 P_2^1(\cos\theta_0) + \frac{6\beta_3}{5} P_3(\cos\theta_0) = 0, \tag{2.21}$$

$$\beta_{l+2}(l+2)\frac{\lambda}{2} P_{l+2}^1(\cos\theta_0) + (2l+3)l\lambda\delta_l P_l^1(\cos\theta_0) - (l+3)(l+4)\frac{\beta_{l+3}}{4l+10} P_{l+3}(\cos\theta_0) \\ - (l+1)(l+2)\delta_{l+1} P_{l+1}(\cos\theta_0) = 0, \quad l = 1, 2, 3, \ldots \tag{2.22}$$

Let us assume that

$$P_l^1 \neq 0. \tag{2.23}$$

Conditions (2.13) can be conveniently rewritten as

$$\beta_1 = 0, \quad \delta_l = -\frac{P_{l+2}^1(\cos\theta_0)}{P_l^1(\cos\theta_0)} \frac{\beta_{l+2}}{4l+6}, \quad l = 1, 2, 3, \ldots \tag{2.24}$$

Taking this into account, condition (2.20) can be rewritten as

$$\delta_0 = \beta_2 P_2(\cos\theta_0), \tag{2.25}$$

and conditions (2.21) and (2.22) can be combined by the formula

$$\beta_{l+2} = \beta_{l+1}\,(4l+6)\,\lambda P_{l+1}^1(\cos\theta_0) \\ \times \left\{(l+2)(l+3) P_{l+2}(\cos\theta_0) - l(l+1)\frac{P_{l+2}^1(\cos\theta_0)}{P_l^1(\cos\theta_0)} P_l(\cos\theta_0)\right\}^{-1}, \\ l = 1, 2, 3, \ldots \tag{2.26}$$

Equation (2.26) is a recurrence relation that allows us to sequentially calculate b_3, b_4, etc., if the coefficient b_2 is known. In particular, we have

$$\beta_3 = 5\beta_2\lambda P_2^1(\cos\theta_0)\left(6P_3(\cos\theta_0) - P_1(\cos\theta_0)\frac{P_3^1(\cos\theta_0)}{P_1^1(\cos\theta_0)}\right)^{-1}, \tag{2.27}$$

$$\beta_4 = 7\beta_3\lambda P_3^1(\cos\theta_0)\left\{10P_4(\cos\theta_0) - 3\frac{P_4^1(\cos\theta_0)}{P_2^1(\cos\theta_0)} P_2(\cos\theta_0)\right\}^{-1}. \tag{2.28}$$

The corresponding coefficients d_l are determined by (2.24). In particular,

$$\delta_1 = -\frac{P_3^1(\cos\theta_0)}{P_1^1(\cos\theta_0)} P_2^1(\cos\theta_0)\frac{\lambda\beta_2}{2}\left(6P_3(\cos\theta_0) - P_1(\cos\theta_0)\frac{P_3^1(\cos\theta_0)}{P_1^1(\cos\theta_0)}\right)^{-1}. \tag{2.29}$$

Let us write down the solutions of the external problem from table 1, writing them out in powers of $1/r$ and explicitly identifying the first terms of the expansion, taking into

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account that, according to (2.24), $\beta_1 = 0$:

$$V_r(r,\theta) = (\delta_0 - \beta_2 P_2)\, r^{-2} - \left\{ \frac{6}{5}\beta_3 P_3 + 2\delta_1 P_1 \right\} r^{-3} - \sum_{l=2}^{\infty} \left\{ \frac{(l+2)(l+3)}{4l+6}\beta_l P_l + l(l+1)\delta_l P_l \right\} r^{-l-2}, \tag{2.30}$$

$$V_\theta(r,\theta) = \left\{ \beta_3 \frac{P_3^1}{10} + \delta_1 P_1^1 \right\} r^{-3} + \sum_{l=2}^{\infty} \left\{ \beta_{l+2} \frac{l P_{l+2}^1}{4l+6} + l\delta_l P_l^1 \right\} r^{-l-2}, \tag{2.31}$$

