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Sharp Threshold of the Gross–Pitaevskii Equation with Trapped Dipolar Quantum Gases

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Abstract. In this paper, we consider the Gross–Pitaevskii equation for the trapped dipolar quantum gases. We obtain the sharp criterion for the global existence and finite time blow-up in the unstable regime by constructing a variational problem and the so-called invariant manifold of the evolution flow.

1 Introduction

In this paper, we consider the time-dependent Gross–Pitaevskii equation in R^3:

$$i\partial_t u + \frac{1}{2}\Delta u = Vu + \lambda_1 |u|^2 u + \lambda_2 (K * |u|^2)u \tag{1.1}$$

with the regular Cauchy data

$$u(0,x) = u_0(x), \quad x \in R^3, \tag{1.2}$$

where λ_1, λ_2 are real constants,

$$V(x) = \frac{1}{2}|x|^2, \tag{1.3}$$

$$K(x) = \frac{x_1^2 + x_2^2 - 2x_3^2}{|x|^5}, \quad x \in R^3, \tag{1.4}$$

and

$$K * |u|^2(x) = \int_{R^3} K(x-y)|u(y)|^2\, dy.$$

The evolution equation of type (1.1) receives a lot of attention because the success of atomic Bose–Einstein condensation (BEC) has stimulated great interest in the properties of trapped quantum gases. The first variant of equation (1.1) was introduced by Yi and You in [8] to describe particles which interact via short-range repulsive forces and long-range (partly attractive) dipolar forces. Physically, according

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to [8], the non-local term $K * |u|^2$ in (1.1) appears when an electric field is introduced along the positive z-axis. Recent developments in the manipulation of such ultra-cold atoms have paved the way towards Bose–Einstein condensation in atomic gases where dipole-dipole interactions between the particles are important. See, for example, [6].

A very important problem for equation (1.1) with Cauchy data is to find the threshold conditions for the initial data. That is, under which conditions on the initial data the solution is global or otherwise blowing up. R. Carles, P. Markowich, and C. Sparber have established in [2] the local wellposedness of the Cauchy problem for equation (1.1). Moreover, they have proved that the solution is global if $\lambda_1 \geq \frac{4\pi}{3}\lambda_2 \geq 0$ (Corollary 2.2 in Section 2). However, when $\lambda_1 < \frac{4\pi}{3}\lambda_2$ (the so-called unstable regime), there is no such result, though global existence of the solution for small data can be obtained in this case. The purpose of our paper is to derive sharp criteria for global existence and blow-up of the solutions of (1.1) in the unstable regime. All of our results are illustrated in Theorems 4.1 and 4.2.

The idea is to construct a constrained variational problem and a so-called invariant set, which was proposed by H. Berestycki and Th. Cazenave [1] and M. I. Weinstein [5] for the nonlinear Schrodinger equations. We shall use the method that was developed by J. Zhang [9] and L. Ma and L. Zhao [4]. For a more general discussion of this method, we refer the readers to [4]. A different consideration of (1.1) has been carried out in our paper [3].

Our paper is organized as follows. In the second section, we give some necessary preliminaries. In the third section, we solve a variational problem. In the last section, we prove the sharp threshold for blowing up and global existence of solutions to (1.1).

2 Preliminaries

The following two important qualities are, at least formally, conserved by the time evolution equation (1.1):

$$\text{(2.1)} \qquad \text{Mass: } M(u) = \|u\|_{L^2}^2 = M(u_0),$$

$$\text{(2.2)} \qquad \text{Energy: } E(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{1}{2}\int_{R^3} |x|^2\,|u|^2 + \frac{\lambda_1}{2}\|u\|_{L^4}^4 + \frac{\lambda_2}{2}\int_{R^3} K * |u|^2\,|u|^2 = E(u_0).$$

The mass and energy conservations lead to the introduction of a natural energy space associated with equation(1.1) in the linear case $\lambda_1 = \lambda_2 = 0$:

$$\text{(2.3)} \qquad \Sigma := \{u \in H^1(R^3) : \|xu\|_{L^2}^2 < \infty\}.$$

It is obvious that Σ is a Hilbert space with the inner product

$$\langle u, v\rangle_\Sigma := \int_{R^3} u\bar{v} + \int_{R^3} \nabla u\overline{\nabla v} + \int_{R^3} |x|^2 u\bar{v}$$

for $u, v \in \Sigma$.

