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EMBEDDING THEOREMS FOR COUNTABLE GROUPS

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1. Introduction. A group P is said to be a CEF-group if, for every countable group G, there is a factor group of P which contains a subgroup isomorphic to G. It was shown by Higman, Neumann, and Neumann [**5**] that the free group of rank two is a CEF-group. More recently, Levin [**6**] proved that if P is the free product of two cyclic groups, not both of order two, then P is a CEF-group. Later, Hall [**3**] gave an alternative proof of Levin's result.

In this paper we give a new proof of Levin's result (Theorem 2). The proof given yields information about the factor group H of P in which a given countable group G is embedded; for example, if G is given by a recursive presentation (this concept is defined in [**4**]), then a recursive presentation is obtained for H, and certain decision problems (in particular, the word problem) are solvable for the recursive presentation obtained for H if and only if they are solvable for the given recursive presentation of G. We also obtain a characterization of the elements of finite order in H. In Theorem 3 a set of sufficient conditions for a factor group of a free product of groups to be a CEF-group is obtained. This result is applied in Theorem 4 to give a generalization of Levin's result. Theorem 4 includes as a special case the result that if P is a free product of cyclic groups as above, and T is a finite subset of P, none of whose elements is conjugate to an element of length one, then there is a positive integer m such that, if $k \geqq m$ and N is the normal closure in P of the set of kth powers of elements of T, then P/N is a CEF-group.

The technique used in the proof of Theorem 2 was suggested by an unpublished proof, due to J. L. Britton, of the theorem of Higman, Neumann, and Neumann mentioned above. Part (i) and the substance of part (iii) of Theorem 2 were also suggested by Britton's work. Miller and Schupp [**9**] have announced a proof of the result that every countable group can be embedded in a two-generator, complete, Hopf group H, and that H can be chosen to be a factor group of the modular group. Schupp informs me that their proof also uses Britton's technique.

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2. Notation and preliminary results. Let $P = G_1 * G_2 * \ldots * G_t$ be the free product of the (non-trivial) groups $G_1, G_2, \ldots, G_t$. Every non-identity

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element u of P has a unique normal form expression $u = a_1a_2 \ldots a_m$, where $m \geqq 1, 1 \neq a_i \in G_{\gamma(i)}$ $(i = 1, 2, \ldots, m)$ and $\gamma(i) \neq \gamma(i+1)$ $(i = 1, 2, \ldots, m-1)$. We call m the length, $l(u)$, of u and define $l(1) = 0$. The elements $a_1a_2 \ldots a_r$ $(r = 1, 2, \ldots, m)$ are called the initial subwords of u. If $l(u) > 1$ and $\gamma(1) \neq \gamma(m)$, then u is said to be cyclically reduced, and the elements $a_ra_{r+1} \ldots a_ma_1 \ldots a_{r-1}$ $(r = 1, 2, \ldots, m)$ are called the cyclic arrangements of u. We say that u is a proper power (in P) if there exists a non-identity element w of P such that $u = w^q$ for some integer $q > 1$. Otherwise u is said to be rootless (in P).

Let u and v be non-identity elements of P with normal forms $a_1a_2 \ldots a_m$ and $b_1b_2 \ldots b_n$, respectively. We define $\zeta(u, v)$ to be the largest integer r such that $a_1a_2 \ldots a_r = b_1b_2 \ldots b_r$. Thus $\zeta(u, v)$ is the length of the "largest" element of P which is an initial subword of both u and v.

Let Q be a subset of P. We put $l(Q) = \operatorname{Min}\{l(u); u \in Q\}$. If

$$Q = \{u_1, u_2, \ldots\},$$

then we use $l\{u_1, u_2, \ldots\}$ as an alternative notation for $l(Q)$. If there is an upper bound for the lengths of elements of Q, we say that Q is bounded and put $L(Q) = \operatorname{Max}\{l(u); u \in Q\}$. For each positive integer k, the set Q_k is defined to consist of the kth powers of elements of Q. If each element of Q is cyclically reduced, then we say that Q is cyclically reduced and denote by $\bar{Q}$ the set consisting of all cyclic arrangements of elements of Q and their inverses. We say that Q is rootless if each element of Q is rootless. We denote by Q^P the normal closure of Q in P and write $Q_k{}^P$ for $(Q_k)^P$.

