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ON REAL HYPERSURFACES OF A COMPLEX PROJECTIVE SPACE WITH RECURRENT RICCI TENSOR

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Abstract. Let M be a real hypersurface of the complex projective space $P_n(\mathbf{C})$. The Ricci tensor S of M is *recurrent* if there exists a 1-form α such that $\nabla S = S \otimes \alpha$. In this paper we show that there are no real hypersurfaces with recurrent Ricci tensor of $P_n(\mathbf{C})$ under the condition that ξ is a principal curvature vector.

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0. Introduction. Let M be a connected real hypersurface of a complex projective space $P_n(\mathbf{C})$, $n \geq 2$ with the Fubini-Study metric of constant holomorphic sectional curvature 4. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from the Kähler structure of $P_n(\mathbf{C})$. It is well-known that there does not exist a real hypersurface M of $P_n(\mathbf{C})$ satisfying the condition that the second fundamental tensor A of M is parallel. We estimated it from another point of view. In [**2**], we considered the condition that the second fundamental tensor A is *recurrent*, i.e., there exists a 1-form α such that $\nabla A = A \otimes \alpha$. We may regard the parallel condition as a special case. We know that the recurrent condition has a close relation to holonomy group ([**8**] and [**14**]). This condition means that the eigenspaces of the shape operator A of M are parallel along any curve γ in M. Here, the eigenspaces of the shape operator A are said to be *parallel* along γ if they are invariant with respect to parallel translation along γ. We proved the nonexistence of real hypersurfaces with recurrent second fundamental tensor of $P_n(\mathbf{C})$ [**11**]. On the other hand, many differential geometers evaluated the real hypersurfaces of $P_n(\mathbf{C})$ paying attention to the Ricci tensor. Cecil and Ryan proved that there are no Einstein real hypersurfaces of $P_n(\mathbf{C})$ [**1**]. Ki showed that the nonexistence of real hypersurfaces of a nonflat complex space form with parallel Ricci tensor [**4**]. In this paper, we investigate the condition that the Ricci tensor S is *recurrent*, i.e., there exists a 1-form α such that $\nabla S = S \otimes \alpha$. We prove the following theorem.

THEOREM. *There are no real hypersurfaces with recurrent Ricci tensor of $P_n(\mathbf{C})$ under the condition that ξ is a principal curvature vector.*

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1. Preliminaries. Let M be a real hypersurface of $P_n(\mathbf{C})$. In a neighborhood of each point, we take a unit normal vector field N in $P_n(\mathbf{C})$. The Riemannian connections $\widetilde{\nabla}$ in $P_n(\mathbf{C})$ and ∇ in M are related by the following formulas for arbitrary vector fields X and Y on M.

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \tag{1.1}$$

$$\widetilde{\nabla}_X N = -AX, \tag{1.2}$$

where g denotes the Riemannian metric of M induced from the Fubini-Study metric G of $P_n(\mathbf{C})$ and A is the second fundamental tensor of M in $P_n(\mathbf{C})$. We denote by TM the tangent bundle of M. An eigenvector X of the second fundamental tensor A is called a *principal curvature vector*. Also an eigenvalue λ of A is called a *principal curvature*. We know that M has an almost contact metric structure induced from the Kähler structure J on $P_n(\mathbf{C})$: We define a $(1, 1)$-tensor field ϕ, a vector field ξ and a 1-form η on M by $g(\phi X, Y) = G(JX, Y)$ and $g(\xi, X) = \eta(X) = G(JX, N)$. Then we have

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0. \tag{1.3}$$

It follows from (1.1) that

$$\nabla_X \xi = \phi AX. \tag{1.4}$$

Let $\widetilde{R}$ and R be the curvature tensors of $P_n(\mathbf{C})$ and M, respectively. From the expression of the curvature tensor $\widetilde{R}$ of $P_n(\mathbf{C})$, we have the following equations of Gauss and Codazzi:

$$\begin{aligned} R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ - 2g(\phi X, Y)\phi Z + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{1.5}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi. \tag{1.6}$$

By the Gauss equation, the Ricci tensor of $(1, 1)$ type of M is given by

$$SX = (2n + 1)X - 3\eta(X)\xi + hAX - A^2 X, \tag{1.7}$$

where h denotes the trace of the shape operator A. We have the differential of the Ricci tensor,

$$\begin{aligned} (\nabla_X S)Y = -3g(\phi AX, Y)\xi - 3\eta(Y)\phi AX + (Xh)AY + h(\nabla_X A)Y \\ - A(\nabla_X A)Y - (\nabla_X A)AY. \end{aligned} \tag{1.8}$$

Now we prepare without proof the following in order to prove our results.

