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THE CHARACTERIZATIONS OF LAPLACIANS IN WHITE NOISE ANALYSIS*

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The Laplacians form a class of the most important differential operators in white noise analysis. The goal of this paper is to give their characterizations. Our main tools are the Fock expansions of operators in terms of integral kernel operators and rotation-invariance. In Section 1, the fundamental setting of white noise analysis is introduced briefly. In Section 2, integral kernel operators and the Fock expansions of operators are given. The characterization theorems for number operator, Gross-Laplacian and Euler operator are given in Sections 3,4 and 5 respectively.

Let $(S) \subset (L^2) \subset (S)^*$ be the Gel'fand triple over white noise $(S'(\mathbf{R}), \mathscr{B}(S'(\mathbf{R})), \mu)$. Let $\mathbf{T} \in \mathscr{L}((S), (S)^*)$. $\mathbf{T}$ is equal to number operator $\mathbf{N}$ up to a constant factor if and only if the following conditions are satisfied: 1) $\mathbf{T} = \mathbf{T}^*$, 2) for all $\varphi, \psi \in (S)$, $\mathbf{T}(\varphi, \psi) = (\mathbf{T}\varphi) : \psi + \varphi : (\mathbf{T}\psi)$, 3) $\mathbf{T}$ is rotation-invariant. $\mathbf{T}$ is equal to Gross-Laplacian Δ_G up to a constant factor if and only if the following conditions are satisfied: 1) for all $\xi \in S(\mathbf{R})$, $[\mathbf{T}, D_\xi] = 0$, 2) $[\mathbf{T}, \mathbf{N}] = 2\mathbf{T}$, 3) $\mathbf{T}$ is rotation-invariant. $\mathbf{T} \in \mathscr{L}((S), (S))$ is equal to Euler operator $\Delta_E = \Delta_G + N$ up to a constant factor if and only if the following conditions are satisfied: 1) for all $\varphi, \psi \in (S)$, $\mathbf{T}(\varphi\psi) = (\mathbf{T}\varphi)\psi + \varphi(\mathbf{T}\psi)$, 2) $\mathbf{T}$ is rotation-invariant.

1. White noise space

We adopt the framework of white noise space set by I. Kubo and S. Takenaka [6] (see also Hida et al. [1] or Yan [9]). Let $S(\mathbf{R})$ be the Schwartz space of rapidly decreasing functions on $\mathbf{R}$. Denote by A the self-adjoint extention of the harmonic oscillator operator on $L^2(\mathbf{R})$:

$$Af(u) = -f''(u) + (1 + u^2)f(u), \quad f \in S(\mathbf{R}).$$

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Put

$$e_n(u) = (-1)^n(\pi^{1/2}2^n n!)^{-1/2} e^{u^2/2} \frac{d^n}{du^n} e^{-u^2}, \quad n \geq 0.$$

Then $e_n \in S(\mathbf{R})$ is the eigenfunction of A, corresponding to eigenvalue $2n+2$. At the same time, $\{e_n, n \geq 0\}$ is an orthonormal basis of $L^2(\mathbf{R})$. Define

$$|f|_{2,p}^2 = |A^p f|_2^2 = \sum_{n=0}^{\infty} (2n+2)^{2p} |\langle f, e_n\rangle|^2, \quad f \in L^2,$$
$$S_p(\mathbf{R}) = \mathscr{D}(A^p) = \{f \in L^2 : |f|_{2,p}^2 < \infty\}, \quad p \geq 0,$$

where $|\cdot|_2$ denotes the norm in $L^2(\mathbf{R})$. With $\{|\cdot|_{2,p}, p \geq 0\}$ $S(\mathbf{R})$ is a nuclear space. Let $S'(\mathbf{R})$ be its dual space. Set

$$S_p(\mathbf{R}) = \left\{f \in S'(\mathbf{R}) : |f|_{2,p}^2 = \sum_{n=0}^{\infty} (2n+2)^{2p} |\langle f, e_n\rangle|^2 < \infty\right\}, \quad p \in \mathbf{R},$$

where $\langle\cdot, \cdot\rangle$ denotes the pairing between $S'(\mathbf{R})$ and $S(\mathbf{R})$. Then

$$S(\mathbf{R}) = \bigcap_{p \in R} S_p(\mathbf{R}), \quad S'(\mathbf{R}) = \bigcup_{p \in R} S_p(\mathbf{R}).$$

By Minlos theorem there exists a unique probability measure μ on $\mathscr{B}(S'(\mathbf{R}))$, the σ-field generated by cylinder sets, such that

$$\int_{S'(\mathbf{R})} e^{i\langle x, \xi\rangle} \mu(dx) = \exp\left\{-\frac{1}{2}|\xi|_2^2\right\}, \quad \xi \in S(\mathbf{R}).$$

The measure μ is called the white noise measure, and the probability space $(S'(\mathbf{R}), \mathscr{B}(S'(\mathbf{R})), \mu)$ is called the white noise space. On the white noise space a Brownian motion $B = \{B_t, -\infty < t < \infty\}$ may be well-defined such that $\mathscr{B}(S'(\mathbf{R})) = \sigma\{B_t, -\infty < t < \infty\}$. Then each $\varphi \in (L^2)$ has chaotic representation:

$$\varphi = \sum_{n=0}^{\infty} \int \cdots \int \varphi_n(t_1, \ldots, t_n) dB_{t_1} \ldots dB_{t_n}, \tag{1.1}$$
$$\|\varphi\|_2^2 = \sum_{n=0}^{\infty} n! |\varphi_n|_2^2,$$

where $\varphi_n \in \hat{L}^2(\mathbf{R}^n)$ (the symmetric subspace of $L^2(\mathbf{R}^n)$). We denote (1.1) also by $\varphi \sim (\varphi_n)$ simply. If for all n, $\varphi_n \in \mathscr{D}(A^{\otimes n})$, and $\sum_{n=0}^{\infty} n! |A^{\otimes n}\varphi_n|_2^2 < \infty$, define

$$\Gamma(A)\varphi \in (L^2), \quad \Gamma(A)\varphi \sim (A^{\otimes n}\varphi_n).$$

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$\Gamma(A)$ is a self-adjoint linear operator in (L^2), and is called the second quantization of A. For $p \geq 0$, set

$$(S)_p = \mathscr{D}(\Gamma(A)^p),$$
$$\| \varphi \|_{2,p}^2 = \| \Gamma(A)^p \varphi \|_2^2 = \sum_{n=0}^{\infty} n! \, | \varphi_n |_{2,p}^2, \quad \varphi \sim (\varphi_n) \in (S)_p.$$
$$(S) = \bigcap_{p \geq 0} (S)_p.$$

With $\{\| \cdot \|_{2,p}, p \geq 0\}$ (S) is also a nuclear space. Each element of (S) is called a test functional. Denote by $(S)_{-p}$ the dual of $(S)_p$, $p \geq 0$. Then the dual of (S) is

$$(S)^* = \bigcup_{p \geq 0} (S)_{-p}.$$

Each element of $(S)^*$ is called a generalized Wiener functional or Hida distribution.