Let Q and R be cyclically reduced subsets of P. Then the following results are easily checked.

(1) $\bar{Q}$ is cyclically reduced, Q_k is cyclically reduced, $l(Q) = l(\bar{Q})$, and $l(Q_k) = kl(Q)$.

(2) $(\bar{Q})_k = (\overline{Q_k})$ (this set will be denoted by $\bar{Q}_k$).

(3) Q is rootless if and only if $\bar{Q}$ is rootless.

(4) $\bar{Q}^P = Q^P$.

(5) If $L(Q)$ is defined, then $L(\bar{Q}) = L(Q)$ and $L(Q_k) = kL(Q)$.

(6) $\overline{Q \cup R} = \bar{Q} \cup \bar{R}$.

The λ-*condition.* Let Q be a cyclically reduced subset of P and λ a positive integer. Q is said to satisfy the λ-condition if

$$\zeta(u, v) + 1 < (1/\lambda)l\{u, v\}$$

whenever u and v are non-equal elements of $\bar{Q}$.

The following theorem is part of the result proved by Britton [**2**].

THEOREM 1. *Let Q be a subset of P which satisfies the* 6-*condition. Then* $Q^P \cap G_i = \{1\}$ $(1 \leqq i \leqq t)$.

Thus, if Q satisfies the 6-condition and ρ is the natural homomorphism from P to P/Q^P, then the restriction of ρ to G_i is an isomorphism of G_i into P/Q^P $(1 \leqq i \leqq t)$.

The following elementary result is essential for what follows.

LEMMA 1. *Let u be a cyclically reduced element of P. Then we have the following:*

(1) *There is a unique rootless element v of P such that $u = v^m$ for some positive integer m. The element v, which is cyclically reduced, is called the least root of u;*

(2) *There are exactly n distinct cyclic arrangements of u, where $n = l(v)$;*

(3) *If $u = w^q$, for some positive integer q, then w is a positive power of v.*

Proof. Let u have normal form $a_1a_2 \ldots a_r$ and put $Q = \overline{\{u\}}$. For each integer s we define a permutation σ_s of Q by

$$\sigma_s(b_1b_2 \ldots b_r) = b_{t+1}b_{t+2} \ldots b_rb_1 \ldots b_t,$$

where $b_1b_2 \ldots b_r \in Q$, $s \equiv t \pmod r$ and $0 \leqq t < r$. It is easy to see that the set of all such permutations is a cyclic group of order n, where n is the least positive integer such that $\sigma_n(u) = u$. It is then easy to check that the lemma holds with $v = a_1a_2 \ldots a_n$.

Note. If $w = \sigma_k(u)$, where $k \neq 0$ and r does not divide k, then we say that w is a non-trivial cyclic arrangement of u. It follows from the lemma that if u is rootless, then u is not equal to any of its non-trivial cyclic arrangements.

LEMMA 2. *Let Q be a cyclically reduced, rootless, and bounded subset of P, with $L(Q) = m$ say. If k and λ are positive integers such that $k \geqq \lambda m + 1$, then Q_k satisfies the λ-condition.*

Proof. Let u and v be non-equal elements of $\bar{Q}$, with $l(u) = r$ and $l(v) = s$. Then $l(u^s) = l(v^r) = rs$, and $u^s \neq v^r$, since otherwise the least roots of u^s and v^r would be equal; but clearly u and v are the least roots of u^s and v^r, respectively, since both u and v are rootless. Therefore, if $k \geqq \lambda m + 1$, we have

$$\zeta(u^k, v^k) + 1 \leqq rs \leqq ml\{u, v\} \leqq \frac{(k-1)}{\lambda} l\{u, v\} < \frac{1}{\lambda} l\{u^k, v^k\}.$$

This proves the lemma.

The following is the key result for our embedding technique.

