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An Integral Formula on Seifert Bundles

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Abstract. We prove an integral formula on closed oriented manifolds equipped with a codimension two foliation whose leaves are compact.

1 Introduction

Foliations are of fundamental importance in differential geometry, particularly in the study of fiber bundles and connections, but, with some exceptions [5, 7], the geometric aspects of foliations have not received considerable attention.

In this paper we consider a foliation $\mathcal{F}$ of codimension two on a closed oriented manifold M. We suppose that all the leaves of $\mathcal{F}$ are compact. The assumption on the codimension of $\mathcal{F}$ implies that all the leaves have finite holonomy [1]. Therefore M is a Seifert fiber space. The leaf space $B = M/\mathcal{F}$ is an orbifold of dimension two and thus can be equipped with a holomorphic structure; the foliation $\mathcal{F}$ is then transversely holomorphic and also Riemannian.

Let g be a Riemannian metric on M, bundle-like with respect to $\mathcal{F}$ and for which all the regular leaves (leaves with trivial holonomy) have the same volume v. See [4]. By Rummler [6], all the leaves are minimal with respect to the metric g. Consider the exact sequence of vector bundles over M: $0 \longrightarrow L \longrightarrow TM \xrightarrow{\Pi} Q \longrightarrow 0$, where L is the tangent bundle to $\mathcal{F}$, $Q \cong L^{\perp}$ (via g) the normal bundle, and Π is the orthogonal projection.

Let T be the second fundamental form of L, $K(L^{\perp})$ the sectional curvature of the plane generated by $L^{\perp}$, and s_{mix} the mixed scalar curvature of L and $L^{\perp}$. See Section 2. We have:

Theorem *Let $\mathcal{F}$ be a foliation of codimension two with compact leaves on a closed oriented manifold M. Let g be a bundle-like metric on M for which all the leaves are minimal and the regular leaves have the same volume v. Then*

$$\int_M \Big[K(L^{\perp}) + \frac{3}{2} s_{\text{mix}} + \frac{3}{4} |T|^2 \Big] \, d\sigma = 2\pi v \chi(B),$$

where $|T|$ is the Hilbert-Shmidt norm of T, $d\sigma$ is the volume form associated with g, and $\chi(B)$ is the Euler-Poincaré characteristic of B.

Corollary *Suppose that $\mathcal{F}$ is 1-dimensional. Then*

$$\int_M \Big[K(L^{\perp}) + \frac{3}{2} \operatorname{Ric}(V) \Big] \, d\sigma = 2\pi v \chi(B),$$

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where V is a unit vector field tangent to $\mathcal{F}$ and $\mathrm{Ric}(V)$ *is the Ricci curvature in the direction V.*

2 Some Notations and Preliminaries

We follow [5]. Let (M, g) be a Riemannian manifold and $\mathcal{F}$ a foliation on M. As in the introduction we let L to be the tangent bundle to $\mathcal{F}$ and $L^\perp$ its normal bundle with respect to the metric g. Recall that the second fundamental form T of L is defined by

$$T(v, w) = \Pi(\nabla_v w) \qquad \text{if } v, w \in L,$$
$$T(v, x) = \Pi^\perp(\nabla_v x) \qquad \text{if } v \in L, x \in L^\perp,$$

where ∇ is the Levi-Civita connection associated with the metric g, and $\Pi^\perp$ is the orthogonal projection onto L. Note the symmetry of T with respect to the arguments v and w; this is due to the integrability of the distribution L. It is well known that $T \equiv 0$ if and only if the foliation $\mathcal{F}$ is totally geodesic.

Consider a local orthonormal frame field $\{v_\alpha\}$ and $\{x_i\}$ adapted to $\mathcal{F}$, that is $v_\alpha \in L$ and $x_i \in L^\perp$, and let $K(x_i, v_\alpha)$ be the sectional curvature of the plane (x_i, v_α). The mixed scalar curvature of L and $L^\perp$ is defined by $s_{\text{mix}} = \sum_{i,\alpha} K(x_i, v_\alpha)$. For each α, consider the endomorphism

$$A^{v_\alpha} : L^\perp \longrightarrow L^\perp, Z \longrightarrow \Pi(\nabla_Z v_\alpha).$$

Assuming M is closed and orientable. We have the following integral formula of Ranjan [5]:

$$(*) \qquad \int_M s_{\text{mix}}\, d\sigma = \int_M \Big[|F|^2 - \sum_\alpha \mathrm{tr}(A^{v_\alpha})^2 \Big]\, d\sigma + \int_M \Big[|H|^2 - \frac{1}{2}|T|^2 \Big]\, d\sigma,$$

where $H = \sum_\alpha T(v_\alpha, v_\alpha)$ is the mean curvature vector of the leaves of $\mathcal{F}$ and $F = \sum_i A(x_i, x_i)$ is the mean curvature vector of the bundle $L^\perp$, A being the second fundamental form of $L^\perp$, and $\mathrm{tr}(A^{v_\alpha})^2$ is the trace of the operator $(A^{v_\alpha})^2$

We now apply $(*)$ to the foliation $\mathcal{F}$ given in the introduction. By a remark made earlier, $H \equiv 0$. Also, $\mathcal{F}$ being Riemannian, the orthogonal distribution $L^\perp$ is totally geodesic, which implies that the symmetrized second fundamental form of $L^\perp$ vanishes; hence $F \equiv 0$. Therefore $(*)$ reduces in our case to

$$(**) \qquad \int_M s_{\text{mix}}\, d\sigma = -\int_M \sum_\alpha \mathrm{tr}(A^{v_\alpha})^2\, d\sigma - \int_M \frac{1}{2}|T|^2\, d\sigma.$$

We now prove the following Gauss-Bonnet style proposition.