For $\xi \in L^2(\mathbf{R})$, the exponential functional $\mathscr{E}(\xi)$ is defined as

$$\mathscr{E}(\xi) = \exp\left\{\langle \cdot, \xi \rangle - \frac{1}{2} | \xi |_2^2\right\} \sim \left(\frac{1}{n!} \xi^{\otimes n}\right).$$

Let $F \in (S)^*$. The S-transform of F is defined as

$$(SF)(\xi) = \langle\langle F, \mathscr{E}(\xi) \rangle\rangle, \quad \xi \in S(\mathbf{R}),$$

where $\langle\langle \cdot, \cdot \rangle\rangle$ denotes the pairing between $(S)^*$ and (S). Each Hida distribution is uniquely determined by its S-transform. For any $F, G \in (S)^*$ there exists a unique Hida distribution, denoted by $F : G$ and called the Wick product of F and G, such that $S(F : G) = S(F) S(G)$.

Let $y \in S'(\mathbf{R})$ and $\varphi \in (S)$. The derivative $D_y \varphi$ of φ in direction y is defined by

$$D_y \varphi = \lim_{t \to 0} \frac{\varphi(\cdot + ty) - \varphi}{t},$$

where the limit is taken in (S). $D_y \in \mathscr{L}((S), (S))$ and its adjoint $D_y^* \in \mathscr{L}((S)^*, (S)^*)$ is defined by

$$\langle\langle D_y^* F, \varphi \rangle\rangle = \langle\langle F, D_y \varphi \rangle\rangle, \quad \forall F \in (S)^*, \quad \varphi \in (S)$$

(see Theorem 2.2 below), where $\mathscr{L}((S), (S))$ (resp. $\mathscr{L}((S)^*, (S)^*)$ denotes the collection of all continuous linear mappings from (S) (resp. $(S)^*$) into itself. Let δ_t be Delta function at a point t. D_{δ_t} and $D_{\delta_t}^*$ are denoted simply by ∂_t and ∂_t^* respectively.

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2. Integral kernel operators

Integral kernel operators are introduced in Hida et al. [3] (see also Hida et al. [2] and Obata [10]). The following lemma is a more precise form of the corresponding result in [3].

LEMMA 2.1. *Let* φ, $\psi \in (S)$. *Set*

(2.1)
$$\iota_{\varphi,\psi}^{(l,m)}(s_1,\ldots,s_l,t_1,\ldots,t_m)=\langle\langle\partial_{s_1}^*\cdots\partial_{s_l}^*\partial_{t_1}\cdots\partial_{t_m}\varphi,\psi\rangle\rangle,\quad l,m\geq 0,\, l+m\geq 1.$$

Then $\iota_{\varphi,\psi}^{(l,m)}\in S(\mathbf{R}^{l+m})$. *Moreover, for any* p, $q\in\mathbf{R}$, *with* $p\geq q$ *and* $\alpha>0$

$$|\iota_{\varphi,\psi}^{(l,m)}|_{2,(-q,p)}\leq C_{\alpha,l,m}\|\varphi\|_{2,p+\alpha}\|\psi\|_{2,-q},\tag{2.2}$$

where $|\cdot|_{2,(-q,p)}$ *is the norm in* $S_{-q}(\mathbf{R}^l)\otimes S_p(\mathbf{R}^m)$, *and*

$$C_{\alpha,l,m}=\sup_n\frac{\sqrt{(l+n)!(m+n)!}}{n!2^{\alpha n}}.\tag{2.3}$$

Let $\kappa\in S'(\mathbf{R}^{l+m})$, l, $m\geq 0$, $l+m\geq 1$. There exists a unique $\Xi_{l,m}(\kappa)\in\mathcal{L}((S),(S)^*)$ (the collection of all continuous linear mappings from (S) into $(S)^*$) such that for all φ, $\psi\in(S)$

$$\langle\langle\Xi_{l,m}(\kappa)\varphi,\psi\rangle\rangle=\langle\kappa,\iota_{\varphi,\psi}^{(l,m)}\rangle.\tag{2.4}$$

If $\kappa\in S_q(\mathbf{R}^l)\otimes S_{-p}(\mathbf{R}^m)$, $p\geq q$, $\alpha>0$, then by (2.2) $\Xi_{l,m}(\kappa)$ is also a linear continuous operator from $(S)_{p+\alpha}$ to $(S)_q$ and

$$\|\Xi_{l,m}(\kappa)\varphi\|_{2,q}\leq C_{\alpha,l,m}|\kappa|_{2,(q,-p)}\|\varphi\|_{2,p+\alpha}.\tag{2.5}$$

The operator $\Xi_{l,m}(\kappa)$ is called an integral kernel operator. For all ξ, $\eta\in S(\mathbf{R})$

$$\langle\langle\Xi_{l,m}(\kappa)\mathscr{E}(\xi),\mathscr{E}(\eta)\rangle\rangle=\langle\kappa,\eta^{\otimes l}\otimes\xi^{\otimes m}\rangle e^{\langle\xi,\eta\rangle}.\tag{2.6}$$

If $\kappa=\sum_{i,j}\kappa_{t_1\cdots t_l j_1\cdots j_m}e_{t_1}\otimes\cdots\otimes e_{t_l}\otimes e_{j_1}\otimes\cdots\otimes e_{j_m}$ is an orthogonal expansion in $S_q(\mathbf{R}^l)\otimes S_{-p}(\mathbf{R}^m)$, then $\Xi_{l,m}(\kappa)$ also has the following strongly convergent expansion.

$$\Xi_{l,m}(\kappa)=\sum_{i,j}\kappa_{t_1\cdots t_l j_1\cdots j_m}D_{e_{i_1}}^*\cdots D_{e_{i_l}}^*D_{e_{j_1}}\cdots D_{e_{j_m}}.\tag{2.7}$$

Denote by $\mathfrak{S}_n$ the permutation group of n letters. For any $\kappa\in S'(\mathbf{R}^n)$ and $\sigma\in\mathfrak{S}_n$ define $\kappa^\sigma\in S'(\mathbf{R}^n)$ by

$$\langle\kappa^\sigma,\xi_1\otimes\cdots\otimes\xi_n\rangle=\langle\kappa,\xi_{\sigma^{-1}(1)}\otimes\cdots\otimes\xi_{\sigma^{-1}(n)}\rangle,\quad\xi_1,\ldots,\xi_n\in S(\mathbf{R}).$$

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For any $\kappa \in S'(\mathbf{R}^{l+m})$ define

$$\hat{\kappa}^{(l,m)} = \frac{1}{l!m!} \sum_{\sigma \in \mathfrak{S}_l \times \mathfrak{S}_m} \kappa^{\sigma}.$$

Since $\iota_{\varphi,\psi}^{(l,m)}(s_1, \ldots, s_l, t_1, \ldots, t_m)$ is symmetric in $(s_1, \ldots, s_l)$ and $(t_1, \ldots, t_m)$ respectively, for all $\varphi, \psi \in (S)$, by definition (2.4) it is easy to see

$$\Xi_{l,m}(\kappa) = \Xi_{l,m}(\hat{\kappa}^{(l,m)}).$$

Moreover, $\hat{\kappa}^{(l,m)}$ is uniquely determined by $\Xi_{l,m}(\kappa)$.

THEOREM 2.2 ([3]). *Let* $\kappa \in S'(\mathbf{R}^{l+m})$.

