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SMALL VALUES AND FORBIDDEN VALUES FOR THE FOURIER ANTI-DIAGONAL CONSTANT OF A FINITE GROUP

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Abstract

For a finite group G, let $\mathrm{AD}(G)$ denote the Fourier norm of the antidiagonal in $G \times G$. The author showed recently in ['An explicit minorant for the amenability constant of the Fourier algebra', *Int. Math. Res. Not. IMRN* **2023** (2023), 19390–19430] that $\mathrm{AD}(G)$ coincides with the amenability constant of the Fourier algebra of G and is equal to the normalized sum of the cubes of the character degrees of G. Motivated by a gap result for amenability constants from Johnson ['Non-amenability of the Fourier algebra of a compact group', *J. Lond. Math. Soc. (2)* **50** (1994), 361–374], we determine exactly which numbers in the interval $[1, 2]$ arise as values of $\mathrm{AD}(G)$. As a by-product, we show that the set of values of $\mathrm{AD}(G)$ does not contain all its limit points. Some other calculations or bounds for $\mathrm{AD}(G)$ are given for familiar classes of finite groups. We also indicate a connection between $\mathrm{AD}(G)$ and the commuting probability of G, and use this to show that every finite group G satisfying $\mathrm{AD}(G) < 61/15$ must be solvable; here, the value $61/15$ is the best possible.

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1. Introduction

1.1. Background context. Given a finite group G, the algebra of complex-valued functions on G (equipped with the pointwise product) only depends on the cardinality of G and does not detect the group structure. However, there is a canonical submultiplicative norm on this algebra, the *Fourier norm*, such that the resulting normed algebra $\mathrm{A}(G)$ characterizes the starting group G up to isomorphism. (More precisely: given finite groups G and H, there is an isometric algebra isomorphism between $\mathrm{A}(G)$ and $\mathrm{A}(H)$ if and only if G and H are isomorphic groups; this is a special case of a result of Walter [14].)

By identifying a subset of G with its indicator function, one can speak of the Fourier norm of a subset of G. Calculating Fourier norms of arbitrary subsets is hard (see [12] for a systematic approach), but there is one case where an exact calculation is possible and gives interesting answers. Consider the set $\{(g, g^{-1}) : g \in G\}$. The Fourier norm of this subset of $G \times G$, denoted by $\mathrm{AD}(G)$ in this paper, is the *Fourier anti-diagonal constant* mentioned in the title. It was recently shown by the author [3, Theorem 1.4] that we have the following explicit formula for $\mathrm{AD}(G)$:

$$\mathrm{AD}(G) = \frac{1}{|G|} \sum_{\varphi \in \mathrm{Irr}(G)} \varphi(\underline{1})^3, \tag{1-1}$$

where $\mathrm{Irr}(G)$ is the set of irreducible complex characters of G and $\varphi(\underline{1})$ is the degree of φ.

Equation (1-1) implies that $\mathrm{AD}(G \times H) = \mathrm{AD}(G)\,\mathrm{AD}(H)$ and that AD is invariant under isoclinism. Additionally, $\mathrm{AD}(H) \leq \mathrm{AD}(G)$ whenever $H \leq G$ (see Proposition 3.5 below). These hereditary properties suggest that $\mathrm{AD}(G)$, viewed as a numerical invariant of G, deserves further study. Furthermore, the sum on the right-hand side of (1-1) already arose in earlier work of Johnson [8] on Fourier algebras of compact groups. The results in [8, Section 4] provide an attractive application of the character theory of finite groups to obtain new (counter-)examples in functional analysis. (For a fuller discussion, see [3, Section 1].)

The following observations, taken from [8, Proposition 4.3], are easy consequences of (1-1):

- if G is abelian, then $\mathrm{AD}(G) = 1$;
- if G is nonabelian, then $\mathrm{AD}(G) \geq \frac{3}{2}$.

Since $\mathrm{AD}(G^n) = \mathrm{AD}(G)^n$, this shows that $\mathrm{AD}(G)$ can take arbitrarily large values. However, to the author's knowledge, nothing further has been done to study the possible values of $\mathrm{AD}(G)$ as G ranges over all nonabelian finite groups. The purpose of the present paper is to make a start on filling this gap.

1.2. Our main new results. The following result has probably been noticed independently by many readers of Johnson's paper, although it is not stated explicitly there. (A proof is given in Section 2 for the sake of completeness.)

PROPOSITION 1.1 (Implicitly folklore). *Let G be a finite group and suppose that $\varphi(\underline{1}) \leq 2$ for all $\varphi \in \mathrm{Irr}(G)$. Then,* $\mathrm{AD}(G) \in \{2 - n^{-1} : n \in \mathbb{N}\}$.

Moreover, every number in $\{2 - n^{-1} : n \in \mathbb{N}\}$ is realized as the AD-constant of some (nonunique) finite group: this can be seen by considering cyclic groups and dihedral groups. Our first main result is that these are the *only* values of AD attained by finite groups in the interval $[1, 2]$. To be precise, we state the following theorem.

THEOREM 1.2 (Possible values of $\mathrm{AD}(G)$ in $[1, 2]$). *Let G be a finite group and suppose that* $\mathrm{AD}(G) \leq 2$. *Then,* $\mathrm{AD}(G) \in \{2 - n^{-1} : n \in \mathbb{N}\}$.

COROLLARY 1.3. *The set* $\{\mathrm{AD}(G) : G \text{ a finite group}\}$ *is not a closed subset of* $[1, \infty)$.

Theorem 1.2 is an immediate consequence of combining Proposition 1.1 with the following lower bound for $\mathrm{AD}(G)$, which appears to be new.

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PROPOSITION 1.4. *Let G be a finite group. If there exists $\varphi \in \mathrm{Irr}(G)$ with $\varphi(\underline{1}) \geq 3$, then* $\mathrm{AD}(G) \geq 2 + |G'|^{-1}$.

The proof of Proposition 1.4 requires some basic character theory, but nothing harder than Frobenius reciprocity. Perhaps surprisingly, while we do need character theory for finite groups, we do not rely on any structure theory (we do not even need the Sylow theorems). In contrast, our other main result requires the classification of finite simple groups with characters of small degree.

THEOREM 1.5 (A threshold ensuring solvability). *Let G be a finite group. If* $\mathrm{AD}(G) < 61/15$*, then G is solvable.*

A direct calculation shows that $\mathrm{AD}(A_5) = 61/15$ and so, in this sense, Theorem 1.5 is sharp. Particular properties of A_5, such as its subgroup structure and its Schur multiplier, play an important role in the proof of Theorem 1.5, since we need to analyse perfect groups that quotient onto A_5.

One difficulty in proving Theorem 1.5 is that AD is not monotone (in either direction) with respect to taking quotients, and so knowing that a group H quotients onto A_5 does not immediately imply that $\mathrm{AD}(H) \geq \mathrm{AD}(A_5)$. Instead, we require a detour through the *commuting probability* $\mathrm{cp}(G)$ (see the start of Section 5 for its definition). Our strategy is inspired by an argument of Tong-Viet in [13] and, indeed, the main work needed to prove Theorem 1.5 lies in establishing the following stronger version of [13, Lemma 2.4].

PROPOSITION 1.6. *Let H be a finite nontrivial perfect group satisfying* $\mathrm{cp}(H) > 1/20$. *Then, $H \cong A_5$ or $H \cong$* $\mathrm{SL}(2, 5)$.

1.3. Outline of this paper. After some preliminary results in Section 2, the proof of Proposition 1.4 is given in Section 3. Since the paper is intended for a general audience, we spell things out in more detail than specialists in group theory would require. In Section 4.1, we calculate the values of $\mathrm{AD}(G)$ for some particular families of groups, some of which are related to calculations in earlier sections; and in Section 4.2, we present some partial results on the general theme that 'small values' of $\mathrm{AD}(G)$ imply that G is close to abelian in some sense. Section 5 is dedicated to the proof of Proposition 1.6 and Theorem 1.5; this is the only part of the paper that makes use of the theory of the cp invariant. In the appendix, we collect some proofs of results that are used in the main body of the paper; these results are special cases or weaker versions of known results, but we take the opportunity to provide some extra details and give more elementary arguments.