The following results have been proved in [2] and we just quote them.

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Proposition 2.1 ([2]) *There exists a $T_* \in R_+ \cup \{\infty\}$ such that* (1.1) *has a unique maximal solution in*

$$\Big\{ u \in C\big([0,T_*);\Sigma\big) : u, \nabla u, xu \in C\big([0,T_*);L^2(R^3)\big) \cap L^{\frac{8}{3}}_{\mathrm{loc}}\big([0,T_*);L^4(R^3)\big) \Big\}$$

such that $u(0) = u_0$. The solution u is maximal in the sense that if $T_* < \infty$, then

$$\|\nabla u(t)\|_{L^2} \to \infty \quad (t \to T_*).$$

Moreover, the quantities defined as (2.1) *and* (2.2) *are conserved for $0 \le t < T_*$.*

As an easy consequence of Proposition 2.1, we have the following corollary.

Corollary 2.2 ([2]) *Under the same assumptions as in Proposition 2.1 and in addition the assumption that $\lambda_1 \ge \frac{4\pi}{3} \ge 0$, we have that $\|\nabla u(t)\|_{L^2}$ is bounded. Thus the solution u is global in time.*

To understand the behavior of the solution at large time, we need this proposition.

Proposition 2.3 ([2]) *Let $u_0 \in \Sigma$ and u be the solution of the Cauchy problem of the equation* (1.1) *in Proposition 2.1. Put $y(t) = \int_{R^3} |x|^2 |u|^2$. Then one has*

$$\ddot{y}(t) = 2\int_{R^3} |\nabla u|^2 + 3\lambda_1 \int_{R^3} |u|^4 + 3\lambda_2 \int_{R^3} K * |u|^2 \, |u|^2 - 2\int_{R^3} |x|^2 |u|^2. \tag{2.4}$$

The following well-known compactness lemma plays an important role in our proof.

Lemma 2.4 ([2]) *The energy space Σ is compact in $L^p(R^3)$ for any $2 \le p < 6$.*

We denote the Fourier transform of f by

$$\hat{f}(\xi) = \int_{R^3} e^{-ix\xi} f(x)\, dx. \tag{2.5}$$

Then the famous Plancherel formula gives us that

$$\int_{R^3} |f|^2 = \frac{1}{(2\pi)^3} \int_{R^3} |\hat{f}|^2. \tag{2.6}$$

The next lemma is also due to R. Carles, P. Markowich, and C. Sparber [2].

Lemma 2.5 *The Fourier transform of K is given by*

$$\hat{K}(\xi) = \frac{4\pi}{3}\Big(3\frac{\xi_3^2}{|\xi|^2} - 1\Big). \tag{2.7}$$

Remark 2.6 We point out that the formula in Lemma 2.5 can also be obtained directly from [7, Theorem 5, p. 73]. Moreover, $\hat{K}$ is a $L^p(R^3) \to L^p(R^3)$ $(1 < p < \infty)$ multiplier by the Calderon–Zygmund theory (see also [2]).

We shall use c to denote various uniform constants.

3 The Variational Problem

From now on, we assume $\sigma > 0$, $\lambda_1 < \frac{4\pi}{3}\lambda_2$.