$$p(r,\theta) = -2\eta\beta_2 P_2(\cos\theta) r^{-3} - 3\eta\beta_3 P_3(\cos\theta) r^{-4} - \eta \sum_{l=4}^{\infty} l\beta_l\, r^{-l-1} P_l(\cos\theta), \tag{2.32}$$

$$\psi(r,\theta) = \sin\theta \frac{\beta_2}{6} P_2^1 - d_0\cos\theta + \sin\theta \left\{ \frac{\beta_3}{10} P_3^1 + \delta_1 P_l^1 \right\} r^{-1} + \sin\theta \sum_{l=2}^{\infty} \left\{ \frac{\beta_{l+2} P_{l+2}^1}{4l+6} + \delta_l P_l^1 \right\} r^{-l}, \tag{2.33}$$

$$\phi(r,\ \theta) = \beta_2 r^{-3} P_2^1(\cos\theta) + \beta_3 r^{-4} P_3^1(\cos\theta) + \sum_{l=4}^{\infty} \beta_l r^{-l-1} P_l^1(\cos\theta). \tag{2.34}$$

Taking into account (2.24) and (2.36), we see that in expressions (2.30)–(2.34) each subsequent term is obtained from the previous one by multiplying by a certain value, which can be represented as $H_l(\lambda/r)$, where H_l is a numerical coefficient of limited magnitude, having a modulus of the order of unity. Thus, if we require satisfying the condition

$$\frac{\lambda}{r} \ll 1, \tag{2.35}$$

then the modulus of each term of the series in relations (2.30)–(2.34) will be much greater than the term following it.

That is, each subsequent term of the series is a small correction to the previous one. According to the d'Alembert convergence criterion, this means that the series (2.30)–(2.34) converges absolutely (Davis 1962). A rough estimate of the approximation error is the value of the first discarded term of the series.

Allowing the terms in the series expansions (2.30)–(2.34) to remain up to the first approximation in λ/r, one can obtain approximate expressions that terminate at the coefficients β_3 and δ_1 that, according to (2.27) and (2.29), include λ in the first power:

$$V_r(r,\theta) \approx (\delta_0 - \beta_2 P_2)\, r^{-2} - \left\{ \frac{6}{5}\beta_3 P_3 + 2\delta_1 P_1 \right\} r^{-3}, \tag{2.36}$$

$$V_\theta(r,\theta) \approx \left\{ \beta_3 \frac{P_3^1}{10} + \delta_1 P_1^1 \right\} r^{-3}, \tag{2.37}$$

$$p(r,\theta) \approx -2\eta\beta_2 P_2 r^{-3} - 3\eta\beta_3 P_3 r^{-4}, \tag{2.38}$$

as well as corresponding relations for $\psi(r,\theta)$ and $\phi(r,\ \theta)$ that can be obtained from (2.33) and (2.34).

Taking into account $P_1(t) = t$, $P_2(t) = (1/2)(3t^2 - 1)$, $P_3(t) = (1/2)(5t^3 - 3t)$, $P_1^1(t) = -\sqrt{1-t^2}$, $P_2^1(t) = -3t\sqrt{1-t^2}$ and $P_3^1(t) = -(3/2)(5t^2-1)\sqrt{1-t^2}$, where $t = \cos\theta$ (Arfken *et al.* 2012), (2.27) and (2.29) give

$$\beta_3 = 2\lambda \frac{\beta_2}{\sin\theta_0}, \tag{2.39}$$

$$\delta_1 = -\frac{3}{20}\beta_3 \left(5\cos^2\theta_0 - 1\right). \tag{2.40}$$

Substituting (2.39) and (2.40) in (2.36)–(2.38), one can obtain the desired solution of the problem:

$$V_r \approx \frac{3\beta_2}{2r^2}\left(\cos^2\theta_0 - \cos^2\theta\right) + \frac{3\beta_2\lambda}{r^3}\frac{\cos\theta}{\sin\theta_0}\left(\cos^2\theta_0 - 2\cos^2\theta + 1\right), \tag{2.39}$$

$$V_\theta \approx \frac{3\beta_2\lambda}{2r^3}\frac{\sin\theta}{\sin\theta_0}\left(\cos^2\theta_0 - \cos^2\theta\right), \tag{2.40}$$

$$p \approx -\frac{\eta\beta_2}{r^3}\left(3\cos^2\theta - 1\right) - \frac{3\lambda\eta}{r^4}\frac{\beta_2}{\sin\theta_0}\left(5\cos^2\theta - 3\right)\cos\theta, \tag{2.41}$$

$$\psi(r,\theta) \approx \frac{\beta_2}{2}\cos\theta\left(\cos^2\theta - 3\cos^2\theta_0\right) + \frac{3\lambda\beta_2}{2r}\frac{\sin^2\theta}{\sin\theta_0}\left(\cos^2\theta_0 - \cos^2\theta\right), \tag{2.42}$$

$$\phi(r,\ \theta) \approx -3\beta_2 \sin\theta\, r^{-3}\left(\cos\theta + \frac{\lambda}{r\sin\theta_0}\left(5\cos^2\theta - 1\right)\right). \tag{2.43}$$

The flow rate of liquid passing through the diffuser, Q, is determined by (2.1). Appendix A, equation (A13), shows that for the solution obtained above, the flow rate is constant in any section of the cone, so that condition (2.2) is indeed satisfied. In this case the flow is determined by the following relation:

$$Q = 2\pi\beta_2 \int_1^{\cos\theta_0} P_2(t)\mathrm{d}t + 2\pi\delta_0 \int_0^{\theta_0} \sin\theta\, \mathrm{d}\theta. \tag{2.44}$$

Carrying out calculations and substituting (2.25), we obtain

$$Q = \pi\beta_2\left[3\cos^2\theta_0 - 2\cos^3\theta_0 - 1\right] = -\pi\beta_2\,(1-\cos\theta_0)^2\,(1+2\cos\theta_0)\,. \tag{2.45}$$

At $\cos\theta_0 = -1/2$ ($\theta_0 = 2\pi/3$) the flow rate given by (2.45) becomes zero, and it is impossible to determine the parameter β_2 via Q. For all other cases, one can write

$$\beta_2 = -\frac{Q}{\pi^2\,(1-\cos\theta_0)^2\,(1+2\cos\theta_0)}. \tag{2.46}$$

Since all other coefficients are sequentially calculated through β_2, taking into account (2.24) and (2.26), the solution can be expressed in terms of the flow rate Q.

Let us express (2.39)–(2.42) through the liquid flow rate by substituting (2.46):

$$V_r \approx 3Q\frac{\cos^2\theta - \cos^2\theta_0 - \dfrac{2\lambda}{r}\dfrac{\cos\theta}{\sin\theta_0}\left(\cos^2\theta_0 - 2\cos^2\theta + 1\right)}{2\pi r^2\,(1-\cos\theta_0)^2\,(1+2\cos\theta_0)}, \tag{2.47}$$

$$V_\theta \approx \frac{3\lambda Q\sin\theta\left(\cos^2\theta - \cos^2\theta_0\right)}{2\pi r^3\sin\theta_0\,(1-\cos\theta_0)^2\,(1+2\cos\theta_0)}, \tag{2.48}$$

$$p \approx \eta Q\frac{3\cos^2\theta - 1 + \dfrac{3\lambda}{r}\dfrac{\cos\theta}{\sin\theta_0}\left(5\cos^2\theta - 3\right)}{\pi r^3\,(1-\cos\theta_0)^2\,(1+2\cos\theta_0)}, \tag{2.49}$$

