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ARTIN RINGS WITH TWO-GENERATED IDEALS

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ABSTRACT. It is shown that each commutative Artin local ring having each of its ideals generated by two elements is the homomorphic image of a one-dimensional local complete intersection ring which also has each of its ideals generated by two elements. It is indicated how this can be applied to show that the property that each ideal is projective over its endomorphism ring does not pass to homomorphic images, and in determining the commutative group rings with the two-generator property.

All rings in this note are commutative Noetherian rings with identity. In [4] Hungerford showed that each local Artinian principal ideal ring is a homomorphic image of a local one-dimensional principal ideal ring. In [1] Bass showed that one-dimensional rings with the two-generator property (that is, each ideal can be generated by two elements) have some properties which are similar to properties of principal ideal domains. For example if such a ring R is reduced and has finite integral closure then each finitely generated torsionfree R-module is isomorphic to a direct sum of ideals of R. Since then many other properties of these rings have been obtained [3, 6, 10, 11, 12], but little is known about Artinian rings with the two-generator property. In this note we show that the analogous result to Hungerford's theorem mentioned above holds for local rings which have the two-generator property. We also indicate two applications of this result. The first of these is to show that stability of rings as defined in [11], [12] is not preserved under homomorphic images, and the other is to the determination of the commutative group rings having the two-generator property. As in [5, p.190] we say that a local ring B is a *complete intersection* if $B \cong R/I$ where R is a regular local ring and I is an ideal of R generated by height(I) elements.

THEOREM. *Let (A, m) be a zero-dimensional local ring with the two-generator property. Then $A \cong R/I$ where R is a one-dimensional local complete intersection with the two-generator property.*

PROOF. Let $m = (x, y)$. Then by [2, Theorems 9 and 12] we can write A as a homomorphic image of a power series ring $\phi : D[[X, Y]] \longrightarrow A$, where $\phi(X) = x$, $\phi(Y) = y$ and D is either a field or a ν-ring, that is a complete principal valuation ring whose maximal ideal is generated by a prime integer p. First assume that D is a field. Since A has the two-generator property, m^2 is generated by two of the elements x^2, xy, y^2. If $m^2 = (xy, y^2)$ let $x^2 = axy + by^2$, $a, b \in R$. Choose preimages $\alpha, \beta \in D[[X, Y]]$ of a and b respectively and let $f = X^2 - \alpha XY - \beta y^2$. Then since the first two powers of the maximal

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ideal of $R = D[[X, Y]]/(f)$ are each generated by two elements, R has the two-generator property by [7, Theorem 1]. Thus R is a local complete intersection of dimension one and maps onto A. The cases $m^2 = (x^2, y^2)$ and $m^2 = (x^2, xy)$ are similar. Thus assume that D is a ν-ring. Then A has mixed characteristic and if p is the characteristic of A/m then $p \in m - \{0\}$. Let $p = sx + ty$, $s, t \in A$, let $\sigma, \tau \in D[[X, Y]]$ be preimages of s and t respectively and let $g = p - \sigma X - \tau Y$. Again consider the case $x^2 = axy + by^2$, $a, b \in A$, the other cases being similar. Choose preimages $\alpha, \beta \in D[[X, Y]]$ of a and b respectively and again let $f = X^2 - \alpha XY - \beta Y^2$. Then as above $R = D[[X, Y]]/(g, f)$ has the two-generator property by [7, Theorem 1], and is clearly a one-dimensional complete intersection and maps onto R.

APPPLICATION 1. In [11] and [12] an ideal I of a ring R is said to be *stable* if I is projective as a module over $\mathrm{Hom}_R(I, I)$ and the ring R is said to be *stable* if each ideal of R is stable. It is shown in [12, Theorem 3.4] that if R is a one-dimensional Macaulay ring in which maximal ideals are not minimal, and R has the two-generator property then R is stable. The converse holds in many cases but not always [12, Example 5.4]. It is known that stable rings, like rings with the two-generator property, have Krull dimension ≤ 1 [12, Proposition 2.1]. Because of this relationship to the two-generator property it is natural to ask if stability passes to homomorphic images. The above theorem together with an example from [12] shows that the homomorphic image of a stable ring need not be stable. Indeed by [12, Example 5.6] there is an Artin local ring (A, m) that has the two generator property which is not stable. By the above theorem we can write $A \cong R/I$ where R is a one-dimensional ring with the two-generator property which is Macaulay. Then R is stable by [12, Theorem 3.4], but its image A is not.

APPLICATION 2. In [8] the commutative one-dimensional monoid rings $R[G]$ which have the two-generator property were determined except in the case that G is a finite abelian group and R is a one-dimensional ring which has some maximal ideals which are also minimal. Since in this case it follows easily that $R \cong R_1 \times R_2$ where maximal ideals of R_1 have height 1 and R_2 is Artinian, and a product of two rings has the two-generator property if and only if each factor does, it remains to determine the commutative Artinian group rings which have the two-generator property. These are determined in [9]. For simplicity we state the characterization from [9] in the local case. Let (R, M) be a local Artinian ring and let G be a finite abelian group of order n. Then $R[G]$ has the two-generator property if and only if R has the two-generator property and

(i) n is a unit in R if R is not a principal ideal ring,

(ii) if R is a principal ideal ring and the characteristic of R/M is a prime integer p which divides n, then the p-Sylow subgroup G_p of G is $\cong Z/p^iZ$ and if $M^2 \neq 0$ then $i = 1$, unless $p = 2$ where the additional case $M = 0$ and $G_2 \cong Z/2^iZ \oplus Z/2Z$ occurs.

The proof that if n is a unit of R and R has the two-generator property then $R[G]$ has the two-generator property proceeds by writing R as a homomorphic image of a

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one-dimensional ring R with the two-generator property as above, and using the characterization in [8] of one-dimensional commutative group rings with the two-generator property.

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