LEMMA 1.1 ([**9**]) *If ξ is a principal curvature vector, then the corresponding principal curvature a is locally constant.*

LEMMA 1.2 ([**9**]) *Assume that ξ is a principal curvature vector and the corresponding principal curvature is a. If $AX = \lambda X$ for $X \perp \xi$, then we have $A\phi X = \overline{\lambda}\phi X$, where $\overline{\lambda} = (a\lambda + 2)/(2\lambda - a)$.*

THEOREM C-R. ([**1**]) *Let M be a connected real hypersurface of $P_n(\mathbf{C})$, $n \geq 3$, whose Ricci tensor S is pseudo-Einstein, i.e. $SX = aX + b\eta(X)\xi$ for any tangent vector*

X on M, where a and b are functions on M. Then M is an open subset of one of the following:

(*a*) *a geodesic hypersphere*

(*b*) *a tube of radius r over a totally geodesic* $P_k(\mathbf{C})$, $0 < k < n-1$, *where* $0 < r < \pi/2$ *and* $cot^2 r = k/(n-k-1)$,

(*c*) *a tube of radius r over a complex quadric* Q_{n-1} *where* $0 < r < \pi/4$ *and* $cot^2 r = n-2$.

THEOREM T. ([**13**]) *Let M be a homogeneous real hypersurface of* $P_n(\mathbf{C})$. *Then M is a tube of some radius r over one of the following Kähler submanifolds:*

(A_1) *hyperplane* $P_{n-1}(\mathbf{C})$, *where* $0 < r < \pi/2$,

(A_2) *totally geodesic* $P_k(\mathbf{C})$ $(1\, k\, n-2)$, *where* $0 < r < \pi/2$,

(B) *complex quadric* Q_{n-1}, *where* $0 < r < \pi/4$,

(C) $P_1(\mathbf{C}) \times P_{(n-1)/2}(\mathbf{C})$, *where* $0 < r < \pi/4$, *and* $n(\geq 5)$ *is odd,*

(D) *complex Grassmann* $G_{2,5}(\mathbf{C})$, *where* $0 < r < \pi/4$ *and* $n = 9$,

(E) *Hermitian symmetric space* $SO(10)/U(5)$, *where* $0 < r < \pi/4$ *and* $n = 15$.

THEOREM K. ([**6**]) *Let M be a real hypersurface of* $P_n(\mathbf{C})$. *Then M has constant principal curvatures and* ξ *is a principal curvature vector if and only if M is locally congruent to a homogeneous real hypersurface.*

THEOREM Ki. ([**4**]) *There are no real hypersurfaces with parallel Ricci tensor of a complex space form* $M_n(c)$, $c \neq 0$.

2. The Ricci tensors of real hypersurfaces of a complex projective space. At first, to prove our theorem, we prepare the following lemma.

LEMMA 2.1. *Let M be a connected real hypersurface of* $P_n(\mathbf{C})$ *with recurrent Ricci tensor S. If all eigenvalues of S are constant, then the Ricci tensor S of M is parallel.*

Proof. We choose a unit eigenvector Y of S with an eigenvalue λ. Then we have

$$\begin{aligned} g((\nabla_X S)Y, Y) &= g(\nabla_X(SY), Y) - g(S\nabla_X Y, Y) \\ &= X\lambda \end{aligned}$$

for any $X \in TM$. On the other hand, from the assumption we obtain

$$\begin{aligned} g((\nabla_X S)Y, Y) &= \alpha(X) g(SY, Y) \\ &= \alpha(X)\lambda. \end{aligned}$$

Since all eigenvalues of S are constant we get $\alpha(X)\lambda = 0$ for any $X \in TM$. So the Ricci tensor S of M is parallel.

In light of (1.7) and the fact that ξ is principal, the principal curvature vectors will also be eigenvectors of S. Thus Ricci tensor of a homogeneous real hypersurface has constant eigenvalues. On the other hand, the hypersurfaces listed in Theorem T do not have parallel Ricci tensor (see for example, [**11**], Corollary 6.6, p.273). Therefore from Lemma 2.1 and Theorem K, we have the following result.