$$\psi(r,\theta) \approx Q\frac{\cos\theta\left(3\cos^2\theta_0-\cos^2\theta\right)+\dfrac{3\lambda}{r}\dfrac{\sin^2\theta}{\sin\theta_0}\left(\cos^2\theta-\cos^2\theta_0\right)}{2\pi\,(1-\cos\theta_0)^2\,(1+2\cos\theta_0)}, \tag{2.50}$$

$$\phi(r,\ \theta) \approx \frac{3Q\sin\theta}{\pi^2 r^3\,(1-\cos\theta_0)^2\,(1+2\cos\theta_0)}\left(\cos\theta+\frac{\lambda}{r\sin\theta_0}\left(5\cos^2\theta-1\right)\right). \tag{2.51}$$

3. Discussion

It has been shown that condition (2.35) has to be satisfied in (2.47)–(2.51). This condition agrees well with the conditions of applicability of the Stokes equations. Indeed, note that to ensure the validity of the Stokes equations, it is necessary to consider a region of the cone that is sufficiently far from its apex. Indeed, the Stokes equations are valid for small Reynolds numbers:

$$Re = \frac{\rho L V}{\eta} \ll 1, \tag{3.1}$$

where ρ is the density of the liquid, V is the characteristic velocity and L is the characteristic size, which for a cone can be considered its cutting radius at a given r. In fact, keeping in mind the approximation (3.1), the characteristic size is the radial coordinate r, measured from the cone apex, which in the most interesting cases has the same order of magnitude as the corresponding radius of the cone funnel for a given r. The ratio of the flow rate to the square of the characteristic size $V \propto Q/r^2$ can serve as an estimate for the velocity. Therefore, instead of (3.1), we can write

$$r \gg \frac{\rho Q}{\eta}. \tag{3.2}$$

In other words, both the Stokes equations and their linear approximation in the expansion in λ work better the further from the cone apex the area of study is. But, generally speaking, the ratio of the parameters $\rho Q/\eta$ and λ can have any value.

For $\lambda = 0$, taking into account (2.48)–(2.50), one can obtain the well-known exact solution of the Stokes equations for the no-slip condition (Happel & Brenner 1983; Harrison 1920; Slezkin 1955):

$$V_r = \frac{3Q\left(\cos^2\theta-\cos^2\theta_0\right)}{2\pi r^2\,(1-\cos\theta_0)^2\,(1+2\cos\theta_0)}, \tag{3.3}$$

$$V_\theta = 0, \tag{3.4}$$

$$p = \frac{\eta Q\left(3\cos^2\theta-1\right)}{\pi r^3\,(1-\cos\theta_0)^2\,(1+2\cos\theta_0)}, \tag{3.5}$$

$$\psi(r,\theta) = \frac{Q\cos\theta\left(3\cos^2\theta_0-\cos^2\theta\right)}{2\pi\,(1-\cos\theta_0)^2\,(1+2\cos\theta_0)}. \tag{3.6}$$

Within the framework of the model under consideration, there are two independent parameters that have the physical dimension of length: $\rho Q/\eta$ and λ. They correspond to two ways to non-dimensionalise the radial coordinate r.

For $Q \neq 0$ one can introduce the dimensionless radial coordinate as follows:

$$\xi = \frac{\eta}{\rho Q} r. \tag{3.7}$$

Then the dimensionless slip length is

$$\bar{\lambda} = \frac{\eta}{\rho Q}\lambda. \tag{3.8}$$

The stream function (2.50) can be rewritten in dimensionless form as

$$\psi(\xi,\theta) = \frac{\psi}{Q} \approx \frac{\cos\theta\left(3\cos^2\theta_0 - \cos^2\theta\right) + \dfrac{3\bar{\lambda}}{\xi}\dfrac{\sin^2\theta}{\sin\theta_0}\left(\cos^2\theta - \cos^2\theta_0\right)}{2\pi\,(1-\cos\theta_0)^2\,(1+2\cos\theta_0)}. \tag{3.9}$$

