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LOCAL EXISTENCE IN TIME OF SMALL SOLUTIONS TO THE ELLIPTIC-HYPERBOLIC DAVEY-STEWARTSON SYSTEM IN THE USUAL SOBOLEV SPACE

by NAKAO HAYASHI and HITOSHI HIRATA

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We study the initial value problem to the Davey–Stewartson system for the elliptic-hyperbolic case in the usual Sobolev space. We prove local existence and uniqueness $H^{5/2}$ with a condition such that the L^2 norm of the data is sufficiently small.

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1. Introduction

We study the initial-value problem for the Davey–Stewartson (DS) systems

$$\left.\begin{aligned} i\partial_t u + c_0\partial_{x_1}^2 u + \partial_{x_2}^2 u &= c_1|u|^2u + c_2 u\partial_{x_1}\varphi, \quad (x,t)\in \mathbf{R}^2\times\mathbf{R},\\ \partial_{x_1}^2\varphi + c_3\partial_{x_2}^2\varphi &= \partial_{x_1}|u|^2, \quad u=u(x,t), \quad \varphi=\varphi(x,t),\\ u(x,0) &= \phi(x), \end{aligned}\right\} \tag{1.1}$$

where $c_0, c_3 \in \mathbf{R}$, $c_1, c_2 \in \mathbf{C}$, u is a complex valued function and φ is a real-valued function. The systems (1.1) for $c_0 < 0$ and $c_3 > 0$ were derived by Davey and Stewartson [**5**] and model the evolution of two-dimensional long waves in a finite-depth liquid. Djordjevic–Redekopp [**6**] showed that the parameter c_3 can become negative when capillary effects are important. For detailed physical background, see [**1, 2, 6**]. When $(c_0, c_1, c_2, c_3) = (1, -1, 2, -1)$, $(-1, -2, 1, 1)$ or $(-1, 2, -1, 1)$ the system (1.1) is referred to as the DSI, DSII defocusing and DSII focusing respectively in the inverse scattering literature. Ghidaglia and Saut [**8**] classified (1.1) as elliptic-elliptic, elliptic-hyperbolic, hyperbolic-elliptic and hyperbolic-hyperbolic according to the respective sign of $(c_0, c_3) : (+,+), (+,-), (-,+)$ and $(-,-)$. For the elliptic-elliptic and hyperbolic-elliptic cases, local and global properties of solutions were studied in [**8**] in the usual Sobolev spaces L^2, H^1 and H^2. In this paper we consider the elliptic-hyperbolic case. In this case, after a rotation in the x_1-x_2 plane and rescaling, the system (1.1) can be written as

$$\left.\begin{aligned} i\partial_t u + \Delta u &= c_1'|u|^2u + c_2' u\partial_{x_1}\varphi + c_3' u\partial_{x_2}\varphi,\\ \partial_{x_1}\partial_{x_2}\varphi &= c_4'\partial_{x_1}|u|^2 + c_5'\partial_{x_2}|u|^2, \end{aligned}\right\} \tag{1.2}$$

where $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$, $c'_1, \ldots, c'_5$ are arbitrary constants. In order to solve the system of equations, one has to assume that $\varphi(\cdot)$ satisfies the radiation condition, namely, we assume that for given functions φ_1 and φ_2

$$\lim_{x_2 \to +\infty} \varphi(x, t) = \varphi_1(x_1, t) \quad \text{and} \quad \lim_{x_1 \to +\infty} \varphi(x, t) = \varphi_2(x_2, t). \tag{1.3}$$

Under the radiation condition (1.3), the system (1.2) can be written as

$$\begin{aligned} i\partial_t u + \Delta u = & d_1 |u|^2 u + d_2 u \int_{x_2}^{\infty} \partial_{x_1} |u(x_1, x'_2, t)|^2 \, dx'_2 \\ & + d_3 u \int_{x_1}^{\infty} \partial_{x_2} |u(x'_1, x_2, t)|^2 \, dx'_1 + d_4 u \partial_{x_1} \varphi_1 + d_5 u \partial_{x_2} \varphi_2 \end{aligned} \tag{1.4}$$

with the initial condition $u(x, 0) = \phi(x)$. Here $d_1 = c'_1 + c'_2 c'_5 + c'_3 c'_4$, $d_2 = -c'_2 c'_4$, $d_3 = -c'_3 c'_5$, $d_4 = c'_2$ and $d_5 = c'_3$.