LEMMA 3. *Let $P = G_1 * G_2 * \ldots * G_t$ and $P_1 = G_1 * G_2 * \ldots * G_{t-1}$ $(t \geqq 3)$. Let Q be a finite or countably infinite subset of P_1 with $u_1, u_2, \ldots$ an enumeration of the distinct elements of Q. Let θ be a map from Q to G_t, with $\theta u_i = g_i$ say $(i = 1, 2, \ldots)$. If the subset R of P consists of the elements $g_1^{-1}u_1$, $g_2^{-1}u_2, \ldots,$ and S is a subset of P_1, then the subset $R \cup S$ of P satisfies the λ-condition provided that the following conditions are satisfied:*

(i) *$Q \cup S$ satisfies the λ-condition and $\bar{Q} \cap \bar{S} = \emptyset$;*

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(ii) *Q satisfies the* $(\lambda + 1)$-*condition;*
(iii) *Q is rootless;*
(iv) $l(Q) \geqq \lambda^2 - 1$;
(v) *No* u_i *is a cyclic arrangement of any* $u_j^{\pm 1}$ (*other than the trivial cyclic arrangement of itself*).

Proof. Suppose that the above conditions are satisfied. Let v, w be non-equal elements of $\bar{R} \cup \bar{S}$ such that

$$\zeta(v, w) + 1 \geqq (1/\lambda)l\{v, w\}. \tag{2.1}$$

Then not both v and w can belong to $\bar{S}$, since (i) holds.

We first suppose that $v \in \bar{R}$ and $w \in \bar{S}$. We then have $v = x_1 g x_2$ say, where $g \in G_t$ and $x_1 x_2 \in \bar{Q}$. Now $\zeta(v, w) \leqq \zeta(x_1 x_2, w)$, since the normal form of w does not involve any element of G_t. Hence, by (i),

$$\zeta(v, w) + 1 \leqq \zeta(x_1 x_2, w) + 1 < (1/\lambda)l\{x_1 x_2, w\} \leqq (1/\lambda)l\{v, w\},$$

since $x_1 x_2, w \in \bar{Q} \cup \bar{S}$. This contradicts (2.1). In the same way a contradiction is obtained if $w \in \bar{R}$ and $v \in \bar{S}$.

Thus we can assume that $v, w \in \bar{R}$. We then have $v = x_1 g x_2$ and $w = y_1 h y_2$ say, where $h, g \in G_t$ and $x_1 x_2, y_1 y_2 \in \bar{Q}$. Since the normal forms of v and w each contain exactly one occurrence of an element of G_t (namely g and h, respectively) it is clear that $\zeta(v, w) \leqq \zeta(x_1 x_2, y_1 y_2) + 1$. Now if $x_1 x_2 \neq y_1 y_2$, then

$$\zeta(x_1 x_2, y_1 y_2) + 1 < \frac{1}{\lambda + 1} l\{x_1 x_2, y_1 y_2\},$$

since (ii) holds. Making use of (iv), we have

$$\begin{aligned}
\zeta(v, w) + 1 &< \frac{1}{\lambda + 1} l\{x_1 x_2, y_1 y_2\} + 1 \\
&= \frac{1}{\lambda} l\{x_1 x_2, y_1 y_2\} - \frac{1}{\lambda(\lambda + 1)} l\{x_1 x_2, y_1 y_2\} + 1 \\
&\leqq \frac{1}{\lambda} l\{v, w\} - \frac{1}{\lambda} - \frac{(\lambda^2 - 1)}{\lambda(\lambda + 1)} + 1 \\
&= \frac{1}{\lambda} l\{v, w\},
\end{aligned}$$

and again (2.1) is contradicted. Thus we can assume that $x_1 x_2 = y_1 y_2$, so that $x_2 x_1$ is a cyclic arrangement of $y_2 y_1$. Now $g x_2 x_1, h y_2 y_1 \in \bar{R}$, and so $x_2 x_1, y_2 y_1 \in \bar{Q}$. Since (v) holds, it follows that $x_2 x_1 = y_2 y_1$ and $g = h$. Now if $x_2 \neq y_2$, then $y_2 y_1$ is a non-trivial cyclic arrangement of $x_2 x_1$ (since $x_1 x_2 = y_1 y_2$); but then, by the note to Lemma 1, we could not have $x_2 x_1 = y_2 y_1$, since $x_2 x_1$ is rootless. Hence $x_2 = y_2$, $x_1 = y_1$, and $g = h$, i.e. $v = w$. This contradicts our assumption that v and w are non-equal, and so the lemma has been proved.