The dimensionless velocity can be obtained from (3.9) by applying the dimensionless variant of the formulae (S.80)–(S.81):

$$U_\xi = \frac{1}{\xi^2\sin\theta}\frac{\partial\psi}{\partial\theta} = \frac{\rho^2 Q}{\eta^2}V_r, \tag{3.10}$$

$$U_\theta = -\frac{1}{\xi\sin\theta}\frac{\partial\psi}{\partial\xi} = \frac{\rho^2 Q}{\eta^2}V_\theta \tag{3.11}$$

or, what is the same, by substituting (3.7) and (3.8) directly into (2.47) and (2.48):

$$U_\xi(\xi,\theta) \approx 3\frac{\cos^2\theta - \cos^2\theta_0 - \dfrac{2\bar{\lambda}}{\xi}\dfrac{\cos\theta}{\sin\theta_0}\left(\cos^2\theta_0 - 2\cos^2\theta + 1\right)}{2\pi\xi^2\,(1-\cos\theta_0)^2\,(1+2\cos\theta_0)}, \tag{3.12}$$

$$U_\theta(\xi,\theta) \approx \frac{3\bar{\lambda}\sin\theta\left(\cos^2\theta - \cos^2\theta_0\right)}{2\pi\xi^3\sin\theta_0\,(1-\cos\theta_0)^2\,(1+2\cos\theta_0)}. \tag{3.13}$$

Similarly, one can obtain dimensionless versions of pressure (2.49) and vorticity (2.51):

$$\beta(\xi,\theta) = \frac{\rho^3 Q^2}{\eta^4}p \approx \frac{3\cos^2\theta - 1 + \dfrac{3\bar{\lambda}}{\xi}\dfrac{\cos\theta}{\sin\theta_0}\left(5\cos^2\theta - 3\right)}{\pi\xi^3\,(1-\cos\theta_0)^2\,(1+2\cos\theta_0)}, \tag{3.14}$$

$$\phi(\xi,\theta) = \frac{\rho^3 Q^2}{\eta^3}\phi \approx \frac{3\sin\theta}{\pi^2\xi^3\,(1-\cos\theta_0)^2\,(1+2\cos\theta_0)}\left(\cos\theta + \frac{\bar{\lambda}}{\xi\sin\theta_0}\left(5\cos^2\theta - 1\right)\right). \tag{3.15}$$

The conditions for the satisfiability of the obtained dimensionless equations, taking into account inequalities (2.35) and (3.2), are written as

$$\xi \gg 1, \quad \xi \gg \frac{\eta\lambda}{\rho Q} \quad \text{and} \quad \frac{\bar{\lambda}}{\xi} \ll 1. \tag{3.16}$$

The second way to non-dimensionalise the radial coordinate r involves using $\lambda \neq 0$, so that the dimensionless coordinate takes the form r/λ. Obviously, the dimensionless equations corresponding to this choice can be obtained from expressions (3.9) and (3.12)–(3.15) by substitution $\bar{\lambda} = 1$, which is a special case of (3.7) and (3.8).

The streamlines with dimensionless slip lengths $\bar{\lambda} = 0.5$ and $\bar{\lambda} = 1$ at polar angles $\theta_0 = \pi/6$ (30°) and $\theta_0 = 8\pi/9$ (160°) constructed on the basis of (3.9) are visualised in figure 2.

Figure 2 shows that a non-zero slip length leads to the occurrence of a vortex in the liquid flow, which increases with increasing parameter $\bar{\lambda}$. The qualitative difference between the flow at $\bar{\lambda} > 0$ and the flow at $\bar{\lambda} = 0$ is that in the first case the polar component of velocity, V_θ, is non-zero, while in the second case (3.4) is satisfied, so that the liquid flows strictly radially from the apex.