We finish this introduction by establishing some conventions and fixing notation. To reduce unnecessary repetition, we adopt the following convention: henceforth, *all groups are assumed to be finite* unless explicitly stated otherwise. The identity element of a group G is denoted by $\underline{1}$, or $\underline{1}_G$ if we wish to avoid ambiguity, and the *derived subgroup* of G (also known as its commutator subgroup) is denoted by G'.

Throughout this article, all representations and characters are taken over the complex field. The basic representation theory and character theory that we need can be found in several introductory texts, such as [7]. We denote the *degree* of a character φ by $\varphi(\underline{1})$; note that this is equal to the dimension of any representation whose trace is φ.

The set of irreducible characters of G is denoted by $\mathrm{Irr}(G)$ and we write $\mathrm{cd}(G)$ for the set $\{\varphi(\underline{1})\colon \varphi \in \mathrm{Irr}(G)\}$ (note that here, we are not counting the multiplicities of the irreducible character degrees). We write $\mathrm{Irr}_n(G)$ for the set of all $\varphi \in \mathrm{Irr}(G)$ that have degree n. If G is nonabelian, we define

$$\mathrm{mindeg}(G) := \min\{d \geq 2\colon\ \mathrm{Irr}_d(G) \text{ is nonempty}\}.$$

For any G (possibly abelian), we define

$$\mathrm{maxdeg}(G) := \max\{\varphi(\underline{1})\colon \varphi \in \mathrm{Irr}(G)\}.$$

Finally, given a group G, we equip $\mathbb{C}^G$ with the following inner product:

$$\langle \varphi, \psi \rangle := \frac{1}{|G|} \sum_{x \in G} \varphi(x)\overline{\psi(x)}.$$

If ψ is a character of G, then it is irreducible if and only if $\langle \psi, \psi \rangle = 1$ [7, Theorem 14.20].

2. Some easy lower bounds on AD

We start by giving a proof of Proposition 1.1, since it also serves as a prototype for later arguments. No novelty is claimed.

Proof of Proposition 1.1. Since $\mathrm{cd}(G) \subseteq \{1, 2\}$,

$$\mathrm{AD}(G) = \frac{1}{|G|}(|\mathrm{Irr}_1(G)| + 8|\mathrm{Irr}_2(G)|).$$

On the other hand, basic character theory tells us that

$$1 = \frac{1}{|G|} \sum_{\varphi \in \mathrm{Irr}(G)} \varphi(\underline{1})^2 = \frac{1}{|G|}(|\mathrm{Irr}_1(G)| + 4|\mathrm{Irr}_2(G)|),$$

and therefore $\mathrm{AD}(G) - 2 = -|\mathrm{Irr}_1(G)|\,|G|^{-1}$.

It is also standard (see for example [7, Theorem 17.11]) that, since $\mathrm{Irr}_1(G)$ can be identified with the (Pontrjagin) dual of the abelian group G/G', we have $|\mathrm{Irr}_1(G)| = |G : G'|$. Hence, $\mathrm{AD}(G) = 2 - |G'|^{-1}$ and since $|G'| \in \mathbb{N}$, the result follows. □

Example 2.1 (Dihedral groups). Let G be a dihedral group of order $2k$, so that $\mathrm{cd}(G) = \{1, 2\}$. If k is odd, then $|G'| = k$, and if k is even, then $|G'| = k/2$. By repeating the calculation in the proof of Proposition 1.1, we see that $\mathrm{AD}(G) = 2 - (1/k)$ when k is odd and $\mathrm{AD}(G) = 2 - (2/k)$ when k is even.

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For general nonabelian G, the proof of Proposition 1.1 still suggests a way to proceed. Informally, since $\varphi(\underline{1})^3 \geq \operatorname{mindeg}(G)\varphi(\underline{1})^2$ for all $\varphi \in \operatorname{Irr}(G) \setminus \operatorname{Irr}_1(G)$, we can add a correction factor to $\operatorname{AD}(G)$ to obtain something bounded below by $\operatorname{mindeg}(G)$, and the size of the correction factor is controlled by the size of $|G'|$. Making this precise leads us to the following lemma, which provides a convenient tool for dealing with 'generic' cases.

LEMMA 2.2 (An all-purpose lower bound). *Let G be nonabelian, and let $m, n \in \mathbb{N}$ satisfy* $\operatorname{mindeg}(G) \geq m$ *and* $|G'| \geq n$. *Then,*

$$\operatorname{AD}(G) \geq 1 + (m-1)\Big(1 - \frac{1}{n}\Big).$$

PROOF. Since $\operatorname{Irr}_j(G)$ is empty whenever $2 \leq j \leq m-1$,

$$\operatorname{AD}(G) - \frac{|\operatorname{Irr}_1(G)|}{|G|} = \frac{1}{|G|}\sum_{n \geq m} n^3 |\operatorname{Irr}_n(G)|$$

and

$$1 - \frac{|\operatorname{Irr}_1(G)|}{|G|} = \frac{1}{|G|}\sum_{n \geq m} n^2 |\operatorname{Irr}_n(G)|.$$

Hence,

$$\operatorname{AD}(G) - \frac{|\operatorname{Irr}_1(G)|}{|G|} \geq m\Big(1 - \frac{|\operatorname{Irr}_1(G)|}{|G|}\Big).$$

As in the proof of Proposition 1.1, $|\operatorname{Irr}_1(G)| = |G : G'|$. Hence, $|\operatorname{Irr}_1(G)||G|^{-1} \leq n^{-1}$. Plugging this into the previous inequality gives

$$\operatorname{AD}(G) \geq m - (m-1)\frac{|\operatorname{Irr}_1(G)|}{|G|} \geq m - \frac{m-1}{n},$$

which completes the proof. □

COROLLARY 2.3 (A sharper form of [8, Proposition 4.3]). *Let G be nonabelian. Then, either* $\operatorname{AD}(G) \geq \frac{5}{3}$, *or* $\operatorname{cd}(G) = \{1, 2\}$ *and* $|G'| = 2$; *in the latter case,* $\operatorname{AD}(G) = \frac{3}{2}$.

PROOF. Note that $\operatorname{mindeg}(G) \geq 2 \iff G$ is nonabelian $\iff |G'| \geq 2$. Therefore, if either $\operatorname{mindeg}(G) \geq 3$ or $|G'| \geq 3$, applying Lemma 2.2 with $(m, n) = (3, 2)$ and $(m, n) = (2, 3)$ yields $\operatorname{AD}(G) \geq \frac{5}{3}$. Otherwise, we must have $\operatorname{cd}(G) = \{1, 2\}$ and $|G'| = 2$, and following the steps in the proof of Lemma 2.2 yields $\operatorname{AD}(G) = \frac{3}{2}$. □

REMARK 2.4. In [3], the present author studied a generalization of $\operatorname{AD}(G)$ to the setting of virtually abelian groups, and showed that $\operatorname{AD}(G) = \frac{3}{2}$ if and only if $|G : Z(G)| = 4$. The proof goes via a version of Corollary 2.3, but substantial work is required since G may be infinite. It is therefore worth noting that when G is finite, there is a much simpler proof of this equivalence; details are given in Appendix A.1.

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We saw in the proof of Corollary 2.3 that if $\mathrm{AD}(G) > \frac{3}{2}$, then either $\mathrm{mindeg}(G) \geq 3$ or $|G'| \geq 3$. The example of S_3 shows that we can have $\mathrm{mindeg}(G) = 2$ and $|G'| = 3$. In contrast, the next result shows that we can never have $\mathrm{mindeg}(G) = 3$ and $|G'| = 2$.