We define the following two functionals on the space Σ:

$$S(u) = \frac{\sigma}{2}\int_{R^3}|u|^2 + \frac{1}{4}\int_{R^3}|\nabla u|^2 + \frac{1}{4}\int_{R^3}|x|^2|u|^2 + \frac{\lambda_1}{4}\int_{R^3}|u|^4 + \frac{\lambda_2}{4}\int_{R^3}K*|u|^2|u|^2 = \frac{\sigma}{2}M(u) + \frac{1}{2}E(u), \tag{3.1}$$

$$R(u) = \frac{1}{2}\int_{R^3}|\nabla u|^2 + \frac{1}{2}\int_{R^3}|x|^2|u|^2 + \frac{3}{4}\lambda_1\int_{R^3}|u|^4 + \frac{3}{4}\lambda_2\int_{R^3}K*|u|^2|u|^2 = E(u) + \frac{\lambda_1}{4}\int_{R^3}|u|^4 + \frac{\lambda_2}{4}\int_{R^3}K*|u|^2|u|^2. \tag{3.2}$$

We now consider the following constrained variational problem:

$$m = \inf\{S(u) : u \in M\}, \tag{3.3}$$

where M is defined as

$$M = \left\{ u \in \Sigma \setminus \{0\} : R(u) = 0 \right\}.$$

At first, we shall prove that M is non-empty, which is asserted by Lemma 3.1 below. Though some arguments of our proof is contained in [2], we give the details for completeness.

Lemma 3.1 *M is non-empty. That is, there exists $u \in \Sigma \setminus \{0\}$ satisfying $R(u) = 0$.*

Proof Let $u_\epsilon(x) = \epsilon^{\frac{\alpha}{2}}v(x_1,x_2)w(\epsilon x_3)$, where v and w are two Schwartz functions for some small constant ϵ to be determined. By direct computation, we get

$$\int_{R^3}|\nabla u_\epsilon|^2 = \epsilon^{\alpha-1}\int_{R^3}|\nabla v(x_1,x_2)|^2|w(x_3)|^2 + \epsilon^{\alpha+1}\int_{R^3}|v(x_1,x_2)|^2|\nabla w(x_3)|^2, \tag{3.4}$$

$$\int_{R^3}|x|^2|u_\epsilon|^2 = \epsilon^{\alpha-3}\int_{R^3}x_3^2|v(x_1,x_2)|^2|w(x_3)|^2 + \epsilon^{\alpha-1}\int_{R^3}(x_1^2+x_2^2)|v(x_1,x_2)|^2|w(x_3)|^2. \tag{3.5}$$

If we define V and W as the Fourier transform of $|v|^2$ and $|w|^2$ respectively, we obtain by the Plancherel formula (2.6):

$$\int_{R^3} |u_\epsilon|^4 = \epsilon^{2\alpha-1}\frac{1}{(2\pi)^3}\int_{R^3} |V|^2\,|W|^2, \tag{3.6}$$

$$\begin{aligned}\int_{R^3} K * |u_\epsilon|^2\,|u_\epsilon|^2 &= \epsilon^{2\alpha-1}\frac{1}{(2\pi)^3}\int_{R^3} 3\frac{\epsilon^2\xi_3^2}{\xi_1^2+\xi_2^2+\epsilon^2\xi_3^2}|V|^2\,|W|^2\\ &\quad -\frac{4\pi}{3}\epsilon^{2\alpha-1}\frac{1}{(2\pi)^3}\int_{R^3}|V|^2\,|W|^2\\ &= o(\epsilon^{2\alpha-1})-\frac{4\pi}{3}\epsilon^{2\alpha-1},\end{aligned} \tag{3.7}$$

where in the last equality we have used the Lebesgue Dominated Convergence theorem and Lemma 2.5.

Therefore we have

$$R(u_\epsilon) \approx \frac{1}{2}\epsilon^{\alpha-1}+\frac{1}{2}\epsilon^{\alpha-3}+\frac{3}{4}\Big(\lambda_1-\frac{4\pi}{3}\lambda_2\Big)\,\epsilon^{2\alpha-1}+o(\epsilon^{2\alpha-1}).$$

Hence we get $R(u_\epsilon) < 0$ provided we choose ϵ small and $\alpha < -2$ (so the leading order term is the third one).