1) $\Xi_{l,m}(\kappa) \in \mathcal{L}((S), (S))$ *if and only if* $\kappa \in S(\mathbf{R}^l) \otimes S'(\mathbf{R}^m)$, *or equivalently, for each* $p \geq 0$ *there exist* $C > 0$ *and* $q \in \mathbf{R}$ *such that for all* $\xi \in S(\mathbf{R}^l)$, $\eta \in S(\mathbf{R}^m)$

$$|\langle \kappa, \xi \otimes \eta \rangle| \leq C |\xi|_{2,-p} |\eta|_{2,q}. \tag{2.8}$$

2) $\Xi_{l,m}(\kappa)$ *can be extended to a continuous linear mapping from* $(S)^*$ *into itself (i.e.* $\Xi_{l,m}(\kappa) \in \mathcal{L}((S)^*, (S)^*)$) *if and only if* $\kappa \in S'(\mathbf{R}^l) \otimes S(\mathbf{R}^m)$.

Let $\mathrm{Tr} \in S'(\mathbf{R}^2)$ be defined as: for all $\xi, \eta \in S(\mathbf{R})$

$$\langle \mathrm{Tr}, \xi \otimes \eta \rangle = \langle \xi, \eta \rangle.$$

$\mathbf{N} = \Xi_{1,1}(\mathrm{Tr})$ is called the number operator, and $-\mathbf{N}$ is called Beltrami-Laplacian. Since $\mathrm{Tr} \in S(\mathbf{R}) \otimes S'(\mathbf{R})$, by Theorem 2.2 1) $\mathbf{N} \in \mathcal{L}((S), (S))$. In fact, if $\varphi \sim (\varphi_n)$, then $\mathbf{N}\varphi \sim (n\varphi_n)$. So we also have $\mathbf{N} \in \mathcal{L}((S)^*, (S)^*)$.

$\Delta_G = \Xi_{0,2}(\mathrm{Tr})$ is called Gross-Laplacian, and by Theorem 2.2 1) $\Delta_G \in \mathcal{L}((S), (S))$.

The following fact is easy.

THEOREM 2.3. *Let* $\mathbf{T} \in \mathcal{L}((S), (S)^*)$. *Then there is a unique* $\mathbf{T}^* \in \mathcal{L}((S), (S)^*)$, *called the adjoint of* $\mathbf{T}$, *such that for all* $\varphi, \psi \in (S)$

$$\langle\langle \mathbf{T}^*\psi, \varphi \rangle\rangle = \langle\langle \mathbf{T}\varphi, \psi \rangle\rangle.$$

Moreover, $\mathbf{T} \in \mathcal{L}((S), (S))$ *(resp.* $\mathcal{L}((S)^*, (S)^*)$) *if and only if* $\mathbf{T}^* \in \mathcal{L}((S)^*, (S)^*)$ *(resp.* $\mathcal{L}((S), (S))$).

In fact, we have $(\mathbf{T}^*)^* = \mathbf{T}$.

If $\kappa \in S'(\mathbf{R}^{l,m})$, $l, m \geq 0$, $l + m \geq 1$, then there exists a unique $\kappa^{*(m,l)} \in S'(\mathbf{R}^{m+1})$ such that for all $\xi \in S(\mathbf{R}^l)$, $\eta \in S(\mathbf{R}^m)$

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$$\langle \kappa^{*(m,l)}, \xi \otimes \eta \rangle = \langle \kappa, \eta \otimes \xi \rangle,$$

and

$$\Xi_{l,m}(\kappa)^* = \Xi_{m,l}(\kappa^{*(m,l)}).$$

THEOREM 2.4 ([10]). *Let* $\mathbf{T} \in \mathcal{L}((S), (S)^*)$. *Then there exists a unique family* $\{\kappa_{l,m} \in S'(\mathbf{R}^{l+m}), l, m \geq 0\}$, *such that* $\hat{\kappa}_{l,m}^{(l,m)} = \kappa_{l,m}$ *and for all* $\varphi \in (S)$

$$\mathbf{T}\varphi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})\varphi, \tag{2.9}$$

where the series converges in $(S)^*$. *Moreover, if* $\mathbf{T} \in \mathcal{L}((S), (S))$ (*resp.* $\mathcal{L}((S)^*, (S)^*)$), *then so are* $\Xi_{l,m}(\kappa_{l,m})$ *for all* $l, m \geq 0$.

The expansion $\mathbf{T} = \sum_{l,m} \Xi_{l,m}(\kappa_{l,m})$ is called Fock expansion of $\mathbf{T}$. Such an expression was first introduced by R. Haag [1](see also Huang [3] and Krée [4]). In fact, we can show the Fock expansion converges is a stronger sense (see the Appendix of the paper).

Let g be a linear homeomorphism from $S(\mathbf{R})$ onto itself. If for all $\xi \in S(\mathbf{R})$

$$|g\xi|_2 = |\xi|_2,$$

g is called a rotation of $S(\mathbf{R})$. g can be extended uniquely to a unitary operator in $L^2(\mathbf{R})$. For any $x \in S'(\mathbf{R})$ define g^*x by

$$\langle g^*x, \xi \rangle = \langle x, g\xi \rangle.$$

It is easy to see that g^* is a linear automorphism of $S'(\mathbf{R})$. Denote by $\mathcal{O}(S(\mathbf{R}))$ the collection of all rotations of $S(\mathbf{R})$, and

$$\mathcal{O}^*(S'(\mathbf{R})) = \{g^* : g \in \mathcal{O}(S(\mathbf{R}))\}.$$

Let $g \in \mathcal{O}(S(\mathbf{R}))$. It is easy to show $g^{\otimes n}$ is also a linear homeomorphism from $S(\mathbf{R}^n)$ onto itself. Then $\Gamma(g)$ is a rotation of (S), i.e., $\Gamma(g)$ is linear homeomorphism from (S) onto itself, and for all $\varphi \in (S)$ $\|\Gamma(g)\varphi\|_2 = \|\varphi\|_2$.

Let $n \geq 1$, $g \in \mathcal{O}(S(\mathbf{R}))$ and $\kappa \in S'(\mathbf{R}^n)$. $(g^{\otimes n})^*\kappa \in S'(\mathbf{R}^n)$ is defined as:

$$\langle (g^{\otimes n})^*\kappa, \xi \rangle = \langle \kappa, g^{\otimes n}\xi \rangle, \quad \xi \in S(\mathbf{R}^n).$$

κ is said to be rotation-invariant, if for all $g \in \mathcal{O}(S(\mathbf{R}))$ $(g^{\otimes n})^*\kappa = \kappa$.

Let $\mathbf{T} \in \mathcal{L}((S), (S)^*)$. $\mathbf{T}$ is said to be rotation-invariant if for all $g \in \mathcal{O}(S(\mathbf{R}))$ $\Gamma(g)^*\mathbf{T}\Gamma(g) = \mathbf{T}$ on (S). In particular, if $\mathbf{T} \in \mathcal{L}((S), (S))$, then $\mathbf{T}$ is

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rotation-invariant if and only if for all $g \in \mathcal{O}(S(\mathbf{R}))\Gamma(g^{-1})\mathbf{T}\Gamma(g) = \mathbf{T}$ on (S). Moreover, let $\mathbf{T} \in \mathcal{L}((S)^*, (S)^*)$. Then $\mathbf{T}$ is said to be rotation-invariant if for all $g \in \mathcal{O}(S(\mathbf{R}))\Gamma(g)^*\mathbf{T}\Gamma(g^{-1})^* = \mathbf{T}$ on $(S)^*$.