LEMMA 2.5. *Let G be a group with $|G'| = 2$. If $\varphi \in \mathrm{Irr}(G)$ and $\varphi(\underline{1}) > 1$, then $\varphi(\underline{1})$ is even.*

Lemma 2.5 follows from more precise results of Miller, stated in [11, Section 1]. His presentation is rather terse and uses the finiteness of G in an essential way. We provide a direct proof of Lemma 2.5 in Appendix A.2, which also works for (irreducible, finite-dimensional, unitary) representations of infinite groups.

PROPOSITION 2.6. *Let G be nonabelian. If $2 \notin \mathrm{cd}(G)$, then $\mathrm{AD}(G) \geq \frac{7}{3}$.*

PROOF. We split into two cases. If $|G'| \geq 3$, then using Lemma 2.2 with $m = 2$ and $n = 3$ gives $\mathrm{AD}(G) \geq \frac{7}{3}$. If $|G'| = 2$, then $3 \notin \mathrm{cd}(G)$ by Lemma 2.5 and so $\mathrm{mindeg}(G) \geq 4$; using Lemma 2.2 with $m = 4$ and $n = 2$ gives $\mathrm{AD}(G) \geq \frac{5}{2} > \frac{7}{3}$. □

In both cases of the proof, the lower bounds are sharp; see Example 4.1 below for details.

3. The proof of Proposition 1.4

For a finite set X and a function $f : X \to \mathbb{C}$, we write $\mathrm{supp}(f)$ for the *support of f*, that is, the set $\{x \in X : f(x) \neq 0\}$.

LEMMA 3.1 (The '$\mathcal{L}$-orbit method' for lower bounds). *Let $\varphi \in \mathrm{Irr}(G)$ and let $n = \varphi(\underline{1})$. Let K be the normal subgroup of G generated by $\mathrm{supp}(\varphi)$. Then,*

$$|\mathrm{Irr}_n(G)| \geq |\mathrm{Irr}_1(G)|\,|G : K|^{-1}.$$

PROOF. To simplify notation, let $\mathcal{L} = \mathrm{Irr}_1(G)$. Then, $\mathcal{L}$ is a group with respect to the pointwise product, and multiplication of characters defines a group action $\mathcal{L} \times \mathrm{Irr}_n(G) \to \mathrm{Irr}_n(G)$ for each n. The $\mathcal{L}$-orbit of φ is a subset of $\mathrm{Irr}_n(G)$ and it has size $|\mathcal{L}|\,|\mathrm{Stab}_{\mathcal{L}}(\varphi)|^{-1}$.

Let $\mathbb{T}$ denote the set of complex numbers of unit modulus, viewed as a group with respect to multiplication. Observe that each $\gamma \in \mathcal{L}$ is $\mathbb{T}$-valued and that

$$\mathrm{Stab}_{\mathcal{L}}(\varphi) = \{\gamma \in \mathcal{L} : \gamma\varphi = \varphi\} = \{\gamma \in \mathcal{L} : \gamma(x) = 1 \text{ for all } x \in \mathrm{supp}(\varphi)\},$$

which is the set of group homomorphisms $G \to \mathbb{T}$ that factor through $G \to G/K$. Therefore, writing A for the abelianization of G/K,

$$|\mathrm{Stab}_{\mathcal{L}}(\varphi)| = |A| \leq |G/K| = |G : K|,$$

and so the $\mathcal{L}$-orbit of φ has at least $|\mathcal{L}|\,|G : K|^{-1}$ elements. The result now follows. □

COROLLARY 3.2. *Let $n \in \mathbb{N}$. If $\mathrm{Irr}_n(G)$ is nonempty, then $|\mathrm{Irr}_n(G)| \geq n^{-2}\,|\mathrm{Irr}_1(G)|$.*

PROOF. Pick some $\varphi \in \mathrm{Irr}_n(G)$ and let K be the normal subgroup of G generated by $\mathrm{supp}(\varphi)$. Since φ is irreducible, $\langle \varphi, \varphi \rangle = 1$. Therefore, since $|\varphi(x)| \le \varphi(\underline{1}) = n$ for all $x \in G$,

$$n^2 |\mathrm{supp}(\varphi)| \ge \sum_{x \in G} |\varphi(x)|^2 = |G|.$$

Hence, $|G : K| \le |G|\,|\mathrm{supp}(\varphi)|^{-1} \le n^2$. Applying Lemma 3.1, the result follows. □

REMARK 3.3. Although the estimates in the proof of Lemma 3.1 are potentially wasteful, the resulting lower bound in Corollary 3.2 is sharp. For if G is an extraspecial group of order 2^{2k+1}, it has exactly 2^{2k} characters of degree 1 and a single irreducible character of degree 2^k. However, it is important later that in certain situations, we can do significantly better (Lemma 3.10 below).

PROPOSITION 3.4. *If G is nonabelian, then* $\mathrm{AD}(G) \ge 2 + (\mathrm{maxdeg}(G) - 3)|G'|^{-1}$.

PROOF. Let $d = \mathrm{maxdeg}(G)$. Since $|G| = \sum_{n=1}^{d} n^2 |\mathrm{Irr}_n(G)|$,

$$\begin{aligned}\mathrm{AD}(G) - 2 &= \frac{1}{|G|} \sum_{n=1}^{d} (n^3 - 2n^2)|\mathrm{Irr}_n(G)| \\ &\ge -\frac{|\mathrm{Irr}_1(G)|}{|G|} + (d^3 - 2d^2)\frac{|\mathrm{Irr}_d(G)|}{|G|}.\end{aligned}$$

Since $\mathrm{Irr}_d(G)$ is nonempty, applying Corollary 3.2 gives the desired inequality. □

The rest of this section deals with cases where $\mathrm{cd}(G) = \{1, 2, 3\}$. We require a property of AD that is not obvious from the definition, but which seems to be crucial to understanding its behaviour.

PROPOSITION 3.5 (Johnson). AD *is monotone with respect to subgroup inclusion. That is, if $H \le G$, then* $\mathrm{AD}(H) \le \mathrm{AD}(G)$.

REMARK 3.6. Proposition 3.5 follows from results in [8, Section 4] concerning 'amenability constants' of Fourier algebras, or from the general theory in [3, Section 2]. One can give a direct proof, based on considering the induction of characters from H to G: see the author's *MathOverflow* question [4] and the comments and answers. It is quite possible that a direct proof along these lines was already known to Johnson.

PROPOSITION 3.7. *Let G be a group such that* $\mathrm{AD}(G) < \frac{7}{3}$ *and let $H \le G$. If $|G : H| = 2$ and* $\mathrm{cd}(G) = \{1, 2, 3\}$, *then* $\mathrm{cd}(H) = \{1, 2, 3\}$.

The proof of Proposition 3.7 requires some general facts, which we state in a separate lemma for convenience.

LEMMA 3.8 (Character degrees of subgroups of index 2). *Let $H \le G$ with $|G : H| = 2$. Then,* $\mathrm{maxdeg}(H) \le \mathrm{maxdeg}(G)$ *and* $\mathrm{cd}(G) \subseteq \mathrm{cd}(H) \cup 2\,\mathrm{cd}(H)$.

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Both parts of the lemma are standard results. For completeness, we quickly sketch their proofs.

PROOF. Given $\psi \in \mathrm{Irr}(H)$, let $\varphi \in \mathrm{Irr}(G)$ be one of the irreducible summands of $\mathrm{Ind}_H^G\psi$. By Frobenius reciprocity, ψ is contained in $\varphi|_H$, so $\psi(\underline{1}) \le \varphi|_H(\underline{1}) = \varphi(\underline{1}) \le \mathrm{maxdeg}(G)$. This proves the first claim.