Proposition 2.2. *The Ricci tensor of a homogeneous real hypersurface of* $P_n(\mathbf{C})$ *cannot be recurrent.*

By Theorem C-R, any pseudo-Einstein real hypersurface is homogeneous one, therefore we check the following:

Corollary 2.3. *The Ricci tensor of a pseudo-Einstein real hypersurface of* $P_n(\mathbf{C})$*, where* $n \geq 3$*, cannot be recurrent.*

Proof of the Theorem. We have the following equation by the assumption that

$$g((\nabla_X S)Y, Z) = \alpha(X)g(SY, Z) = (2n+1)\alpha(X)g(Y, Z) - 3\alpha(X)\eta(Y)\eta(Z) + h\alpha(X)g(AY, Z) - \alpha(X)g(A^2 Y, Z).$$

Using (1.8), we obtain

$$\begin{aligned}(2n+1)\alpha(X)g(Y, Z) - 3\alpha(X)\eta(Y)\eta(Z) + h\alpha(X)g(AY, Z) - \alpha(X)g(A^2 Y, Z)\\ +3\eta(Z)g(\phi AX, Y) + 3\eta(Y)g(\phi AX, Z) - (Xh)g(AY, Z) - hg((\nabla_X A)Y, Z)\\ + g(A(\nabla_X A)Y, Z) + g((\nabla_X A)AY, Z) = 0,\end{aligned} \tag{2.1}$$

for arbitrary tangent vectors X, Y and Z.

If we put $Y = \xi$ and $Z = \phi X$ in (2.1), then we have

$$\begin{aligned}h\alpha(X)g(A\xi, \phi X) - \alpha(X)g(A^2\xi, \phi X) + 3g(AX, X) - 3\eta(AX)\eta(X)\\ - (Xh)g(A\xi, \phi X) - hg((\nabla_X A)\xi, \phi X) + g(A(\nabla_X A)\xi, \phi X) + g((\nabla_X A)A\xi, \phi X) = 0\end{aligned} \tag{2.2}$$

We may assume that $A\xi = a\xi$. Then by Lemma 1.1, a is constant. We get

$$(\nabla_X A)\xi = a\phi AX - A\phi AX. \tag{2.3}$$

Using (2.3) in the equation (2.2), we have

$$\begin{aligned}3g(AX, X) - 3a(\eta(X))^2 - hag(\phi AX, \phi X) + hg(A\phi AX, \phi X)\\ -g(A\phi AX, A\phi X) + a^2 g(\phi AX, \phi X) = 0,\end{aligned}$$

for any tangent vector X on M. We choose X as a unit principal curvature vector orthogonal to ξ and by the Lemma 1.2, we have

$$AX = \lambda X \quad \text{and} \quad A\phi X = \overline{\lambda} X,$$

where $\overline{\lambda} = (a\lambda + 2)/(2\lambda - a)$. Therefore we obtain the following equation:

$$\lambda(\overline{\lambda}^2 - h\overline{\lambda} - (a^2 - ha + 3)) = 0. \tag{2.4}$$

This formula also holds with λ and $\overline{\lambda}$ exchanged, so we get

$$(\lambda - \overline{\lambda})(\lambda\overline{\lambda} + (a^2 - ha + 3)) = 0. \tag{2.5}$$

On the other hand, from Lemma 1.2, the relationship between λ and $\overline{\lambda}$ can be written

$$\lambda\overline{\lambda} = \frac{\lambda + \overline{\lambda}}{2} a + 1. \tag{2.6}$$

If $\lambda = \overline{\lambda}$, this becomes

$$\lambda^2 = a\lambda + 1. \tag{2.7}$$

If 0 occurs as a principal curvature (for a principal vector orthogonal to ξ), then (2.6) shows that all principal curvatures must be constant.

Next assuming that 0 is not a principal curvature (again we consider only directions orthogonal to ξ), formula (2.4) shows that there are at most two distinct principal curvatures. If λ and $\overline{\lambda}$ are distinct, we have

$$\lambda + \overline{\lambda} = h \quad \text{and} \quad \lambda\overline{\lambda} = -(a^2 - ha + 3)$$

which yields

$$-(a^2 - ha + 3) = \frac{ha}{2} + 1,$$

i.e.

$$a^2 - \frac{ha}{2} + 4 = 0.$$

Thus the coefficients in (2.4) are constant and hence so are λ and $\overline{\lambda}$. The final possibility is that all principal curvatures (with principal vectors orthogonal to ξ) satisfy (2.7) and are again constant.

By Theorem K and Proposition 2.2, the proof is concluded.

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