Let $\mathbf{T} \in \mathcal{L}((S), (S)^*)$. Then $\mathbf{T}$ is rotation-invariant if and only if so is $\mathbf{T}^*$. Moreover, if $\mathbf{T}$ is a rotation-invariant continuous linear mapping from (S) into itself, then $\mathbf{T}^*$, as a continuous linear mapping from $(S)^*$ into itself, is rotation-invariant, too.

We shall use the following results on the rotation-invariance.

THEOREM 2.5 ([9]). *Let* $\mathbf{T} \in \mathcal{L}((S), (S)^*)$ *with Fock expansion* $\mathbf{T} = \sum_{l,m} \Xi_{l,m}(\kappa_{l,m})$. *Then* $\mathbf{T}$ *is rotation-invariant if and only if so are all* $\Xi_{l,m}(\kappa_{l,m})$, $l, m \geq 0$.

THEOREM 2.6 ([9]). *Let* $\kappa \in S'(\mathbf{R}^{l+m})$, $l, m \geq 0$,. *If* $l + m$ *is odd, then* $\Xi_{l,m}(\kappa)$ *is rotation-invariant if and only if* $\Xi_{l,m}(\kappa) = 0$. *If* $l + m$ *is even, then* $\Xi_{l,m}(\kappa)$ *is rotation-invariant if and only if it is a linear combination of* $(\Delta_G^p)^*\mathbf{N}^q\Delta_G^r$ *with* p, q, r *being non-negative integers such that* $p + q + r \leq (l + m)/2$. *In addition,* $\Xi_{l,m}(\kappa)$ *maps* (S) *into* (S) *if and only if it is a linear combination of* $\mathbf{N}^q\Delta_G^r$ *with* $q + r \leq (l + m)/2$.

COROLLARY 2.7 ([3]). *Let* $\kappa \in S'(\mathbf{R}^2)$ *and* $\mathbf{T} = \Xi_{1,1}(\kappa)$ (*resp.* $\Xi_{0,2}(\kappa)$) *be rotation-invariant. Then* $\mathbf{T}$ *is equal to* $\mathbf{N}$ (*resp.* Δ_G) *up to a constant factor.*

3. The characterization of number operator

THEOREM 3.1. *Let* $\mathbf{T} \in \mathcal{L}((S), (S)^*)$. *Then*

$$\mathbf{T}(\varphi : \phi) = (\mathbf{T}\varphi) : \phi + \varphi : (\mathbf{T}\phi), \quad \forall \varphi, \phi \in (S), \tag{3.1}$$

holds if and only if the Fock expansion of $\mathbf{T}$ *has the form*:

$$\mathbf{T} = \sum_{l} \Xi_{l,1}(\kappa_{l,1}). \tag{3.2}$$

Proof. By the density of $\{\mathcal{E}(\xi), \xi \in S(\mathbf{R})\}$ in (S), (3.1) is equivalent to that for all $\xi_1, \xi_2, \eta \in S(\mathbf{R})$

$$\begin{aligned} \langle\langle \mathbf{T}(\mathcal{E}(\xi_1) : \mathcal{E}(\xi_2)), \mathcal{E}(\eta)\rangle\rangle &= \langle\langle \mathbf{T}(\mathcal{E}(\xi_1 + \xi_2)), \mathcal{E}(\eta)\rangle\rangle \\ &= (\mathbf{T}\mathcal{E}(\xi_1)) : \mathcal{E}(\xi_2), \mathcal{E}(\eta)\rangle\rangle + \langle\langle \mathcal{E}(\xi_1) : (\mathbf{T}\mathcal{E}(\xi_2)), \mathcal{E}(\eta)\rangle\rangle. \end{aligned} \tag{3.3}$$

Let $\mathbf{T} = \sum_{l,m} \Xi_{l,m}(\kappa_{l,m})$ be the Fock expansion of $\mathbf{T}$. (3.3) is just

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$$(3.4)\qquad \sum_{l,m} \langle \kappa_{l,m}, \eta^{\otimes l} \otimes (\xi_1 + \xi_2)^{\otimes m} \rangle = \sum_{l,m} \langle \kappa_{l,m}, \eta^{\otimes l} \otimes (\xi_1^{\otimes m} + \xi_2^{\otimes m}) \rangle.$$

Replacing ξ_1, ξ_2 and η by $s\xi_1$, $s\xi_2$ and $t\eta$ in (3.4) respectively and comparing the coefficients of power series, from (3.4) we conclude that for all $l, m \geq 0$

$$(3.5)\qquad \langle \kappa_{l,m}, \eta^{\otimes l} \otimes (\xi_1 + \xi_2)^{\otimes m} \rangle = \langle \kappa_{l,m}, \eta^{\otimes l} \otimes (\xi_1^{\otimes m} + \xi_2^{\otimes m}) \rangle.$$

Replacing ξ_2 by $t\xi_2$ in (3.5) yields

$$\sum_{r=0}^{m} \langle \kappa, \eta^{\otimes l} \otimes \xi_1^{\otimes m-r} \otimes \xi_2^{\otimes r} \rangle \binom{m}{r} t^r = \langle \kappa, \eta^{\otimes l} \otimes \xi_1^{\otimes m} \rangle + \langle \kappa, \eta^{\otimes l} \otimes \xi_2^{\otimes m} \rangle t^m.$$

If $\kappa_{l,m} \neq 0$, it must be $m = 1$. Hence the necessity is verified. Reversing the reasoning yields the sufficiency. □

THEOREM 3.2. *Let* $\mathbf{T} \in \mathscr{L}((S), (S)^*)$. *Then* $\mathbf{T}$ *is equal to* $\mathbf{N}$ *up to a constant factor if and only if the following conditions are satisfied*:

1) $\mathbf{T} = \mathbf{T}^*$,

2) *for all* $\varphi, \psi \in (S)\mathbf{T}(\varphi : \psi) = (\mathbf{T}\varphi) : \psi + \varphi : (\mathbf{T}\psi)$,

3) $\mathbf{T}$ *is rotation-invariant.*

Proof. The necessity is well-known. We need only to show the sufficiency. Let $\mathbf{T} = \sum_{l,m} \Xi_{l,m}(\kappa_{l,m})$ be the Fock expansion of $\mathbf{T}$, and put $\mathbf{T}_{l,m} = \Xi_{l,m}(\kappa_{l,m})$. By Lemmas 3.1 and 3.2 from condition 2) we deduce that for all $m \neq 1$ $\mathbf{T}_{l,m} = 0$.

The condition 1) is equivalent to that for all $l, m \geq 0$

$$\mathbf{T}_{l,m} = \mathbf{T}^*_{m,l}.$$

Hence for $l \neq 1$ $\mathbf{T}_{l,m} = \mathbf{T}^*_{m,l} = 0$. Therefore,

$$\mathbf{T} = \mathbf{T}_{1,1} = \Xi_{1,1}(\kappa_{1,1}).$$

At last, by making use of rotation-invariance of $\mathbf{T}$ and Corollary 2.7 we conclude that $\mathbf{T}$ is equal to $\mathbf{N}$ up to a constant factor. □

We give three examples to show that in order to characterize $\mathbf{N}$ anyone of the three conditions in Theorem 3.2 cannot be deleted.