For the second claim, let $\varphi \in \mathrm{Irr}(G)$. If $\varphi|_H$ is irreducible, then $\varphi(\underline{1}) \in \mathrm{cd}(H)$. If not, then it follows from Clifford theory (or direct arguments using Frobenius reciprocity) that $\varphi|_H$ splits as the sum of two irreducible characters, say β_1 and β_2, which satisfy $\varphi = \mathrm{Ind}_H^G\beta_1 = \mathrm{Ind}_H^G\beta_2$. In particular, $\varphi(\underline{1}) = 2\beta_1(\underline{1}) \in 2\,\mathrm{cd}(H)$. □

PROOF OF PROPOSITION 3.7. By monotonicity of AD (Proposition 3.5), $\mathrm{AD}(H) \le \mathrm{AD}(G) < \frac{7}{3}$. Hence, by Lemma 3.8, $\mathrm{maxdeg}(H) \le 3$ and $3 \in \mathrm{cd}(H)$. Since H is nonabelian and $\mathrm{AD}(H) < \frac{7}{3}$, the contrapositive of Proposition 2.6 implies that $2 \in \mathrm{cd}(H)$. □

We now observe that two-dimensional irreducible representations of G can be used to produce three-dimensional representations with useful properties. In what follows, ε denotes the constant function 1, regarded as the trivial representation of the group.

LEMMA 3.9. *Let π be a two-dimensional irreducible representation of G and let π^* denote its contragredient.*

(i) *The representation ε occurs in $\pi \otimes \pi^*$ with multiplicity* 1.
(ii) *Let ρ be the summand in $\pi \otimes \pi^*$ complementary to ε. Suppose that ρ is reducible. Then, G has a subgroup of index* 2.

This is surely not a new observation, but since we are unaware of a precise reference, a full proof is given below.

PROOF. Part (i) follows from Schur's lemma. (Alternatively, let $\psi = \mathrm{Tr}\,\pi$; then the multiplicity of ε in $\pi \otimes \pi^*$ is equal to $\langle \psi\overline{\psi}, \varepsilon\rangle = |G|^{-1}\sum_{x\in G}\psi(x)\overline{\psi(x)} = 1$.)

For part (ii), let $\varphi = \mathrm{Tr}\,\rho$; by part (i), φ is real-valued and $\langle \varphi, \varepsilon\rangle = 0$. We claim that there exists a real-valued character on G of degree 1 occurring as a summand of φ. Assuming such a character exists, it may be viewed as a group homomorphism $\sigma : G \to \{\pm 1\}$. Since ε is not a summand of φ, we know that $\sigma \ne \varepsilon$ and so $\ker\sigma$ has index 2 in G, as required.

To prove the claim, note that since φ has degree 3 and is reducible, its decomposition into irreducible characters includes at least one $\gamma \in \mathrm{Irr}_1(G)$. If γ is real-valued, we are done. If not, then $\overline{\gamma} \ne \gamma$ and $\langle \varphi, \overline{\gamma}\rangle = \overline{\langle \varphi, \gamma\rangle} \ge 1$. Hence, γ and $\overline{\gamma}$ occur in φ with multiplicity 1, and $\varphi = \gamma + \overline{\gamma} + \sigma$, where $\sigma \in \mathrm{Irr}_1(G)$ is real-valued. □

If G has no subgroups of index 2 and $2 \in \mathrm{cd}(G)$, then by Lemma 3.9, for each $\psi \in \mathrm{Irr}_2(G)$, the character $\beta := \psi\overline{\psi} - \varepsilon$ must be irreducible; and because β is a 'small perturbation' of a nonnegative character, we can obtain improved lower bounds on $|\mathrm{supp}(\beta)|$, allowing us to apply Lemma 3.1 more effectively. It turns out that the relevant

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estimates have nothing to do with group structure, so we present them as a separate lemma.

LEMMA 3.10. *Let X be a finite nonempty set and let $d \geq 1$. Suppose that $f : X \to [-1, d]$ has mean 0 and variance 1, that is,*

$$\sum_{x\in X} f(x) = 0 \quad \text{and} \quad \sum_{x\in X} f(x)^2 = |X|.$$

Then, $|\mathrm{supp}(f)| \geq d^{-1}|X|$.

PROOF. Fix some 'threshold value' $c \in [0, d]$, to be determined later, and partition $\mathrm{supp}(f)$ as $N \cup P \cup R$ where:

- $N := \{x \in X : -1 \leq f(x) < 0\}$;
- $P := \{x \in X : 0 < f(x) \leq c\}$;
- $R := \{x \in X : c < f \leq d\}$.

Then, since $\sum_{x\in \mathrm{supp}(f)} f(x)^2 = |X|$,

$$\begin{aligned} |X| &= \sum_{x\in N} f(x)^2 + \sum_{x\in P} f(x)^2 + \sum_{x\in R} f(x)^2 \\ &\leq \sum_{x\in N} |f(x)| + c \sum_{x\in P} f(x) + d \sum_{x\in R} f(x). \end{aligned} \tag{$*$}$$

On the other hand, since $\sum_{x\in \mathrm{supp}(f)} f(x) = 0$,

$$\sum_{x\in P} f(x) = \sum_{x\in N} |f(x)| - \sum_{x\in R} f(x),$$

and substituting this into (*) yields

$$\begin{aligned} |X| \leq (c+1) \sum_{x\in N} |f(x)| + (d-c) \sum_{x\in R} f(x) &\leq (c+1)|N| + d(d-c)|R| \\ &\leq \max(c+1, d(d-c))\, |\mathrm{supp}(f)|. \end{aligned}$$

Taking $c = d - 1$ gives $|X| \leq d|\mathrm{supp}(f)|$ as required. □

PROPOSITION 3.11. *Suppose that G has no subgroups of index 2, but has an irreducible representation of degree 2. Then,* $|\mathrm{Irr}_3(G)| \geq \frac{1}{3}|\mathrm{Irr}_1(G)|$.

PROOF. Let $\psi \in \mathrm{Irr}_2(G)$ and let $\beta = \psi\overline{\psi} - \varepsilon$. We observe that:

- β takes values in $[-1, 3]$, since $0 \leq |\psi(x)|^2 \leq 4$ for all $x \in G$;
- $\langle \beta, \varepsilon \rangle = 0$, by Lemma 3.9(i);
- $\langle \beta, \beta \rangle = 1$, since β is irreducible by Lemma 3.9(ii).

Hence, by Lemma 3.10, $|\mathrm{supp}(\beta)| \geq \frac{1}{3}|G|$, and applying Lemma 3.1 completes the proof. □

REMARK 3.12. In general, the bound in Proposition 3.11 cannot be improved. To see this, take $G = \mathrm{SL}(2,3)$. Then, $\mathrm{cd}(G) = \{1, 2, 3\}$ and $|\mathrm{Irr}_1(G)| = 3 = 3|\mathrm{Irr}_3(G)|$, while $|G : G'| = 3$ (so that G cannot quotient onto the two-element group).

PROOF OF PROPOSITION 1.4. Let G be a group with $\mathrm{maxdeg}(G) \geq 3$. If $\mathrm{maxdeg}(G) \geq 4$, then $\mathrm{AD}(G) \geq 2 + |G'|^{-1}$ by Proposition 3.4. So we assume henceforth that $\mathrm{maxdeg}(G) = 3$. Note that this implies $|G'| \geq 3$, by Lemma 2.5. Moreover, if $\mathrm{cd}(G) = \{1, 3\}$, then by Proposition 2.6, $\mathrm{AD}(G) \geq \frac{7}{3} \geq 2 + |G'|^{-1}$.