For such u_ϵ, we define $u_\epsilon^\mu = \mu u_\epsilon$. Then we obtain

$$R(u_\epsilon^\mu) \approx \mu^2+\Big(\lambda_1-\frac{4\pi}{3}\lambda_2\Big)\,\mu^4.$$

Hence for $\mu < 1$ small, we have $R(u_\epsilon^\mu) > 0$. It is obvious that $R(\mu u_\epsilon)$ is continuous with respect to the variable μ. Then for some $\mu_0 \in (0,1)$, $R(u_\epsilon^{\mu_0}) = 0$. This implies that M is non-empty. ∎

After proving Lemma 3.1, we turn to prove some positivity property of the functional S. More precisely, we have the following.

Lemma 3.2 *$S(u)$ is uniformly bounded from below; that is, for any $u \in M$, $S(u) \geq c > 0$ for some uniform constant $c > 0$.*

Proof For $u \in M$, we have $R(u) = 0$ and then

$$\int_{R^3} |\nabla u|^2 + \int_{R^3} |x|^2\,|u|^2 = -\frac{3}{2}\lambda_1\int_{R^3}|u|^4 - \frac{3}{2}\lambda_2\int_{R^3} K * |u|^2\,|u|^2. \tag{3.8}$$

By the Holder inequality, we have

$$\Big|\int_{R^3} K * |u|^2\,|u|^2\Big| \leq \big\|K * |u|^2\big\|_{L^2(R^3)}\,\big\||u|^2\big\|_{L^2(R^3)}. \tag{3.9}$$

Remark 2.6 implies

$$\big\|K * |u|^2\big\|_{L^2(R^3)} \leq \big\||u|^2\big\|_{L^2(R^3)} = \|u\|^2_{L^4(R^3)}. \tag{3.10}$$

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Combining the inequalities (3.8), (3.9) and (3.10), we get

$$\int_{R^3} |\nabla u|^2 + \int_{R^3} |x|^2 |u|^2 \le C \int_{R^3} |u|^4, \tag{3.11}$$

which, via the Gagliardo–Nirenberg inequality, leads to

$$\int_{R^3} |\nabla u|^2 + \int_{R^3} |x|^2 |u|^2 \le C\|u\|_{L^2} \|\nabla u\|_{L^2}^3. \tag{3.12}$$

Recall that by the uncertainty principle,

$$\int_{R^3} |u|^2 \le C\Big(\int_{R^3} |x|^2 |u|^2\Big)^{\frac{1}{2}} \Big(\int_{R^3} |\nabla u|^2\Big)^{\frac{1}{2}}. \tag{3.13}$$

We then have by the Hölder inequality that

$$\int_{R^3} |\nabla u|^2 + \int_{R^3} |x|^2 |u|^2 \le C\Big(\int_{R^3} |\nabla u|^2 + \int_{R^3} |x|^2 |u|^2\Big)^2, \tag{3.14}$$

which insures that

$$\int_{R^3} |\nabla u|^2 + \int_{R^3} |x|^2 |u|^2 \ge c > 0 \tag{3.15}$$

for some constant $c > 0$.

When $R(u) = 0$, $S(u)$ is reduced to

$$S(u) = \frac{\sigma}{2} \int_{R^3} |u|^2 + \frac{1}{12} \int_{R^3} |\nabla u|^2 + \frac{1}{12} \int_{R^3} |x|^2 |u|^2. \tag{3.16}$$

Hence

$$S(u) \ge c > 0 \tag{3.17}$$

for some uniform constant $c > 0$. ■

Lemma 3.2 shows that $m > 0$ and the following lemma will show that m can be attained for some $u \in M$, that is, $m = \min\{S(u) : u \in M\}$.

Lemma 3.3 *There exists at least one $u \in M$ for which $m = S(u)$.*

Proof We firstly claim that

$$m = \inf\big\{ \tilde{S}(u) : u \in \Sigma \setminus \{0\}, R(u) \le 0 \big\}, \tag{3.18}$$

where

$$\tilde{S}(u) := \frac{\sigma}{2} \int_{R^3} |u|^2 + \frac{1}{12} \int_{R^3} |\nabla u|^2 + \frac{1}{12} \int_{R^3} |x|^2 |u|^2. \tag{3.19}$$

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In fact, if we denote the right hand side of (3.18) by $\tilde{m}$, then we have that $m \le \tilde{m}$.