1) $\Delta_G^* \mathbf{N} = \Xi_{3,1}(\mathrm{Tr} \otimes \mathrm{Tr})$ satisfies conditions 2) and 3) by Lemma 3.2 and Theorem 2.5, but not condition 1).

2) $\Delta_G^* \mathbf{N} \Delta_G = \Xi_{3,3}(\mathrm{Tr} \otimes \mathrm{Tr} \otimes \mathrm{Tr})$ satisfies conditions 1) and 3) by (2.8) and Theorem 2.5, but not condition 2).

3) Take $\kappa \in S'(\mathbf{R}_2)$ such that $\kappa = \kappa^{*(1,1)}$ and $\kappa \neq \mathrm{Tr}$, for instance, $\kappa = e_1 \otimes$

e_1. Then $\Xi_{1,1}(\kappa)$ satisfies conditions 1) and 2) by (2.8) and Lemma 3.2, but not condition 3) by Corollary 2.7.

THEOREM 3.3. *Let* $\mathbf{T} \in \mathscr{L}((S), (S))$ *satisfy the following conditions:*

1) *for all* $\varphi, \psi \in (S)\mathbf{T}(\varphi : \psi) = (\mathbf{T}\varphi) : \psi + \varphi : (\mathbf{T}\psi)$,

2) $\mathbf{T}$ *is rotation-invariant.*

Then $\mathbf{T}$ *is equal to* $\mathbf{N}$ *up to a constant factor.*

Proof. We continue to adopt the notations in Theorem 3.2. From condition 1) we know for $m \neq 1$ $\mathbf{T}_{l,m} = 0$. By Theorem 2.6 (see[9]) from condition 2) we know that $\mathbf{T}_{l,1} = 0$ for even l, and for odd l

$$\mathbf{T}_{l,1} = \sum_{p+q \leq (l+1)/2} C_{p,q}\mathbf{N}^p \Delta_G^q = \sum_{p+q \leq (l+1)/2} C_{p,q}\Xi_{p,p+2q}(\tau_p \otimes \mathrm{Tr}^{\otimes p}),$$

where $C_{p,q}$ are constants, and $\tau_p = \sum_{j_1,\ldots,j_p} e_{j_1} \otimes \cdots \otimes e_{j_p} \otimes e_{j_1} \otimes \cdots \otimes e_{j_p} \in S(\mathbf{R}^{2p})$ such that $\Xi_{p,p}(\tau_p) = \Pi_{j=0}^{p-1}(\mathbf{N} - j)$. In order to have $p + 2q = 1$ it must be $p = 1$ and $q = 0$. Thus $\mathbf{T} = \mathbf{T}_{1,1}$ is equal to $\mathbf{N}$ up to a factor by Corollary 2.7. □

4. The characterization of Gross-Laplacian

For $\mathbf{T}_1, \mathbf{T}_2 \in \mathscr{L}((S), (S)^*)$, let

$$[\mathbf{T}_1, \mathbf{T}_2] = \mathbf{T}_1\mathbf{T}_2 - \mathbf{T}_2\mathbf{T}_1,$$

if the right side makes sense on (S).

LEMMA 4.1. *Let* $\kappa \in S'(\mathbf{R}^{l+m})$. *Then*

$$[\mathbf{N}, \Xi_{l,m}(\kappa)] = (l - m)\Xi_{l,m}(\kappa). \tag{4.1}$$

Proof. Let $\zeta, \eta, \eta_i, \ldots \in S(\mathbf{R})$, $i = 1, \ldots$. It is well known that on (S)

$$[D_\zeta, D_\eta^*] = \langle \zeta, \eta \rangle,$$
$$[D_\zeta, D_\eta^* \cdots D_{\eta_l}^*] = \sum_{i=1}^{l} \langle \zeta, \eta_i \rangle D_{\eta_1}^* \cdots D_{\eta_{i-1}}^* D_{\eta_{i+1}}^* \cdots D_{\eta_l}^*.$$

From (2.7) it is not difficult to get

$$[D_\zeta, \Xi_{l,m}(\kappa)] = l\,\Xi_{l-1,m}(\kappa \otimes_{(1,0)} \zeta), \tag{4.2}$$

where $\kappa \otimes_{(1,0)} \zeta \in S'(\mathbf{R}^{l+m-1})$ is defined by

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$$\langle \kappa \otimes_{(1,0)} \zeta, \eta \rangle = \langle \kappa, \zeta \otimes \eta \rangle, \quad \forall \eta \in S(\mathbf{R}^{l+m-1}).$$

Analogously, we have

$$(4.3) \qquad [D_\zeta^*, D_{\eta_1} \cdots D_{\eta_m}] = -\sum_{i=1}^{m} \langle \zeta, \eta_i \rangle D_{\eta_1} \cdots D_{\eta_{i-1}} D_{\eta_{i+1}} \cdots D_{\eta_m}$$
$$[D_\zeta^*, \Xi_{l,m}(\kappa)] = -m\, \Xi_{l,m}(\kappa \otimes_{(0,1)} \zeta),$$

where $\kappa \otimes_{(0,1)} \zeta \in S'(\mathbf{R}^{l+m-1})$ is defined by

$$\langle \kappa \otimes_{(1,0)} \zeta, \eta \rangle = \langle \kappa, \eta \otimes \zeta \rangle, \quad \forall \eta \in S(\mathbf{R}^{l+m-1}).$$

Hence

(4.4)
$$\begin{aligned}
[D_\zeta^* D_\zeta, \Xi_{l,m}(\kappa)] &= D_\zeta^* D_\zeta \Xi_{l,m}(\kappa) - \Xi_{l,m}(\kappa) D_\zeta^* D_\zeta \\
&= D_\zeta^* D_\zeta \Xi_{l,m}(\kappa) - D_\zeta^* \Xi_{l,m}(\kappa) D_\zeta + D_\zeta^* \Xi_{l,m}(\kappa) D_\zeta - \Xi_{l,m}(\kappa) D_\zeta^* D_\zeta \\
&= l D_\zeta^* \Xi_{l-1,m}(\kappa \otimes_{(1,0)} \zeta) - m\, \Xi_{l,m-1}(\kappa \otimes_{(0,1)} \zeta) D_\zeta \\
&= l \Xi_{l-1,m}(\zeta \otimes \kappa \otimes_{(1,0)} \zeta) - m\, \Xi_{l,m}(\kappa \otimes_{(0,1)} \zeta \otimes \zeta).
\end{aligned}$$

Since $\mathbf{N} = \sum_i \partial_i^* \partial_i$ and

$$\kappa = \sum_i e_i \otimes \kappa \otimes_{(1,0)} e_i = \sum_i \kappa \otimes_{(1,0)} e_i \otimes e_i,$$

(4.1) follows from (4.4). □

LEMMA 4.2. *Let* $\mathbf{T} \in \mathscr{L}((S), (S)^*)$, *and* $r \geq 0$ *be an integer. Then*

$$(4.5) \qquad [\mathbf{T}, \mathbf{N}] = r\mathbf{T}$$

if and only if the Fock expansion of $\mathbf{T}$ *is*

$$(4.6) \qquad \mathbf{T} = \sum_{l=0}^{\infty} \Xi_{l,l+r}(\kappa_{l,l+r}).$$

Proof. Let $\mathbf{T} = \sum_{l,m} \Xi_{l,m}(\kappa_{l,m})$ be the Fock expansion of $\mathbf{T}$, and $\xi, \eta \in S(\mathbf{R})$.