It only remains to deal with the cases where $\mathrm{cd}(G) = \{1, 2, 3\}$. If $\mathrm{AD}(G) \geq \frac{7}{3}$, then we are done, as before. So we may assume that $\mathrm{cd}(G) = \{1, 2, 3\}$ and $\mathrm{AD}(G) < \frac{7}{3}$. Put $H_0 = G$ and apply the following recursive procedure: if $n \in \mathbb{N}$ and H_{n-1} has a subgroup of index 2, choose H_n to be such a subgroup; otherwise, stop. Note that at each stage, Proposition 3.7 ensures that $\mathrm{cd}(H_n) = \{1, 2, 3\}$.

Since G is finite, this procedure must terminate; let H be the last subgroup in this sequence. Since $\mathrm{cd}(H) = \{1, 2, 3\}$,

$$\mathrm{AD}(H) = \frac{1}{|H|}(|\mathrm{Irr}_1(H)| + 8|\mathrm{Irr}_2(H)| + 27|\mathrm{Irr}_3(H)|) \quad \text{and}$$
$$1 = \frac{1}{|H|}(|\mathrm{Irr}_1(H)| + 4|\mathrm{Irr}_2(H)| + 9|\mathrm{Irr}_3(H)|).$$

Hence, $\mathrm{AD}(H) = 2 - |H|^{-1}|\mathrm{Irr}_1(H)| + 9|H|^{-1}|\mathrm{Irr}_3(H)|$. Since H has no subgroups of index 2, it satisfies the hypotheses of Proposition 3.11, and so

$$\mathrm{AD}(H) \geq 2 + \frac{2|\mathrm{Irr}_1(H)|}{|H|} = 2 + \frac{2}{|H'|}.$$

As $\mathrm{AD}(G) \geq \mathrm{AD}(H)$ (Proposition 3.5) and $|G'| \geq |H'|$, we conclude that $\mathrm{AD}(G) \geq 2 + 2|G'|^{-1}$, which completes the proof of Proposition 1.4. □

4. Further examples and implications of small values

4.1. Values of AD for particular groups. We present three families of groups with rather different properties (nilpotent, solvable with trivial centre and quasi-simple), where one obtains rather simple formulae for the AD-constants in each family. In each case, the ratio $\mathrm{AD}(G)\,\mathrm{maxdeg}(G)^{-1}$ converges to 1 as $|G| \to \infty$.

EXAMPLE 4.1 (Extraspecial p-groups). Let p be a prime and let $n \in \mathbb{N}$. If G is an extraspecial p-group of order p^{2n+1}, then the degrees of its irreducible characters and their multiplicities are well documented. Namely, G has exactly p^{2n} characters of degree 1 and exactly $p - 1$ irreducible characters of degree p^n. Hence,

$$\mathrm{AD}(G) = \frac{p^{2n} \cdot 1^3 + (p-1)p^{3n}}{p^{2n+1}} = p^{n-1}(p-1) + \frac{1}{p}.$$

We note two particular cases, relevant to Proposition 2.6. If $p = 2$ and $n = 2$, then $\mathrm{cd}(G) = \{1, 4\}$ and $\mathrm{AD}(G) = \frac{5}{2}$. If $p = 3$ and $n = 1$, then $\mathrm{cd}(G) = \{1, 3\}$ and $\mathrm{AD}(G) = \frac{7}{3}$.

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EXAMPLE 4.2 (Affine groups of finite fields). For q a prime power ≥ 3, let $\mathbb{F}_q$ denote the finite field with q elements and consider the natural semidirect product $\mathbb{F}_q \rtimes \mathbb{F}_q^\times$ (sometimes referred to as the affine group or '$ax + b$ group' of $\mathbb{F}_q$). This group has exactly $q - 1$ characters of degree 1 and a single irreducible character of degree $q - 1$. Hence,

$$\mathrm{AD}(\mathbb{F}_q \rtimes \mathbb{F}_q^\times) = \frac{(q-1)\cdot 1^3 + (q-1)^3}{q(q-1)} = q - 2 + \frac{2}{q}.$$

Note that when $q = 3$, this group is isomorphic to the dihedral group of order 6, and its AD-constant is $\frac{5}{3}$; this matches the calculation in Example 2.1.

EXAMPLE 4.3 (Special linear groups of degree 2). Let q be a prime power and let $\mathrm{SL}(2, q)$ denote the special linear group of degree 2 over the finite field with q elements; this has order $q^3 - q$.

For q even, put $q = 2r$; then $\mathrm{Irr}(\mathrm{SL}(2, q))$ is the union of four pairwise disjoint sets X_1, X_{q-1}, X_q and X_{q+1}, where each member of X_j has degree j, and

$$|X_1| = 1; |X_{2r-1}| = r; |X_{2r}| = 1; |X_{2r+1}| = r - 1.$$

For q odd, put $q = 2r + 1$; then $\mathrm{Irr}(\mathrm{SL}(2, q))$ is the union of six pairwise disjoint sets X_1, X_r, X_{r+1}, X_q and X_{q+1}, where each member of X_j has degree j, and

$$|X_1| = 1; |X_r| = 2; |X_{r+1}| = 2; |X_{2r}| = r; |X_{2r+1}| = 1; |X_{2r+2}| = r - 1.$$

By brute-force calculation, we eventually obtain

$$\mathrm{AD}(\mathrm{SL}(2, q)) = \begin{cases} \dfrac{q^3 - 3}{q^2 - 1} = q - \dfrac{1}{q-1} + \dfrac{2}{q+1} & \text{for } q \text{ even,} \\ \dfrac{2q^3 - q^2 - 9}{2(q^2 - 1)} = q - \dfrac{1}{2} - \dfrac{2}{q-1} + \dfrac{3}{q+1} & \text{for } q \text{ odd.} \end{cases}$$

The next set of examples was suggested to the author by P. Levy.

EXAMPLE 4.4 (Finite subgroups of SO(3) and SU(2)). We ignore the cyclic groups and dihedral groups, and their double covers inside SU(2), since these are covered by previous results. So there are only three new examples to consider. In the following list, when we refer to the 'character degrees' of a group H, we mean 'the degrees of its irreducible characters, listed with multiplicity'.

(a) The alternating group A_4 has character degrees $1, 1, 1, 3$. Its double cover is the binary tetrahedral group $2T \cong \mathrm{SL}(2, 3)$, whose character degrees are $1, 1, 1, 2, 2, 2, 3$. Thus,

$$\mathrm{AD}(A_4) = \tfrac{30}{12} = \tfrac{5}{2} \quad \text{and} \quad \mathrm{AD}(2T) = \tfrac{54}{24} = \tfrac{9}{4} < \mathrm{AD}(A_4).$$

(b) The symmetric group S_4 has character degrees $1, 1, 2, 3, 3$. Its double cover is the binary octahedral group $2O$, whose character degrees are $1, 1, 2, 2, 2, 3, 3, 4$.

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Thus,

$$\mathrm{AD}(S_4) = \tfrac{64}{24} = \tfrac{8}{3} \quad \text{and} \quad \mathrm{AD}(2O) = \tfrac{144}{48} = 3 > \mathrm{AD}(S_4).$$

(c) The alternating group A_5 has character degrees $1, 3, 3, 4, 5$. Its double cover is the binary icosahedral group $2I \cong \mathrm{SL}(2, 5)$, whose character degrees are $1, 2, 2, 3, 3, 4, 4, 5, 6$. Thus,

$$\mathrm{AD}(A_5) = \tfrac{244}{60} = \tfrac{61}{15} \quad \text{and} \quad \mathrm{AD}(2I) = \tfrac{540}{120} = \tfrac{9}{2} > \mathrm{AD}(A_5).$$

REMARK 4.5. It is already known that although AD cannot increase when passing to subgroups, it can increase when passing to a quotient. For instance, in a 'note added in proof' in [10], it is observed that the Schur cover of A_6 has an AD-constant strictly smaller than that of the triple cover of A_6. However, Example 4.4(a) shows that there exists a much smaller example.