On the other hand, for $u \in \Sigma$ with $R(u) < 0$, we let $u^{\mu} = \mu u$. Then

$$R(u^{\mu}) = \mu^2 \frac{1}{2} \int_{R^3} |\nabla u|^2 + \mu^2 \frac{1}{2} \int_{R^3} |x|^2 |u|^2 + \mu^4 \frac{3}{4} \lambda_1 \int_{R^3} |u|^4 + \mu^4 \frac{3}{4} \lambda_2 \int_{R^3} K * |u|^2 |u|^2, \tag{3.20}$$

which can be written as

$$R(u^{\mu}) \approx \mu^2 - \mu^4. \tag{3.21}$$

Then $R(u^{\mu})$ takes values from positive to negative as μ varies from 0 to 1. Hence for some $0 < \mu_0 < 1$,

$$R(u^{\mu_0}) = 0.$$

From the definition of m, we have

$$m \le S(u^{\mu_0}) = \tilde{S}(u^{\mu_0}) < \tilde{S}(u).$$

We then deduce that $\tilde{m} \le m$.

Now we choose u_n satisfying

$$R(u_n) \le 0, \quad \tilde{S}(u_n) \to m \quad (n \to \infty).$$

From (3.19), we know that u_n is bounded in Σ. Then the compactness lemma (Lemma 2.4) implies that up to a subsequence,

$$u_n \to u \tag{3.22}$$

in $L^p(R^3)$ for $2 \le p < 6$. Hence we have

$$\int_{R^3} |u_n|^4 \to \int_{R^3} |u|^4, \tag{3.23}$$

$$\int_{R^3} K * |u_n|^2 |u_n|^2 \to \int_{R^3} K * |u|^2 |u|^2. \tag{3.24}$$

By the lower semi-continuity of the Σ norm, we have

$$\int_{R^3} |\nabla u|^2 \le \liminf_{n \to \infty} \int_{R^3} |\nabla u_n|^2, \tag{3.25}$$

$$\int_{R^3} |x|^2 |u|^2 \le \liminf_{n \to \infty} \int_{R^3} |x|^2 |u_n|^2. \tag{3.26}$$

From (3.23), (3.24), (3.25), (3.26), we deduce that

$$R(u) \le 0$$

and

$$\tilde{S}(u) \leq \liminf_{n\to\infty} \tilde{S}(u_n) = m. \tag{3.27}$$

We claim that $u \neq 0$. Assume we have proved this claim. By considering (3.18), we get that

$$\tilde{S}(u) = m; \tag{3.28}$$

moreover,

$$R(u) = 0, \tag{3.29}$$

since otherwise a similar argument as before shows there exists $0 < \mu_0 < 1$ such that $R(u^{\mu_0}) = 0$, $m \leq S(u^{\mu_0}) = \tilde{S}(u^{\mu_0}) < \tilde{S}(u) = m$, which leads to a contradiction. Thus u is a minimizer of the variational problem 3.3.

We now prove the claim. If not, we then have, by (3.22), $u_n \to 0$ in $L^p(R^3)$, $2 \leq p < 6$. Then (3.11) implies

$$\int_{R^3} |\nabla u_n|^2 + \int_{R^3} |x|^2 |u_n|^2 \to 0, \tag{3.30}$$

which contradicts (3.15). Thus the proof is complete. ■

4 Sharp Threshold Result

In this section, we shall prove the sharp threshold result for the equation (1.1).

We now introduce two sets

$$K^+ = \{u \in \Sigma : S(u) < m, R(u) > 0\}, \tag{4.1}$$

$$K^- = \{u \in \Sigma : S(u) < m, R(u) < 0\}. \tag{4.2}$$

It is not hard to see, by considering $S(\mu u)$ and $R(\mu u)$ (for fixed $u \in \Sigma$) as smooth functions of the variable μ, that the set $\{u \in \Sigma : S(u) < m\}$ (and K_-) is non-empty.