From (4.1) we have

$$\langle\langle [\mathbf{T}, \mathbf{N}] \mathscr{E}(\xi), \mathscr{E}(\eta) \rangle\rangle = \sum_{l,m=0}^{\infty} (m-l) \langle \kappa_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle e^{\langle \xi, \eta \rangle}.$$

Hence we know that (4.5) holds if and only if for all $\xi, \eta \in S(\mathbf{R})$

$$\sum_{l,m=0}^{\infty} \langle \kappa_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle [m - l - r] = 0.$$

Replace ξ, η by $s\xi$, $t\eta$ respectively, and by the uniqueness of the coefficients of power series we get

$$\forall\, l, m \geq 0, \quad (m - l - r)\kappa_{l,m} = 0,$$

i.e. (4.6) holds. □

LEMMA 4.3. *Let* $\mathbf{T} \in \mathscr{L}((S), (S)^*)$. *Then* $\forall\, \zeta \in S(\mathbf{R})$, $[\mathbf{T}, D_\zeta] = 0$ *if and only if the Fock expansion of* $\mathbf{T}$ *is*

$$\mathbf{T} = \sum_{m=0}^{\infty} \Xi_{0,m}(\kappa_{0,m}). \tag{4.7}$$

Proof. Let $\mathbf{T} = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})$ be the Fock expansion of $\mathbf{T}$, $\xi, \eta, \zeta \in S(\mathbf{R})$. From (4.2) it is easy to see

$$\langle\langle [D_\zeta, \mathbf{T}]\mathscr{E}(\xi), \mathscr{E}(\eta) \rangle\rangle = \sum_{l=1}^{\infty} \sum_{m=0}^{\infty} l \langle \kappa_{l,m}, \eta^{\otimes l-1} \otimes \zeta \otimes \xi^{\otimes m} \rangle e^{\langle \xi, \eta \rangle}.$$

Therefore, for all $\zeta \in S(\mathbf{R})$ $[\mathbf{T}, D_\zeta] = 0$ if and only if for all ξ, η, $\zeta \in S(\mathbf{R})$

$$\sum_{l=1}^{\infty} \sum_{m=0}^{\infty} l \langle \kappa_{l,m}, \eta^{\otimes l-1} \otimes \zeta \otimes \xi^{\otimes m} \rangle = 0.$$

Replace ξ, η, ζ and $s\xi$, $t\eta$, $w\zeta$ respectively, and by the uniqueness of the coefficients of power series we get

$$\kappa_{l,m} = 0, \quad l \geq 1, \quad m \geq 0,$$

i.e. (4.7) holds. □

Remark. If $\mathbf{T} \in \mathscr{L}((S), (S))$, in a similar way we can see that $\forall\, t$, $[\mathbf{T}, \partial_t] = 0$ if and only if the Fock expansion of $\mathbf{T}$ is given as in (4.7). This has been pointed out in Luo [8].

THEOREM 4.4. *Let* $\mathbf{T} \in \mathscr{L}((S), (S)^*)$. *Then* $\mathbf{T}$ *is equal to* Δ_G *up to a constant factor if and only if the following conditions are satisfied:*

1) *for all* $\xi \in S(\mathbf{R})$, $[\mathbf{T}, D_\xi] = 0$,
2) $[\mathbf{T}, \mathbf{N}] = 2\mathbf{T}$,
3) $\mathbf{T}$ *is rotation-invariant.*

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Proof. The necessity is well-known. We need only to show the sufficiency. From the condition 1) and Lemma 4.3 we know the Fock expansion of $\mathbf{T}$ has the form $\mathbf{T} = \sum_{m=0}^{\infty} \Xi_{0,m}(\kappa_{0,m})$. Furthermore, from the condition 2) and Lemma 4.2 we get $\mathbf{T} = \Xi_{0,2}(\kappa_{0,2})$. At last, by the condition 3) and Corollary 2.7 we arrive at the conclusion. □

By the above remark we see that the condition 1) in Theorem 4.4 can be replaced by the following conditions: $\mathbf{T} \in \mathscr{L}((S), (S))$ and for all t, $[\mathbf{T}, \partial_t] = 0$.

5. The characterization of Euler operator

The Euler operator $\Delta_E = \Delta_G + \mathbf{N}$ is introduced by Liu and Yan [7]. It is the counterpart of finite-dimensional Euler operator in the setting of white noise analysis. In fact, let $\varphi \in (S)$ and $\lambda \in \mathbf{R}$. Then for all $0 \neq t \in \mathbf{R}$ and $x \in S'(\mathbf{R})$ $\varphi(tx) = t^{\lambda}\varphi(x)$ if and only if $\Delta_E \varphi = \lambda\varphi$.

$\mathbf{T} \in \mathscr{L}((S), (S)^*)$ is called a derivation if for all $\varphi, \phi \in (S)$

$$\mathbf{T}(\varphi\phi) = (\mathbf{T}\varphi)\phi + \varphi(\mathbf{T}\phi). \tag{5.1}$$

It is well-known that the Euler operator $\Delta_E = \Delta_G + \mathbf{N}$ is a derivation. For the characterization of derivations and the following lemma one may refer to Obata [10].

LEMMA 5.1. *Let $\mathbf{T} \in \mathscr{L}((S), (S)^*)$, then the following statements are equivalent:*

i) *$\mathbf{T}$ is a derivation;*

ii) *For all $\xi, \eta, \zeta \in S(\mathbf{R})$*

$$\begin{aligned} &\langle\langle \mathbf{T}\mathscr{E}(\xi + \eta), \mathscr{E}(\zeta)\rangle\rangle e^{\langle \xi, \eta\rangle} \\ &\qquad = \langle\langle \mathbf{T}\mathscr{E}(\xi), \mathscr{E}(\eta + \zeta)\rangle\rangle e^{\langle \eta, \zeta\rangle} + \langle\langle \mathbf{T}\mathscr{E}(\eta), \mathscr{E}(\xi + \zeta)\rangle\rangle e^{\langle \xi, \zeta\rangle}. \end{aligned} \tag{5.2}$$

iii) *The Fock expansion of $\mathbf{T} = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})$ satisfies the following conditions:*

1) $\kappa_{l,0} = 0, \quad l \geq 0,$

2) *for all, $l, m \geq 0$ and $\xi, \eta \in S(\mathbf{R})$*

$$\langle \kappa_{l,m+1}, \eta^{\otimes l} \otimes \xi^{\otimes m+1}\rangle = \binom{l+m}{l} \langle \kappa_{l+m,1}, (\eta^{\otimes l} \otimes \xi^{\otimes m}) \otimes \xi\rangle. \tag{5.3}$$