3. The main results. We can now reprove Levin's result as follows. We take G_1 and G_2 to be groups with presentations $\{a|\, a^r = 1\}$ and $\{b|\, b^s = 1\}$, respectively, where r and s are non-negative integers with $r \neq 1$ and $s \neq 1, 2$. We put $P_1 = G_1 * G_2$, so that P_1 has presentation

$$(3.1) \qquad \{a, b|\, a^r = b^s = 1\}.$$

Let $G = G_3$ be a countable group given by the presentation

$$(3.2) \qquad \{g_1, g_2, \ldots | R_1(g_1, g_2, \ldots) = 1, R_2(g_1, g_2, \ldots) = 1, \ldots\}.$$

We put $P = G_1 * G_2 * G_3$. The subset Q of P_1 is defined to consist of the elements $u_1, u_2, \ldots$ of P_1, where

$$(3.3) \quad u_k = ab(ab^2)^{100k+1}ab(ab^2)^{100k+2} \ldots ab(ab^2)^{100(k+1)} \quad (k = 1, 2, \ldots).$$

The subset R of P is to consist of the elements $g_k^{-1}u_k$ of P $(k = 1, 2, \ldots)$. Then it is easy to see that the conditions of Lemma 3 are satisfied with $\lambda = 6$ and S the empty set. Thus the subset R of P satisfies the 6-condition. From Theorem 1 it follows that P/R^P contains an isomorphic copy of G_3.

Now,

$$(3.4) \quad \{a, b; g_1, g_2, \ldots |\, a^r = b^s = 1; R_1(g_1, g_2, \ldots) = 1, \\ R_2(g_1, g_2, \ldots) = 1, \ldots; g_1 = u_1, g_2 = u_2, \ldots\}$$

is a presentation of P/R^P, so that

$$(3.5) \quad \{a, b|\, a^r = b^s = 1; R_1(u_1, u_2, \ldots) = 1, R_2(u_1, u_2, \ldots) = 1, \ldots\}$$

is also a presentation of P/R^P. Since (3.5) is a presentation of a factor group of the group P_1, we have the following result.

THEOREM 2. *Let P_1 be the group with presentation* (3.1). *Then P_1 is a CEF-group. In fact, if G is a countable group with presentation* (3.2), *then the factor group P/R^P of P_1 with presentation* (3.5), *where the u_k are given by* (3.3), *contains an isomorphic copy of G as a subgroup. Moreover, the following results hold:*

(i) *If* (3.2) *has a finite number m of defining relations, then* (3.5) *has $m + 2$ defining relations;*

(ii) *If an element w of P/R^P has finite order n, then there is an element v of P of order n such that $\rho(v) = w$, where ρ is the natural homomorphism from P to P/R^P;*

(iii) *If* (3.2) *is a recursive presentation, then* (3.5) *is a recursive presentation, and then each of the word problem, the order problem, and the power problem is solvable for the presentation* (3.5) *of P/R^P if and only if it is solvable for the presentation* (3.2) *of G* (*the order problem is said to be solvable for G if given any element of G we can determine its order; the power problem is solvable for G if given any pair h, g of elements of G we can determine whether or not h belongs to the cyclic subgroup of G generated by g*);

(iv) *If* (3.2) *is a recursive presentation with solvable word problem, then there exists an effective process to determine, given a word in the generators of* (3.5), *whether or not this word represents an element of the copy of G embedded in the group with presentation* (3.5).

Proof. Part (i) is obvious. Part (ii) follows from [**7**, Corollary 1]. If the presentation (3.2) is recursive, then it is clear, from the way in which the u_k are defined, that the presentation (3.5) is also recursive. The "if" part of (iii) follows from [**8**, Theorems 3 and 4]. The converse follows from the fact that the mapping $g_k \to u_k$ $(k = 1, 2, \ldots)$, extends to an isomorphism of G into the group given by the presentation (3.5).

To prove (iv) it is clearly sufficient to prove that such a process exists for the presentation (3.4). Let w be a word in the generators of (3.4). Then, in the notation of [**8**] (with $R = \Omega$), we can find a word w' such that $w' \approx w$ and w' is either Ω-reduced or is the identity. Now if w represents an element of G, then $w \approx g$, where g is a word in the generators of G. Hence $w' \approx g$, and, from [**8**, Corollary 1], we must have $w' = g$; that is, w' must be a word in the generators of G which represents the same element of G as the word g. This is enough to prove (iv), since the word problem is solvable for G.