4.2. Structural consequences for G of upper bounds on AD.

PROPOSITION 4.6 (A cheap lower bound for p-groups). *Let p be a prime. If G is a nonabelian p-group, then $\mathrm{AD}(G) \geq p - 1 + 1/p$. Equality is attained by an extraspecial p-group of order p^3.*

PROOF. Since G is a p-group, both $\mathrm{mindeg}(G)$ and $|G'|$ are powers of p. Therefore, both are $\geq p$, since G is nonabelian. The rest follows from Lemma 2.2 and the calculation in Example 4.1. □

A similar idea can be used to control (sub)groups of odd order whose AD-constants are small. The next result is a slightly stronger version of an observation by G. Robinson (personal communication).

LEMMA 4.7. *If $\mathrm{AD}(G) < \frac{7}{3}$, then every odd order subgroup of G is abelian.*

PROOF. We prove the contrapositive. Suppose that G has a nonabelian subgroup H that has odd order. Since $\varphi(\underline{1})$ divides $|H|$ for each $\varphi \in \mathrm{Irr}(H)$, we have $\mathrm{mindeg}(H) \geq 3$; since $|H'|$ divides $|H|$, we have $|H'| \geq 3$. Therefore, by monotonicity (Proposition 3.5) and Lemma 2.2,

$$\mathrm{AD}(G) \geq \mathrm{AD}(H) \geq 1 + (3-1)\tfrac{2}{3} = \tfrac{7}{3},$$

as required. □

COROLLARY 4.8. *If G is nilpotent and $\mathrm{AD}(G) < \frac{7}{3}$, then G is the product of a 2-group and an abelian group of odd order.*

PROOF. If p is an odd prime, then by Lemma 4.7, each p-Sylow subgroup of G is abelian. However, since G is finite and nilpotent, it factorizes as the direct product of its Sylow subgroups. □

REMARK 4.9. We can show by relatively elementary arguments that $\mathrm{AD}(G) > 4$ whenever G is nonabelian and simple; since $\mathrm{AD}(A_5) = 61/15$, this is already quite close to the optimal result. Although we obtain a stronger result in Section 5, we include the proof of the weaker result here as an illustration of our earlier method.

The main idea is similar to the proof of Lemma 2.2. Let $m = \mathrm{mindeg}(G)$. Since m divides $|G| = \sum_{\varphi\in\mathrm{Irr}(G)} \varphi(\underline{1})^2$ and since $|\mathrm{Irr}_1(G)| = 1$,

$$\sum_{\varphi\in\mathrm{Irr}(G),\varphi(\underline{1})>m} \varphi(\underline{1})^2 \equiv -1(\mathrm{mod}\ \ m).$$

Hence, there is at least one $\sigma \in \mathrm{Irr}(G)$ with $\sigma(\underline{1}) \geq m+1$. Therefore,

$$\begin{aligned}\mathrm{AD}(G) - m &= \sum_{\varphi\in\mathrm{Irr}(G)} \frac{(\varphi(\underline{1})-m)\varphi(\underline{1})^2}{|G|}\\ &\geq -\frac{m-1}{|G|} + \frac{(\sigma(\underline{1})-m)\sigma(\underline{1})^2}{|G|} \geq -\frac{m-1}{|G|} + \frac{(m+1)^2}{|G|} > 0.\end{aligned}$$

If $m \geq 4$, this immediately gives $\mathrm{AD}(G) > 4$. So, it only remains to deal with cases where $m = 3$. The finite simple subgroups of $\mathrm{PGL}(3,\mathbb{C})$ were determined by Blichfeldt in [2] and, using his classification, one can show that the only simple groups with $m = 3$ are A_5 and $\mathrm{PSL}(2,7)$ (some further explanation is given in Appendix A.3). We see in Example 4.4(c) that $\mathrm{AD}(A_5) = 61/15 > 4$, and since $\mathrm{PSL}(2,7)$ has character degrees $1,3,3,6,7,8$, we find that $\mathrm{AD}(\mathrm{PSL}(2,7)) = 563/84 > 6$.

5. A sharp lower bound on AD for nonsolvable groups

Our aim in this section is to prove Theorem 1.5: if G is nonsolvable, then $\mathrm{AD}(G) \geq 61/15$. One difficulty, if we rely on our existing tools, is that although AD behaves well with respect to taking subgroups, it does not behave well with respect to taking quotients (see Remark 4.5).

Instead, our proof is inspired by techniques used in [13] to prove an analogous 'threshold' result for the quantity

$$f(G) := \frac{1}{|G|}\sum_{\varphi\in\mathrm{Irr}(G)} \varphi(\underline{1}).$$

The key in [13] is to exploit the inequality $f(G)^2 \leq \mathrm{cp}(G)$, where the *commuting probability* $\mathrm{cp}(G)$ is equal to $|G|^{-1}|\mathrm{Irr}(G)|$. (Strictly speaking, this is not the original definition of cp, but its equivalence with the original definition is well known.) The inequality relating f with cp is immediate from the Cauchy–Schwarz inequality; in our setting, we can use Hölder's inequality to obtain an analogous relationship between AD and cp, but in the opposite direction.

PROPOSITION 5.1. *For every G, we have $1 \leq \mathrm{AD}(G)^2\,\mathrm{cp}(G)$. Equality is strict if G is nonabelian.*

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PROOF. Applying Hölder's inequality with conjugate exponents $\frac{3}{2}$ and 3 gives

$$|G| = \sum_{\varphi\in\mathrm{Irr}(G)} \varphi(\underline{1})^2 \cdot 1 \leq \Big(\sum_{\varphi\in\mathrm{Irr}(G)} \varphi(\underline{1})^3 \Big)^{2/3} \Big(\sum_{\varphi\in\mathrm{Irr}(G)} 1^3 \Big)^{1/3},$$

and the inequality is strict unless every $\varphi \in \mathrm{Irr}(G)$ has the same degree, that is, unless G is abelian. The result now follows by dividing both sides by $|G|$ and then cubing. $\square$

REMARK 5.2. The invariant cp has been intensively studied and, in particular, it is shown in [6, Theorem 11] that if $1/12 > \mathrm{cp}(G) > 3/40$, then G is solvable. Although this result itself is not strong enough to imply Theorem 1.5, the ideas in its proof can be seen (refracted through the prism of [13]) in what follows.

The key advantage of working with cp, compared with either f or AD, is that it behaves well with respect to both taking subgroups and taking quotients. In particular, we make crucial use of the following result.

LEMMA 5.3 (Gallagher, [5]). *Suppose that $N \trianglelefteq G$. Then,*

$$\min(\mathrm{cp}(G/N), \mathrm{cp}(N)) \geq \mathrm{cp}(G/N)\,\mathrm{cp}(N) \geq \mathrm{cp}(G).$$

It was observed by Dixon that the largest value of cp on simple nonabelian groups is attained at A_5. We need some information about the second largest value attained by cp on this class of groups. The following is a slightly stronger version of [13, Lemma 2.3].

LEMMA 5.4. *There exists $1/28 \leq \delta_0 \leq 1/20$ with the following property: if S is a finite nonabelian simple group and $\mathrm{cp}(S) > \delta_0$, then $S \cong A_5$ (and $\mathrm{cp}(S) = 1/12$).*

By consulting the classification of finite simple groups (CFSG) and considering the minimal degrees of nontrivial irreducible characters, it can be shown that one can take $\delta_0 = 1/28$. (This is best possible since $\mathrm{cp}(\mathrm{PSL}(2, 7)) = 1/28$.) We can show without resorting to the full CFSG that $\delta_0 = 1/20$ works. Details are given in Appendix A.3: our approach invokes parts of the classification of finite subgroups of $\mathrm{PGL}(3, \mathbb{C})$ and $\mathrm{PGL}(4, \mathbb{C})$, given by Blichfeldt in the 1900s [1, 2].