THEOREM 5.2. *Let $\mathbf{T} \in \mathscr{L}((S), (S))$. Then $\mathbf{T}$ is equal to Δ_E up to a constant factor if and only if it is a derivation and rotation-invariant.*

Proof. We need only to show the sufficiency. Since $\mathbf{T}$ is rotation-invariant, the Fock expansion of $\mathbf{T}$ is given as

$$\mathbf{T} = \sum_{l+m=\text{even}} \Xi_{l,m}(\kappa_{l,m}),$$

and each $\Xi_{l,m}(\kappa_{l,m})$ is also rotation-invariant by Theorem 2.5. By Theorem 2.6 (see Obata [9])

$$\Xi_{l,m}(\kappa_{l,m}) = \sum_{2\alpha+\beta=l, 2\gamma+\beta=m} c_{\alpha,\beta,\gamma}(\Delta_G^{\alpha})^* \Xi_{\beta,\beta}(\tau_\beta)\Delta_G^{\gamma}, \tag{5.4}$$

where $c_{\alpha,\beta,\gamma}$ are constants. Furthermore, since $\Xi_{l,m}(\kappa_{l,m}) \in \mathscr{L}((S), (S))$, the terms with Δ_G^* to not appear in the right side of (5.4). Thus we have

$$\Xi_{l,m}(\kappa_{l,m}) = \begin{cases} c_{l,m}\Xi_{l,l}(\tau_l)\Delta_G^{(m-l)/2} & m \geq l, \\ 0, & m < l, \end{cases} \tag{5.5}$$

where $c_{l,m}$ is a constants. By Lemma 5.1 1) $\kappa_{l,0} = 0$, $l \geq 0$. By (5.5) $\kappa_{l,1} = 0$, $l > 1$. Then by (5.3) $\kappa_{l,m+1} = 0$, $l + m > 1$. Thus except for $\kappa_{0,2}$ and $\kappa_{1,1}$ all $\kappa_{l,m} = 0$. By Corollary 2.7 $\kappa_{0,2} = c_0\mathrm{Tr}$, $\kappa_{1,1} = c_1\mathrm{Tr}$, where c_0, c_1 are constants. Again by (5.3)

$$\langle \kappa_{0,2}, \xi^{\otimes 2}\rangle = \langle \kappa_{1,1}, \xi \otimes \xi\rangle, \quad \xi \in S(\mathbf{R}).$$

Hence $c_0 = c_1 = c$. At last, $\mathbf{T} = c(\Xi_{0,2}(\mathrm{Tr}) + \Xi_{1,1}(\mathrm{Tr})) = c(\Delta_G + \mathbf{N}) = c\Delta_E$. □

In Theorem 5.2 the assumption $\mathbf{T} \in \mathscr{L}((S), (S))$ cannot be weakened to $\mathbf{T} \in \mathscr{L}((S), (S)^*)$. In fact, take

$$\mathbf{T} = \Delta_G^*\mathbf{N} + \Delta_G^*\Delta_G + 2\mathbf{N}(\mathbf{N} - 1) + 3\mathbf{N}\Delta_G + \Delta_G^2.$$

Obviously, $\mathbf{T}$ is rotation-invariant. And $\mathbf{T}$ is a derivation. In fact, it is straightward to check (5.2) holds for $\mathbf{T}$, noting that for all ξ, $\eta \in S(\mathbf{R})$

$$\begin{aligned}
\langle\langle \Delta_G^*\mathbf{N}\mathscr{E}(\xi), \mathscr{E}(\eta)\rangle\rangle &= \langle \xi, \eta\rangle |\eta|^2 e^{\langle \xi,\eta\rangle}, \\
\langle\langle \Delta_G^*\Delta_G\mathscr{E}(\xi), \mathscr{E}(\eta)\rangle\rangle &= |\xi|^2 |\eta|^2 e^{\langle \xi,\eta\rangle}, \\
\langle\langle \mathbf{N}(\mathbf{N}-1)\mathscr{E}(\xi), \mathscr{E}(\eta)\rangle\rangle &= \langle \xi, \eta\rangle^2 e^{\langle \xi,\eta\rangle}, \\
\langle\langle \mathbf{N}\Delta_G\mathscr{E}(\xi), \mathscr{E}(\eta)\rangle\rangle &= \langle \xi, \eta\rangle |\xi|^2 e^{\langle \xi,\eta\rangle}, \\
\langle\langle \Delta_G^*\mathscr{E}(\xi), \mathscr{E}(\eta)\rangle\rangle &= |\xi|^4 e^{\langle \xi,\eta\rangle}, \\
\langle\langle \mathbf{T}\mathscr{E}(\xi), \mathscr{E}(\eta)\rangle\rangle &= \langle \xi, \xi + \eta\rangle |\xi + \eta|^2 e^{\langle \xi,\eta\rangle}.
\end{aligned}$$

Since $\mathbf{T}$ is a derivation, according to Obata [10], it is a first order differential operator. In fact,

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$$\mathbf{T} = \int \Phi_t(x)\partial_t dt, \quad \Phi_t(x) = \langle :x^{\otimes 3}:, \delta_t \otimes \mathrm{Tr}\rangle.$$

Appendix. The convergence of Fock expansion

Let $\mathbf{T} \in \mathscr{L}((S), (S)^*)$, there exist $p, q \in \mathbf{R}$ with $p \geq q$, such that $\mathbf{T}$ can be extended to a continuous linear mapping from $(S)_p$ into $(S)_q$. Now $\langle\langle \mathbf{T}\varphi, \phi\rangle\rangle$ is a continuous bilinear functional on $(S)_p \times (S)_{-q}$, there exists a constant $C_1 > 0$ such that

(6.1) $$|\langle\langle \mathbf{T}\varphi, \phi\rangle\rangle| \leq C_1 \|\varphi\|_{2,p} \|\phi\|_{2,-q}.$$

For all $\xi, \eta \in S(\mathbf{R})$

(6.2)

$$\begin{aligned}
&e^{-(\xi,\eta)} \langle\langle \mathbf{T}\mathscr{E}(\xi), \mathscr{E}(\eta)\rangle\rangle \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (\xi, \eta)^k \sum_{l,m=0}^{\infty} \frac{1}{l!m!} \langle\langle \mathbf{T}\, I_m(\xi^{\otimes m}), I_l(\eta^{\otimes l})\rangle\rangle \\
&= \sum_{l,m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(l+k)!(m+k)!} \binom{l+k}{k} \binom{m+k}{k} (\xi^{\otimes k}, \eta^{\otimes k}) \langle\langle \mathbf{T}\, I_m(\xi^{\otimes m}), I_l(\eta^{\otimes l})\rangle\rangle \\
&= \sum_{l,m=0}^{\infty} \frac{1}{l!m!} \sum_{k=0}^{l\wedge m} (-1)^k k! \binom{l}{k} \binom{m}{k} (\xi^{\otimes k}, \eta^{\otimes k}) \langle\langle \mathbf{T}\, I_{m-k}(\xi^{\otimes m-k}), I_{l-k}(\eta^{\otimes l-k})\rangle\rangle.
\end{aligned}$$