Theorem 4.1 *K^+ and K^- are two invariant sets of the flow generated by the equation (1.1). More precisely, for any $u_0 \in K^+$, if u is the solution to (1.1) with the initial data u_0, then $u(t) \in K^+$ for any $t \in I$, where I is the maximal existence time interval of the solution u. So is K^-.*

Proof By Proposition 2.1, we have

$$S\big(u(t)\big) = S(u_0). \tag{4.3}$$

Thus from $S(u_0) < m$, it follows that

$$S\big(u(t)\big) < m. \tag{4.4}$$

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To check $u(t) \in K^+$, we need to show that

$$R\big(u(t)\big) > 0. \tag{4.5}$$

If this is not true, then by the continuity of u and the fact that $R\big(u(0)\big) = R(u_0) > 0$, we have for some t_1

$$R\big(u(t_1)\big) = 0. \tag{4.6}$$

It follows that $u(t_1) \in M$. At the same time, we have

$$S\big(u(t_1)\big) < m. \tag{4.7}$$

This is impossible from the definition of m. Similarly we can prove that K^- is invariant under the flow generated by the equation (1.1). ∎

With the help of Theorem 4.1, we obtain the following sharp global existence and blow-up criterion for (1.1).

Theorem 4.2 *Assume that $\lambda_1 < \frac{4\pi}{3}\lambda_2$. If $u(t) \in \Sigma$ is the maximal solution to* (1.1) *with the initial data $u_0 \in \Sigma$ satisfying $S(u_0) < m$, then we have that*
(i) *u is global for the initial data $u_0 \in K^+$;*
(ii) *u blows up at finite time for the initial data $u_0 \in K^-$.*

Proof (i) If $u_0 \in K^+$, then by Theorem 4.1, we conclude that $u(t) \in K^+$. That is, $R\big(u(t)\big) > 0$, which implies that

$$\frac{\lambda_2}{4}\int_{R^3} K * |u|^2\,|u|^2 > -\frac{1}{6}\int_{R^3} |\nabla u|^2 - \frac{1}{6}\int_{R^3} |x|^2\,|u|^2 - \frac{1}{4}\lambda_1 \int_{R^3} |u|^4. \tag{4.8}$$

By Theorem 4.1, we have

$$S\big(u(t)\big) < m. \tag{4.9}$$

From (4.8) and (4.9), we conclude that

$$\frac{\sigma}{2}\int |u|^2 + \frac{1}{12}\int_{R^3} |\nabla u|^2 + \frac{1}{12}\int_{R^3} |x|^2\,|u|^2 < m. \tag{4.10}$$

This, by Proposition 2.1, implies the global result of the solution flow $u(t)$.

(ii) If $u_0 \in K^-$, then by Theorem 4.1, we conclude that $u(t) \in K^-$, that is, $R\big(u(t)\big) < 0$. Then for any fixed t, there exists $0 < \mu < 1$ such that

$$R\big(\mu u(t)\big) = 0, \tag{4.11}$$

which implies via the definition of m that

$$S(\mu u) \geq m. \tag{4.12}$$

We compute by the inequality (4.11) that

$$S(u) - S(\mu u) = \frac{1}{2}R(u) + \frac{\sigma}{2}(1-\mu^2)\int_{R^3}|u|^2. \tag{4.13}$$

By the viral identity (Proposition 2.3), we have

$$\frac{d^2}{dt^2}\int_{R^3}|x|^2\,|u|^2 = 4R(u) - 4\int_{R^3}|x|^2\,|u|^2. \tag{4.14}$$

Then we have that

$$\frac{d^2}{dt^2}\int_{R^3}|x|^2\,|u|^2 < 4R(u). \tag{4.15}$$

From (4.13), (4.15) and (4.12), we conclude

$$\frac{d^2}{dt^2}\int_{R^3}|x|^2\,|u|^2 < 8\big(S(u) - m\big) = 8\big(S(u_0) - m\big) < 0. \tag{4.16}$$

This implies the blow-up of the solution flow $u(t)$ at some finite time. This completes the proof of Theorem 4.2. ∎

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