The following lemma is presumably standard knowledge, but it seems quicker to give an explanation than to look up a reference.

LEMMA 5.5. *Let H be a perfect group.*

(a) *If W is a solvable group and $\theta : H \to W$ is a homomorphism, then $\theta(H) = \{\underline{1}_W\}$.*
(b) *If X is any set on which H acts, then each H-orbit in X either has size 1 or size ≥ 5.*

PROOF. Part (a) follows by induction on the derived series of W. For part (b), observe that an H-orbit of size n defines a homomorphism $\alpha : H \to S_n$ whose image acts transitively on the original orbit. If $n \leq 4$, then S_n is solvable and so $\alpha(H)$ is trivial by part (a); this is only possible if $n = 1$. $\square$

We now turn to the proof that the only perfect groups with commuting probability greater than $1/20$ are A_5 and $\mathrm{SL}(2, 5)$ (Proposition 1.6). Our argument is patterned

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on the proof of [13, Lemma 2.4], but since we need better bounds than those in Tong-Viet's paper, we take the opportunity to make some simplifications and give a more streamlined approach.

PROOF OF PROPOSITION 1.6. Let S be the quotient of H by any maximal proper normal subgroup. Then, S is simple (by maximality), nontrivial (by properness) and nonabelian (since H is perfect). By Lemma 5.3, $\mathrm{cp}(S) \geq \mathrm{cp}(H) > 1/20$, so by Lemma 5.4, $S \cong A_5$.

Thus, we have a surjective homomorphism $H \to A_5$, with kernel N, say. If N is trivial, there is nothing to prove; so henceforth, we assume $|N| \geq 2$ and aim to prove that $H \cong \mathrm{SL}(2,5)$.

By definition, H is an extension of A_5 by the group N. Suppose that we can show it is a *central* extension; then, since H is perfect, it must be a quotient of the Schur cover of A_5, which is isomorphic to $\mathrm{SL}(2,5)$. Since $|\mathrm{SL}(2,5)| = 2|A_5| \leq |H|$, the quotient map from $\mathrm{SL}(2,5)$ onto H must be injective, and we are done.

Therefore, it suffices to prove that $N \subseteq Z(H)$. Let $k_H(N)$ denote the number of H-conjugacy classes contained in N. As in the proof of [13, Lemma 2.4], we have the inequality

$$12\,\mathrm{cp}(H) \leq \frac{k_H(N)}{|N|}. \tag{5-1}$$

(We briefly sketch how this works. If M is any finite group and $N \trianglelefteq M$, then [6, Lemma 1(iii)], which is actually proved in [9, Remark A2'], tells us that

$$|\mathrm{Irr}(M)| \leq k_M(N) \sup_{B \leq M/N} |\mathrm{Irr}(B)|.$$

We then apply this inequality with $M = H$, noting that $|H : N| = 60$, and appeal to the fact that each subgroup of A_5 has at most five distinct irreducible characters.)

Since $\mathrm{cp}(H) > 1/20$, it follows from (5-1) that $k_H(N) > \frac{3}{5}|N|$. By definition $k_H(N)$ counts the number of orbits for the conjugation action of H on N. By Lemma 5.5, the size of each nonsingleton orbit is at least 5. Therefore, if F denotes the set of fixed points of the action,

$$|N| \geq 5(k_H(N) - |F|) + |F| = 5k_H(N) - 4|F|,$$

and combining this with the previous lower bound on $k_H(N)$ gives

$$|F| \geq \tfrac{1}{4}(5k_H(N) - |N|) > \tfrac{1}{2}|N|.$$

Now observe that $F = Z(H) \cap N$. So by the previous inequality, $|N : Z(H) \cap N| < 2$, which is only possible if $Z(H) \cap N = N$, and this completes the proof. □

We can now show that on the class of finite perfect groups, the AD-constant is minimized at A_5. In fact, a more precise statement can be made.

COROLLARY 5.6. *Let H be a nontrivial perfect group which satisfies* $\mathrm{AD}(H) \leq 2\sqrt{5} \approxeq 4.47$. *Then, $H \cong A_5$ and* $\mathrm{AD}(H) = 61/15 \approxeq 4.07$.

PROOF. By Proposition 5.1, $\operatorname{cp}(H) > \operatorname{AD}(H)^{-2} \geq 1/20$. Hence, by Proposition 1.6, H is isomorphic to either A_5 or $\operatorname{SL}(2,5)$. However, the second possibility is excluded, since we saw in Example 4.4(c) that $\operatorname{AD}(\operatorname{SL}(2,5)) = \frac{9}{2} > 2\sqrt{5}$. □

PROOF OF THEOREM 1.5. Since G is not solvable, its derived series stabilizes at some subgroup $H \leq G$ that is perfect and nontrivial. By monotonicity, $\operatorname{AD}(G) \geq \operatorname{AD}(H)$; and by Corollary 5.6, we have $\operatorname{AD}(H) \geq \min(2\sqrt{5}, 61/15) = 61/15$. □

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Appendix. Easier proofs of some known results

A.1. Finite groups with two character degrees and derived subgroup of order 2. The groups described in the title are, by Corollary 2.3, those finite nonabelian groups where AD attains its minimum value. In this section, we give a quick proof that these groups are precisely those in which the centre has index 4.

Let G be a finite group with $\operatorname{cd}(G) = \{1, 2\}$ and $|G'| = 2$. We have $\operatorname{Irr}(G) = \operatorname{Irr}_1(G) \cup \operatorname{Irr}_2(G)$; let $l = |\operatorname{Irr}_1(G)|$ and $m = |\operatorname{Irr}_2(G)|$. Also, since every conjugacy class injects into G' and $|G'| = 2$, each conjugacy class in G has size 1 or 2. Let $s = |Z(G)|$ and let n be the number of conjugacy classes of size 2.

Note that $|G| = \sum_{\varphi\in\operatorname{Irr}(G)} \varphi(\underline{1})^2 = l + 4m$. Since $|G'| = 2$, we have $|G| = 2l$, and so $l = 4m$. Moreover, $|G| = s + 2n$, while $s + n = l + m$ since the character table is square. Therefore,

$$s + 2n = 8m, \quad s + n = 5m.$$

Solving for s and n yields $s = 2m$ and $n = 3m$. In particular, we conclude that

$$|G : Z(G)| = \frac{8m}{2m} = 4.$$

Conversely, suppose that $|G : Z(G)| = 4$. The argument that follows is essentially the same as in [3], but we include the details for the sake of completeness.

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Note that $G/Z(G)$ cannot be cyclic (otherwise, by lifting the generator, we would find that G is abelian) and hence it is isomorphic to $C_2 \times C_2$. Pick two generators for $G/Z(G)$ and lift them to $x, y \in G$. Then, x^2, y^2 and $[x, y]$ all belong to $Z(G)$. Since $G = Z(G) \cup xZ(G) \cup yZ(G) \cup xyZ(G)$, a short case-by-case analysis shows that every commutator in G equals either $\underline{1}$ or $[x, y]$. In particular, $|G'| = 2$.

Moreover, $A = Z(G) \cup xZ(G)$ is an abelian subgroup of G with index 2. Hence, as shown in the proof of Lemma 3.8, $\mathrm{cd}(G) \subseteq \{1, 2\}$. Since G is nonabelian, this inclusion of sets is an equality.

A.2. A self-contained proof of Lemma 2.5. We give a proof of Lemma 2.5, which works even for infinite groups. Thus, for this subsection only, we let G be a not-necessarily-finite group and we suppose that $|G'| = 2$.

Let π be a finite-dimensional, unitary, irreducible representation of G with dimension $d \geq 2$. (When G is finite, every irreducible representation of G is automatically finite-dimensional and is equivalent to a unitary representation.) Our aim is to show that d is even.