Higman [**4**] has shown that there exists a finitely presented group H which contains a copy of every recursively presented group; embedding H in a factor group of the group with presentation $\{a, b | a^r = b^s = 1\}$ as above, we have the following result.

COROLLARY 1. *Let r, s be non-negative integers with $r \neq 1$ and $s \neq 1, 2$. Let P_1 be the group with presentation $\{a, b| a^r = b^s = 1\}$. Then there exists a finitely presented factor group of P_1 which contains a copy of every recursively presented group.*

Lemma 3 and Theorem 2 combine to give the following general result.

THEOREM 3. *Let $P_1 = G_1 * \ldots * G_{t-1}$ $(t \geqq 3)$, and let S be a subset of P_1 which satisfies the 6-condition. Then the factor group P_1/S^{P_1} is a CEF-group if there exists a subset $Q = \{u_1, u_2\}$ of P_1, such that Q and S satisfy the conditions of Lemma 3, with $\lambda = 6$.*

Proof. Suppose that such a subset Q exists. Let H be a given countable group and let G_t be a 2-generator group containing an isomorphic copy of H. We prove the theorem by showing that some factor group of P_1/S^{P_1} is a CEF-group.

Let G_t be generated by the elements g_1, g_2. We put $P = G_1 * G_2 * \ldots * G_t$ and define the subset R of P by $R = \{g_1^{-1}u_1, g_2^{-1}u_2\} \cup S$. Then by Lemma 3, R satisfies the 6-condition. From Theorem 1 it follows that P/R^P contains an isomorphic copy of G_t. Now it is easy to see, using an argument involving presentations as in the proof of Theorem 2, that P/R^P is isomorphic to a factor group of P_1/S^{P_1}. This proves the theorem.

We now give an application of Theorem 3.

THEOREM 4. *Let $P_1 = G_1 * G_2$, where G_2 is not the cyclic group of order 2. If T is a cyclically reduced and bounded subset of P_1, then there is a positive integer m such that $P_1/T_k^{P_1}$ is a CEF-group whenever $k \geqq m$.*

Proof. Let S be the set of least roots of elements of T. Then S is a cyclically reduced, rootless, and bounded subset of P_1, and $L(T) \geqq L(S)$. It follows from Lemma 2 that S_k satisfies the 6-condition if $k \geqq 6L(S) + 1$. We note that $T_k^{P_1} \subset S_k^{P_1}$ for all positive integers k.

Let a_1 be a non-identity element of G_1, and b_1, c_1 non-identity elements of G_2 with $b_1 \neq c_1$. We define the elements u_n, v_n ($n = 27, 28, \ldots$) of P_1 by

$$u_n = a_1b_1(a_1c_1)a_1b_1(a_1c_1)^3 \ldots a_1b_1(a_1c_1)^{2n-1}$$

and

$$v_n = a_1b_1(a_1c_1)^2a_1b_1(a_1c_1)^4 \ldots a_1b_1(a_1c_1)^{2n}.$$

We put $Q(n) = \{u_n, v_n\}$. Let w be an element of P_1 of maximum length such that w is an initial subword of two non-equal elements of $\overline{Q(n)}$.Then, from the normal forms of u_n and v_n, we have

$$w^{\pm 1} = (a_1c_1)^{2n-3}a_1b_1(a_1c_1)^{2n-1}a_1,$$

so that $l(w) = 8n - 5$. Since $l\{\overline{Q(n)}\} = l(u_n) = 2n(n + 1)$, it follows that $Q(n)$ satisfies the 7-condition, since $n \geqq 27$ and so

$$\frac{8n - 4}{2n(n + 1)} < \frac{1}{7}.$$

We wish to choose k and n so that $S_k \cup Q(n)$ satisfies the 6-condition. First of all we impose the condition $k \geqq 6L(S) + 1$ to ensure that S_k satisfies the 6-condition. We note that if z is a cyclically reduced and rootless element of P_1 such that z^q ($q > 2$) is an initial subword of an element of $\overline{Q(n)}$, then it is easy to see, from the normal forms of u_n and v_n, that z must be a cyclic arrangement of $(a_1c_1)^{\pm 1}$.