For all $\eta_1, \ldots, \eta_l, \xi_1, \ldots, \xi_m \in S(\mathbf{R})$ define

(6.3)
$$\begin{aligned}
\langle \rho_{l,m}, \eta_1 \otimes \cdots \otimes \eta_l \otimes \xi_1 \otimes \cdots \otimes \xi_m\rangle
&= \frac{1}{l!m!} \sum_{k=0}^{l\wedge m} (-1)^k k! \binom{l}{k}\binom{m}{k} (\xi_1, \eta_1) \cdots (\xi_k, \eta_k) \\
&\quad \times \langle\langle \mathbf{T}\, I_{m-k}(\xi_{k+1} \otimes \cdots \otimes \xi_m), I_{l-k}(\eta_{k+1} \otimes \cdots \otimes \eta_l)\rangle\rangle.
\end{aligned}$$

Let $\alpha > 1/\log 2$ such that $2e \sum_i (2i+2)^{-2\alpha} < 1$ (for example, $\alpha \geq 2$). Now by (6.1) we have

(6.4)
$$\begin{aligned}
&\sum_{i_1,\ldots,i_l, j_1,\ldots,j_m} |\langle \rho_{l,m}, A^{q-\alpha} e_{i_1} \otimes \cdots \otimes A^{q-\alpha} e_{i_l} \\
&\qquad\qquad \otimes A^{-(p+\alpha)} e_{j_1} \otimes \cdots \otimes A^{-(p+\alpha)} e_{j_m}\rangle|^2 \\
&\leq \frac{l\wedge m + 1}{(l!m!)^2} \sum_{i_1,\ldots,i_l, j_1,\ldots,j_m} \sum_{k=0}^{l\wedge m} \left[k! \binom{l}{k}\binom{m}{k}\right]^2 \prod_{s=1}^{k} (A^{q-\alpha} e_{i_s}, A^{-(p+\alpha)} e_{j_s})^2 \\
&\qquad \times C_1^2 (l-k)!(m+k)! \,|A^{q-\alpha} e_{i_{k+1}} \otimes \cdots \otimes A^{q-\alpha} e_{i_l}|_{2,-q}^2 \\
&\qquad\qquad \times |A^{(p+\alpha)} e_{j_{k+1}} \otimes \cdots \otimes A^{-(p+\alpha)} e_{j_m}|_{2,p}^2
\end{aligned}$$

$$\leq C_1^2 \frac{l \wedge m + 1}{l!m!} \sum_{i_1, \cdots, i_l, j_1, \cdots, j_m} \sum_{k=0}^{l \wedge m} \binom{l}{k} \binom{m}{k} (2i_1 + 2)^{-2\alpha} \cdots (2i_l + 2)^{-2\alpha}$$
$$\times (2j_1 + 2)^{-2\alpha} \cdots (2j_m + 2)^{-2\alpha}$$
$$\leq C_1^2 \frac{l \wedge m + 1}{l!m!} \left[\binom{2l}{l} \binom{2m}{m} \right]^{1/2} \left[\sum_i (2i + 2)^{-2\alpha} \right]^{l+m} < \infty.$$

Thus $\rho_{l,m}$ can be extended to an element of $S_{q-\alpha}(\mathbf{R}^l) \otimes S_{-(p+\alpha)}(\mathbf{R}^{l+m})$. Set $\kappa_{l,m} = \hat{\rho}_{l,m}^{(l,m)}$. Then

$$(6.5) \qquad | \kappa_{l,m} |^2_{2,(q-\alpha,-(p+\alpha))} \leq | \rho_{l,m} |^2_{2,(q-\alpha,-(p+\alpha))}$$
$$\leq C_1^2 \frac{l \wedge m + 1}{l!m!} \left[\binom{2l}{l} \binom{2m}{m} \right]^{1/2} \left[\sum_i (2i + 2)^{-2\alpha} \right]^{l+m}.$$

From (6.2) and (6.3) we know that for all ξ, $\eta \in S(\mathbf{R})$

$$(6.6) \qquad \langle\langle \mathbf{T}\mathscr{E}(\xi), \mathscr{E}(\eta) \rangle\rangle = \sum_{l,m=0}^{\infty} \langle\langle \Xi_{l,m}(\kappa_{l,m}) \mathscr{E}(\xi), \mathscr{E}(\eta) \rangle\rangle.$$

By (2.5) for all φ, $\psi \in (S)$ and l, $m \geq 0$

$$(6.7) \qquad \| \Xi_{l,m}(\kappa_{l,m}) \varphi \|_{2,q-\alpha} \leq (C_{\alpha,l,m} | \kappa_{l,m} |_{2,(q-\alpha,-(p+\alpha))}) \| \varphi \|_{2,p+2\alpha}.$$

All $\Xi_{l,m}(\kappa_{l,m})$ may be viewed as continuous linear operators from $(S)_{p+2\alpha}$ into $(S)_{q-\alpha}$. We are going to show

$$(6.8) \qquad \sum_{l,m} C_{\alpha,l,m} | \kappa_{l,m} |_{2,(q-\alpha,-(p+\alpha))} < \infty.$$

Then we know the Fock expansion $\mathbf{T} = \sum_{l,m} \Xi_{l,m}(\kappa_{l,m})$ converges with respect to the operator norm in $\mathscr{L}((S)_{p+2\alpha}, (S)_{q-\alpha})$.

From (2.3) we know that for $\alpha > 1/\log 2$

$$C_{\alpha,l,m} = \sup_{n \geq 0} \frac{\sqrt{(l+n)!(m+n)!}}{n! 2^{\alpha n}}$$
$$\leq \sup_{n \geq 0} \sqrt{\frac{(l+n)!}{n!}} 2^{-\alpha n/2} \sup_{n \geq 0} \sqrt{\frac{(m+n)!}{n!}} 2^{-\alpha n/2}$$
$$\leq \sup_{n \geq 0} (l+n)^{l/2} 2^{-\alpha n/2} \sup_{n \geq 0} (m+n)^{m/2} 2^{-\alpha n/2} \leq l^{l/2} m^{m/2},$$

since by elementary calculus it is easy to see that the function $f(t) = (l+t)^{l/2} 2^{-\alpha t/2}$ is decreasing on $[0, \infty)$ for any fixed l and $\alpha > 1/\log 2$. Then by (6.5) there is a constant $C_2 > 0$ such that

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$$\sum_{l,m} C_{\alpha,l,m} | \kappa_{l,m} |_{2,(q-\alpha,-(p+\alpha))}$$
$$\leq C_2 \sum_{l,m} l^{l/2} m^{m/2} \left[\frac{\sqrt{(l+1)(m+1)}}{l!m!} \left(\binom{2l}{l} \binom{2m}{m} \right)^{1/2} \right]^{1/2} \left[\sum_i (2i+2)^{-2\alpha} \right]^{\frac{l+m}{2}}$$
$$= C_2 \left[\sum_l \left(l^l \frac{\sqrt{l+1}}{l!} \binom{2l}{l}^{1/2} \left[\sum_i (2i+2)^{-2\alpha} \right]^l \right)^{1/2} \right]^2.$$

By Stirling's formula we have

$$l^l \frac{\sqrt{l+1}}{l!} \binom{2l}{l}^{1/2} (\sum_i (2i+2)^{-2\alpha})^l \leq C_3 (2e \sum_i (2i+2)^{-2\alpha})^l,$$

where C_3 is a constant. Thus (6.8) holds.

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