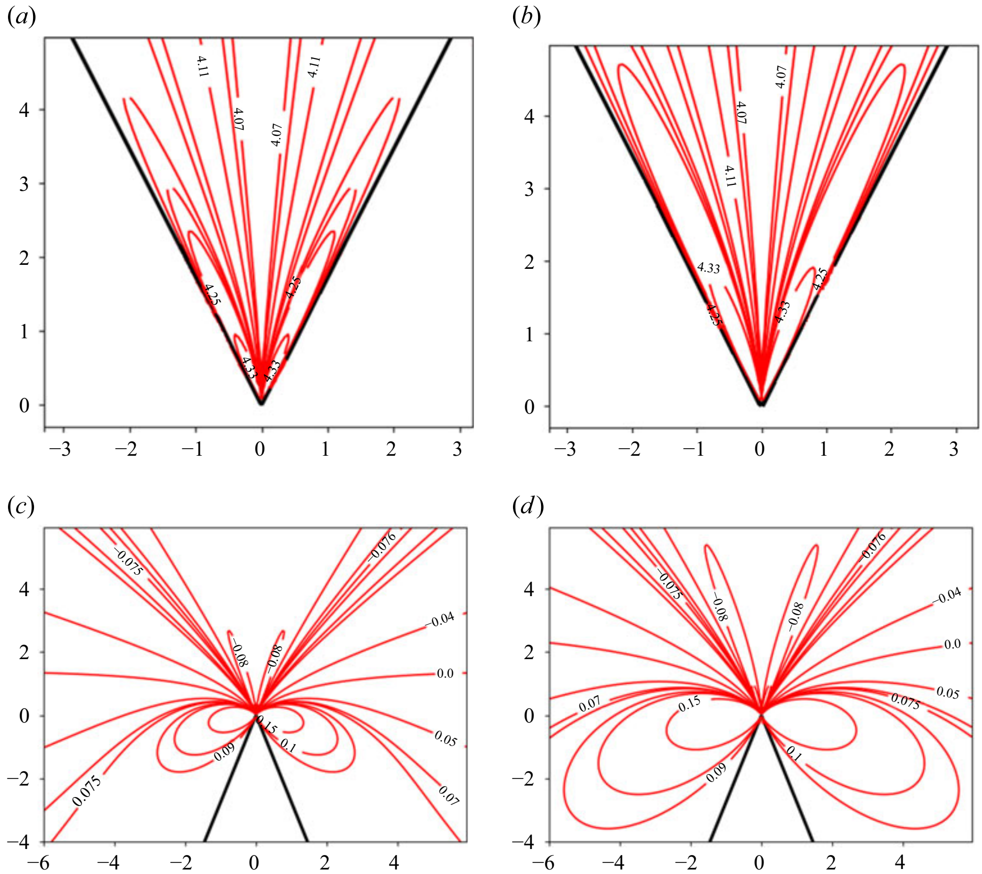

Figure 2. Streamlines in a cone given by dimensionless stream function (3.9) for polar angle $\theta_0 = \pi/6$ (a,b) and $\theta_0 = 8\pi/9$ (c,d), with dimensionless slip length $\bar{\lambda} = 0.5$ (a,c) and $\bar{\lambda} = 1$ (b,d).

In addition, the streamlines determined without removing the units of measure from (2.47), (2.48) and (2.50) are visualised in supplementary material II.

To obtain a solution suitable for $r \sim \lambda$ ($\xi \sim \bar{\lambda}$), it is necessary to take into account the following terms in the expansion of the velocity and pressure in a series in λ/r. To do this, it is necessary to use the recurrence relations (2.24)–(2.26) to consequentially determine the coefficients β_l and δ_l, corresponding to increasingly higher values of l.

An experiment with a conical diffuser having a polar angle θ_0 greater than 90° showed that the direction of fluid motion at angles θ_0 of about 90° differs slightly from straight lines; however, for angles $\theta_0 \sim 141° - 160°$ the flow at large distances from the cone apex has a significantly vortex structure (Bond 1925). Our study suggests that one of the probable reasons for this may be the violation of no-slip conditions on the working surface of the cone.