Suppose that $a_1c_1 \in \bar{S}$. Let w be an initial subword of both a cyclic arrangement x of $(a_1c_1)^{\pm k}$ and an element y of $\overline{Q(n)}$. We have $l(x) = 2k$ and $l(y) \geqq 2n(n + 1)$. Now if x and y are chosen so that w has maximum length, then $w^{\pm 1} = (a_1c_1)^{2n}a_1$, provided that $k > 2n$. Thus, in any case, $l(w) \leqq 4n + 1$. Hence, if $k > 12n + 6$, then $\zeta(x, y) + 1 < \frac{1}{6}l\{x, y\}$.

We now choose n to be the least positive integer such that $n \geqq 27$ and

$$2n(n + 1) > 18L(S). \tag{3.6}$$

With this choice of n we choose k to be any positive integer such that

$$k > \mathrm{Max}\{6L(S) + 1, 12n + 6\} \tag{3.7}$$

if $a_1c_1 \in \bar{S}$, or

$$k > \mathrm{Max}\{6L(S) + 1, 19\} \tag{3.8}$$

otherwise.

We now show that $Q(n) \cup S_k$ satisfies the 6-condition. In view of the choices of n and k, it is sufficient to show that if w is an initial subword of both x and y, where $x \in \bar{S}_k$ and $y \in \overline{Q(n)}$, then $l(w) + 1 < \frac{1}{6}l\{x, y\}$. We have $x = z^k$ say, where $z \in \bar{S}$.

We first suppose that $l(w) + 1 \geqq \frac{1}{6}l(x)$. Then

$$l(w) + 1 \geqq \frac{k}{6}\, l(z) > \frac{19}{6}\, l(z) > 3l(z),$$

so that z^3 is an initial subword of w. Hence z^3 is an initial subword of both x and y, and so z must be a cyclic arrangement of $(a_1c_1)^{\pm 1}$. Thus $a_1c_1 \in \bar{S}$, and so $k > 12n + 6$. This contradicts the result we obtained previously under the supposition that $a_1c_1 \in \bar{S}$.

We now suppose that $l(w) + 1 \geqq \frac{1}{6}l(y)$. Now $l(y) \geqq 2n(n + 1)$, so that, from our choice of n,

$$l(w) + 1 > \frac{18}{6}\, L(S) \geqq 3l(z).$$

This yields a contradiction as above. Thus we have shown that $Q(n) \cup S_k$ satisfies the 6-condition.

It follows from Theorem 3 that $P_1/S_k{}^{P_1}$ is a CEF-group. Since this group is a factor group of $P_1/T_k{}^{P_1}$, we have therefore shown that $P_1/T_k{}^{P_1}$ is a CEF-group.

Notes. (i) The above proof gives an upper bound for the value of the integer m, and this upper bound can be computed if $L(S)$ is known.

(ii) The condition that T be cyclically reduced and bounded can be replaced by the condition that T is bounded and does not contain a conjugate of an element of length one. This follows easily from the fact that any non-identity element of P_1 which is not conjugate to an element of length one is conjugate to a cyclically reduced element of P_1.

Examples. (i) The group $\{x, y|\ x^2 = y^3 = (xy)^k = 1\}$ is a CEF-group if $k > 330$. To show this we take $G_1 = \{x|\ x^2 = 1\}$, $G_2 = \{y|\ y^3 = 1\}$, and $T = \{xy\} = S$. We choose, in the notation of the theorem, $a_1 = x$, $b_1 = y$, and $c_1 = y^2$. Then $\bar{S} = \{a_1b_1, b_1a_1, a_1c_1, c_1a_1\}$. Since $a_1c_1 \in \bar{S}$ we have to use (3.7). We can take $n = 27$ and $k > 12.27 + 6$.

(ii) The group $\{x, y|\ x^2 = y^r = (xy)^k = 1\}$ is a CEF-group if $r > 3$ and $k > 19$. Here we take $G_1 = \{x|\ x^2 = 1\}$, $G_2 = \{y|\ y^r = 1\}$ and $T = \{xy\} = S$. We choose $a_1 = x$, $b_1 = y$, and $c_1 = y^2$, so that $\bar{S} = \{a_1b_1, b_1a_1, b^{r-1}a_1, a_1b_1{}^{r-1}\}$. Since $a_1c_1 \notin \bar{S}$ we can use (3.8). We can take $n = 27$ and $k > 19$.

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