LEMMA A.1. *Let $G' = \{\underline{1}, z\}$. Then, $z \in Z(G)$.*

PROOF. If $\alpha \in \mathrm{Aut}(G)$, then $\alpha(G') = G'$ and $\alpha'(\underline{1}) = \underline{1}$, and hence $\alpha(z) = z$. Now take α to be an arbitrary inner automorphism of G. □

LEMMA A.2. *Let $g \in G$. Then, $g^2 \in Z(G)$.*

PROOF. Let $g, x \in G$. Then, $gxg^{-1} = [g, x]x$. Since $[g, x]$ is central (by Lemma A.1) and $[g, x]^2 = \underline{1}$,

$$g^2xg^{-2} = g([g, x]x)g^{-1} = [g, x](gxg^{-1}) = [g, x][g, x]x = x.$$

Thus, g^2 is central. □

PROOF THAT d IS EVEN. Recall that every nontrivial commutator in G is equal to z. Since π is irreducible and $d \geq 2$, $\pi(G)$ is not abelian, and hence $\pi(z) \neq I_\pi$. By Lemma A.1 and Schur's lemma, $\pi(z)$ is a scalar multiple of I_π; since $z^2 = \underline{1}$, it follows that $\pi(z) = -I_\pi$.

Since G is nonabelian, there exist $x, y \in G$ that do not commute. Since $xyx^{-1} = zy$, we have $\pi(x)\pi(y)\pi(x)^{-1} = -\pi(y)$. However, by Lemma A.2 and Schur's lemma, $\pi(y^2)$ is a scalar multiple of I_π. Pick $\lambda \in \mathbb{T}$ such that $\pi(y)^2 = \lambda^2 I_\pi$; then $U := \lambda^{-1}\pi(y)$ is an involution in $\mathrm{Lin}(H_\pi)$ and U is conjugate to $-U$.

Since U is an involution, it has exactly d eigenvalues counted with multiplicity, and these eigenvalues belong to $\{-1, 1\}$; moreover, since U is conjugate to $-U$, the eigenvalues -1 and 1 must occur with equal multiplicity, m say. Thus, $d = 2m$. (Alternatively, observe that $\frac{1}{2}(I_d + U)$ is an idempotent that has trace equal to $d/2$, which again forces d to be even.) □

A.3. A proof of Lemma 5.4 with $\delta_0 = 1/20$. We follow the general strategy seen in the proofs of [13, Lemma 2.3] and [6, Theorem 11]. Suppose that S is simple and

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nonabelian. Writing m for mindeg(S),

$$|S| - 1 \geq \sum_{\varphi \in \mathrm{Irr}(G), \varphi(\underline{1})>1} \varphi(\underline{1})^2 \geq (|\mathrm{Irr}(S)| - 1)m^2 = m^2 \operatorname{cp}(S)|S| - m^2,$$

and rearranging gives $m^2 - 1 \geq (m^2 \operatorname{cp}(S) - 1)|S|$. If we are given an explicit $\delta_0 > 0$ such that $\operatorname{cp}(S) > \delta_0$, it follows that

$$\frac{1}{|S|} \geq \frac{m^2 \operatorname{cp}(S) - 1}{m^2 - 1} > \frac{m^2\delta_0 - 1}{m^2 - 1}. \tag{A.1}$$

Provided that $m^2\delta_0 > 1$, the inequality (A.1) gives an explicit upper bound on $|S|$.

Thus, in cases where m is sufficiently large, S belongs to some small list of known examples and, in each case, we can see by inspection that m is actually small (giving a contradiction). Separate *ad hoc* arguments are then needed to deal with the cases where m is 'small'.

In [13, Lemma 2.3], this strategy is used with $\delta_0 = 16/225$ and so the easy part of the argument works for all $m \geq 4$; the only remaining cases are those with $m = 3$, and these are covered by the following result.

THEOREM A.3 (Blichfeldt, implicitly). *Let S be a finite simple group with an irreducible representation of degree* 3. *Then, $S \cong A_5$ or $S \cong$* PSL(2, 7).

Inspecting the proof of [13, Lemma 2.3], the 'threshold value' stated there can be improved from 16/225 to 1/15, provided that we know the simple groups of order ≤ 225. However, the methods in that paper cannot reach 1/16 (since we require $m^2\delta_0 > 1$), and for our eventual application to Proposition 1.6, we require $\delta_0 \leq 1/20$. We therefore need the following additional result.

THEOREM A.4 (Blichfeldt, implicitly). *Let S be a finite simple group with an irreducible representation of degree* 4. *Then, $S \cong A_5$.*

For the reader who wishes to consult the original papers, we provide some details of how the theorems stated above are derived from the results stated in [1, 2].

PROOFS OF THEOREM A.3 AND A.4. Let $d \in \{3, 4\}$ and let S be a finite simple group with an irreducible representation of degree d. Then, the image of S under this representation can be identified with a finite subgroup $\widetilde{S} \leq SU(d)$ that acts irreducibly on $\mathbb{C}^d$. Let S_0 be the image of $\widetilde{S}$ in $\mathrm{PGL}_d(\mathbb{C})$. Of course, $S \cong \widetilde{S} \cong S_0$.

In the language of [2, page 553] and [1, page 205], S_0 is primitive: this follows from the fact that simple groups cannot act nontrivially on sets of size ≤ 4, see Lemma 5.5.

The case $d = 3$. The primitive simple finite subgroups of $\mathrm{PGL}_3(\mathbb{C})$ are determined up to isomorphism in [2, Section 24] (relying on previous work of Maschke): any such subgroup must be isomorphic to A_5, PSL(2, 7) or A_6. Moreover, it is observed

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that in the last case, A_6 cannot be lifted from $\mathrm{PGL}_3(\mathbb{C})$ up to $\mathrm{GL}_3(\mathbb{C})$; thus $\widetilde{S}$ must be isomorphic to either A_5 or $\mathrm{PSL}(2,7)$, and this completes the proof of Theorem A.3.

The case $d = 4$. The primitive simple finite subgroups of $\mathrm{PGL}_4(\mathbb{C})$ are determined up to isomorphism in [1, Section III]; the list appears as items 22°–27° on pages 225–226 of that paper, and consists (in modern notation) of A_5, A_6, A_7, $\mathrm{PSL}(2,7)$ and $\mathrm{PSp}(4,3)$. Blichfeldt does not state for which of these S_0 the corresponding 'lift' in $\mathrm{GL}(4,\mathbb{C})$ is simple, but if we invoke known character tables for these groups, then we see that the only possibility for $\widetilde{S}$ is A_5 (none of the others have irreducible representations of degree 4), and this completes the proof of Theorem A.4. □

Proof of Lemma 5.4 with $\delta_0 = 1/20$. Let S be nonabelian and simple, and let $m = \mathrm{mindeg}(S)$. Suppose that $\mathrm{cp}(S) > 1/20$. We start by showing that this forces $m \leq 4$. For, if $m \geq 5$, taking $\delta_0 = 1/20$ in (A.1) gives

$$\frac{1}{|S|} > \left(\frac{m^2}{20} - 1\right)\frac{1}{m^2 - 1} \geq \left(\frac{25}{20} - 1\right)\frac{1}{24} = \frac{1}{96}.$$

However, up to isomorphism, the only nonabelian simple group of order < 96 is A_5, which we know has $m = 3$, and this gives a contradiction.

Therefore, $m \in \{2, 3, 4\}$. It is well documented that finite simple groups have no irreducible representations of degree 2 (see for example [7, Corollary 22.13] for an elementary proof), and it follows from Theorem A.4 that $m \neq 4$. The only remaining possibility is that $m = 3$. By Theorem A.3, this implies that $S \cong A_5$ or $S \cong \mathrm{PSL}(2,7)$; and since $\mathrm{cp}(\mathrm{PSL}(2,7)) = 1/28 < 1/20$, the second case is ruled out. We conclude that $S \cong A_5$, as required. □

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