4. Conclusion

An alternative form of the general solution of the linearised, stationary, axisymmetric Navier–Stokes equations for an incompressible fluid in spherical coordinates has been obtained (table 1). The previously published solution of this problem (Sampson 1891;

Savic 1953; Haberman & Sayre 1958; Happel & Brenner 1983) is given in terms of the stream function, which leads to formulae that are quite complex for practical application. From a mathematical point of view, it is more 'natural' to use the representation of velocity through a vector potential, since this allows one to directly apply the well-developed apparatus of polynomials and associated Legendre functions. At the same time, both approaches are mathematically equivalent, which is proven by both general considerations and specific calculations.

The new form of the solution is applied to the problem of fluid flow through a conical diffuser under boundary conditions of partial slip for an arbitrary slip length. For the first time, an analytical solution has been derived for Stokes flow through a conical diffuser under the condition of partial slip. Recurrent relations are obtained that allow determination of the velocity, pressure and stream function for a certain slip length λ. The solution has been analysed in the first order of decomposition with respect to a small dimensionless parameter λ/r. In particular, for $\lambda = 0$ we obtain the limiting case, the well-known solution by Harrison to the problem of a diffuser with no-slip boundary conditions (Happel & Brenner 1983; Harrison 1920; Slezkin 1955; Batchelor 2000).

The considered example shows that the solution given in table 1 and obtained in supplementary material I allows us successfully to solve quite complex boundary problems. It is easy to verify that the proposed general solution, when substituted into the corresponding boundary conditions, allows us to obtain correct expressions for such test cases as the Stokes drag force, the Hadamard–Rybczynski equation (describing the motion of a spherical drop of liquid in another external liquid), etc. Successful verification of these equations allows table 1 to be recommended for further use.

The generalisation of the stream function to the three-dimensional non-axisymmetric region is the vector potential. Thus, the first step towards obtaining a general solution to the Stokes equations would be to obtain a general solution to the axisymmetric problem in representation of the vector potential. This is exactly what is done in the proposed work and can be found in supplementary material I.

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Appendix A. Proof of the constancy of the flow rate Q in each section of the cone

The flow rate of liquid passing through the diffuser is determined by the formula

$$Q = 2\pi r^2 \int_0^{\theta_0} V_r(r, \theta) \sin\theta \, \mathrm{d}\theta. \tag{A1}$$

The flow rate must be constant in any section of the cone: $Q = \text{const}$. Then

$$\int_0^{\theta_0} V_r(r, \theta) \sin\theta \, \mathrm{d}\theta \sim \frac{1}{r^2}. \tag{A2}$$

Let us verify that this constancy of the flow rate across the cone cross-section actually takes place.

Substituting V_r from table 1 in (A1), one can obtain

$$Q = 2\pi r^2 \sum_{l=1}^{\infty} l(l+1) \left\{ \frac{\beta_l}{4l-2} r^{-l} + \delta_l r^{-l-2} \right\} \int_1^{\cos\theta_0} P_l(t)\mathrm{d}t + 2\pi\delta_0 \int_0^{\theta_0} \sin\theta \, \mathrm{d}\theta, \tag{A3}$$

where $\int_1^{\cos\theta_0} P_l(t)\mathrm{d}t$, in general, are some non-zero constants.

Let us break the sum on the right-hand side of (A3) into terms as follows:

$$\sum_{l=1}^{\infty} l(l+1)\left\{\frac{\beta_l r^{-l}}{4l-2} + \delta_l r^{-l-2}\right\}\int_1^{\cos\theta_0} P_l(t)\mathrm{d}t = \frac{\beta_1}{r}\int_1^{\cos\theta_0} P_1(t)\mathrm{d}t + \beta_2 r^{-2}\int_1^{\cos\theta_0} P_2(t)\mathrm{d}t$$
$$+\sum_{l=1}^{\infty}(l+2)(l+3)\frac{\beta_{l+2} r^{-l-2}}{4l+6}\int_1^{\cos\theta_0} P_{l+2}(t)\mathrm{d}t + \sum_{l=1}^{\infty} l(l+1)\delta_l r^{-l-2}\int_1^{\cos\theta_0} P_l(t)\mathrm{d}t. \tag{A4}$$