Proposition *Let $c_1(Q)$ be the first Chern class of the line bundle Q and χ be the volume form on the leaves. Then,*

$$\int_M c_1(Q) \wedge \chi = 2\pi v \chi(B).$$

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Proof Let D be the adapted connection on Q defined by

$$D_v x = \Pi[v, x] \quad \text{for } v \in L, x \in Q,$$
$$D_x y = \Pi(\nabla_x y \quad \text{for } x, y \in Q.$$

The foliation $\mathcal{F}$ being Riemannian, let U be a simple open set of M such that $\mathcal{F}$ is locally defined by a Riemannian submersion $p\colon U \longrightarrow U/\mathcal{F}$. We define a local orthonormal frame field on U as follows: $v_1, v_2, \dots, v_p$ are tangent to $\mathcal{F}$ (p=dimension of $\mathcal{F}$), and x_1, x_2 are the horizontal lifts of an orthonormal frame on $U/\mathcal{F}$. We have $D_{v_\alpha} x_i = 0$, $i = 1, 2$ and $\alpha = 1, 2, \dots, p$. On the other hand, $D_{x_i} x_j$ is the transversal Levi-Civita connection that is the Riemannian connection on $U/\mathcal{F}$ [7] equipped with the metric $p_* g$. Consequently if ω is the connection form associated with the frame $v_1, v_2, \dots, v_p, x_1, x_2$, then ω is a basic form, that is $i_V \omega = 0$, and $\theta(V)\omega = 0$ for $V \in L$; here $i_V, \theta(V)$ are respectively the interior product and the Lie derivative in the direction V, see [7]. Therefore the 2-form $c_1(Q) = \frac{1}{2\pi i} d\omega$ is also basic.

Now if F is the generic compact fibre of M, we have

$$\int_M c_1(Q) \wedge \chi = \int_B \Big(\int_F c_1(Q) \wedge \chi \Big) = \int_B c_1(Q) \wedge (\int_F \chi)$$
$$= \int_B \nu c_1(Q) = \nu \int_B c_1(Q) = 2\pi \nu \chi(B)$$

by Satake [8]. See also [2].

3 Proof of the Theorem

Using the notations of the proposition, the form $c_1(Q) = \frac{1}{2\pi i} d\omega$ descends to the local quotient to the curvature form $\Omega = K d\lambda$, where K is the Gaussian curvature of the open $U/\mathcal{F}$ and $d\lambda$ the volume form. A theorem of O'Neill applied to the Riemannian submersion $p\colon U \longrightarrow U/\mathcal{F}$ implies that $K = K(x_1, x_2) + \frac{3}{4} |\Pi^\perp [x_1, x_2]|^2$. See [3, p. 127].

Recall that the endomorphism A^{v_α}, $\alpha = 1, 2, \dots, p$ is defined by

$$A^{v_\alpha}\colon L^\perp \longrightarrow L^\perp, Z \longrightarrow \Pi(\nabla_Z v_\alpha).$$

The foliation $\mathcal{F}$ being transversely holomorphic, A^{v_α} is $\mathcal{C}$-linear [2], hence represented by a matrix of the form $\left(\begin{smallmatrix} C_\alpha & D_\alpha \\ -D_\alpha & C_\alpha \end{smallmatrix} \right)$. Moreover since $\mathcal{F}$ is Riemannian we have $C_\alpha \equiv 0$, $(\alpha = 1, 2, \dots, p)$, consequently $\operatorname{tr}(A^{v_\alpha})^2 = -2D_\alpha^2$. On the other hand, elementary computations show that $|\Pi^\perp [x_1, x_2]|^2 = 4 \sum_\alpha D_\alpha^2$. This shows that the Chern class $c_1(Q)$ is represented by

$$[K(x_1, x_2) - \frac{3}{2} \sum_\alpha \operatorname{tr}(A^{v_\alpha})^2]\, d\lambda$$

($d\lambda = *\chi$, where $*$ is the Hodge Star operator). Therefore,

$$\int_M c_1(Q) \wedge \chi = \int_M \big[K(x_1, x_2) - \frac{3}{2} \sum_\alpha \operatorname{tr}(A^{v_\alpha})^2 \big]\; d\sigma.$$

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Using $(**)$, we see that

$$\int_M c_1(Q) \wedge \chi = \int_M \left[K(L^\perp) + \frac{3}{2} s_{\text{mix}} + \frac{3}{4} |T|^2 \right] d\sigma = 2\pi \nu \chi(B)$$

by the proposition. The theorem is proved.

Proof of the corollary Observe that the generic fibre in this case is a compact Lie group. Therefore, one can choose the metric g so that the fibres are geodesics. Hence, $T \equiv 0$. On the other hand, $\mathrm{Ric}(V)$, here, is the sum of all sectional curvatures of planes containing V which is s_{mix} and the corollary follows.

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