In the penultimate sum on the right-hand side of (A4) the starting point for l is shifted from 3 to 1 to synchronise it with the last sum. Combining the sums, we have

$$\sum_{l=1}^{\infty} l(l+1)\left\{\frac{\beta_l r^{-l}}{4l-2} + \delta_l r^{-l-2}\right\}\int_1^{\cos\theta_0} P_l(t)\mathrm{d}t = \frac{\beta_1}{r}\int_1^{\cos\theta_0} P_1(t)\mathrm{d}t + \beta_2 r^{-2}\int_1^{\cos\theta_0} P_l(t)\mathrm{d}t$$
$$+\sum_{l=1}^{\infty}\left\{(l+2)(l+3)\frac{\beta_{l+2}}{4l+6}\int_1^{\cos\theta_0} P_{l+2}(t)\mathrm{d}t + l(l+1)\delta_l \int_1^{\cos\theta_0} P_l(t)\mathrm{d}t\right\} r^{-l-2}. \tag{A5}$$

Obviously, to ensure (A2), the following relations must be satisfied:

$$\beta_1 \int_1^{\cos\theta_0} P_1(t)\mathrm{d}t = 0, \tag{A6}$$

$$(l+2)(l+3)\frac{\beta_{l+2}}{4l+6}\int_1^{\cos\theta_0} P_{l+2}(t)\mathrm{d}t + l(l+1)\delta_l \int_1^{\cos\theta_0} P_l(t)\mathrm{d}t = 0. \tag{A7}$$

To calculate $\int_1^{\cos\theta_0} P_l(t)\mathrm{d}t$, one can use the differential equation for Legendre polynomials (Arfken *et al.* 2012):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(1-t^2\right)\frac{\mathrm{d}P_l(t)}{\mathrm{d}t}\right] = -l(l+1)P_l(t). \tag{A8}$$

Therefore,

$$\int_1^{\cos\theta_0} P_l(t)\mathrm{d}t = -\frac{(1-t^2)}{l(l+1)}\left.\frac{\mathrm{d}P_l(t)}{\mathrm{d}t}\right|_1^{\cos\theta_0}. \tag{A9}$$

Let us use the definition of the first-order associated Legendre polynomials (Arfken *et al.* 2012):

$$P_l^1(t) = -\sqrt{1-t^2}\frac{\mathrm{d}P_l(t)}{\mathrm{d}t}. \tag{A10}$$

Substituting (A10) in (A9), one can obtain

$$\int_1^{\cos\theta_0} P_l(t)\mathrm{d}t = \frac{\sqrt{1-t^2}}{l(l+1)}\left. P_l^1(t)\right|_1^{\cos\theta_0} = \frac{\sin\theta_0}{l(l+1)}P_l^1\left(\cos\theta_0\right). \tag{A11}$$

Substituting (2.14) into conditions (A6) and (A7), we obtain (for $\sin\theta_0 \neq 0$) exactly the conditions (2.13), which were already taken into account in solving the problem and mean the impermeability of the cone wall. Hence (A4) takes the form

$$\sum_{l=1}^{\infty} l(l+1)\left\{\frac{\beta_l r^{-l}}{4l-2} + \delta_l r^{-l-2}\right\}\int_1^{\cos\theta_0} P_l(t)\mathrm{d}t = \beta_2 r^{-2}\int_1^{\cos\theta_0} P_2(t)\mathrm{d}t. \tag{A12}$$

Substituting (A12) into (A3), one can get

$$Q = 2\pi\beta_2\int_1^{\cos\theta_0} P_2(t)\mathrm{d}t + 2\pi\delta_0\int_0^{\theta_0}\sin\theta\,\mathrm{d}\theta. \tag{A13}$$

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