By using inverse scattering methods several results were obtained for the DSI system ($d_1 = 0$, $d_2 = d_3 = 1/2$, and $d_4 = d_5 = 1$ in (1.4)). In [7] A. S. Fokas and L. Y. Sung showed that if the initial function ϕ is in the Schwartz class and if $\partial_{x_1} \varphi_1(t, x_1)$ and $\partial_{x_2} \varphi_2(t, x_2)$ are also in the Schwartz class with respect to the spatial variables and continuous in t, then the DSI system has a unique solution global in t which, for each fixed t, belongs to the Schwartz class in the spatial variables. Furthermore it is known that the DSI system has the localized-soliton-type exact solutions which are called dromion (for the study of the dromion solutions, see, e.g., [11], [17]).

In order to state some known results and our results, we define some notation. We let $\partial = (\partial_{x_1}, \partial_{x_2})$, $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| = \alpha_1 + \alpha_2$ and $\alpha_1, \alpha_2 \in \mathbf{R} \cup \{0\}$. We define the weighted Sobolev space as follows:

$$H^{m,l} = \{ f \in L^2; \, \|(1 - \partial_{x_1}^2 - \partial_{x_2}^2)^{m/2} (1 + |x_1|^2 + |x_2|^2)^{l/2} f\| < \infty \},$$

$$H^{m,l}(\mathbf{R}_{x_j}) = \{ f \in L^2(\mathbf{R}_{x_j}); \, \|(1 - \partial_{x_j}^2)^{m/2} (1 + |x_j|^2)^{l/2} f\|_{L^2(\mathbf{R}_{x_j})} < \infty \},$$

where $\|\cdot\|$ denotes the usual L^2 norm. We denote the usual L^p norm by $\|\cdot\|_p$. For any Banach space E, $L^p(A; E)$ means the set of E-valued L^p functions on A, where $A = [0, T]$, $A = \mathbf{R}^2$ or $A = \mathbf{R}_{x_j}$ and $C([0, T]; E)$ means the set of E-valued continuous functions on $[0, T]$. We write $L^p([0, T]; E) = L^p_T E$, $L^p(\mathbf{R}_{x_j}; E) = L^p_{x_j} E$ which makes the notation simple. For example $L^{p_1}(\mathbf{R}_{x_1}; L^{p_2}([0, T]; L^{p_3}(\mathbf{R}_{x_2})))$ can be denoted as $L^{p_1}_{x_1} L^{p_2}_T L^{p_3}_{x_2}$. We also write $H^{s,0} = H^s$ and $H^{s,0}(\mathbf{R}_{x_j}) = H^s(\mathbf{R}_{x_j}) = H^s_{x_j}$ for simplicity.

For the local existence of small solutions to (1.4), Linares and Ponce [16], Chihara [4] and Hayashi [9] obtained the following results.

Proposition 1 [16, Theorem B]. *We assume that* $\phi \in H^s \cap H^{6,6} \equiv W_s$, $s \geq 12$, $\varphi_1 = \varphi_2 \equiv 0$ *and* $\|\phi\|_{H^{12}} + \|\phi\|_{H^{6,6}}$ *is sufficiently small. Then there exists a positive constant* $T > 0$ *and a unique solution* u *of* (1.4) *such that* $u \in C([0, T]; W_s)$.

Proposition 2 [**4**, Theorem 1.1]. *We assume that $\phi \in H^s$, where s is a sufficiently large integer, $\varphi_1 = \varphi_2 \equiv 0$ and $\|\phi\| < 1/(2\sqrt{\max\{|d_2|, |d_3|\}}e)$. Then there exists a positive constant $T > 0$ and a unique solution u of* (1.4) *such that $u \in C_w([0, T]; H^s) \cap C([0, T]; H^{s-1})$, where $C_w([0, T]; H^s)$ is the set of H^s valued weak continuous functions on* $[0, T]$.

Proposition 3 [**9**, Theorem 1]. *We assume that $\phi \in H^\delta \cap H^{0,\delta} \equiv Z_\delta$, $\delta \geq \delta_0 > 1$, $\partial_{x_1}\varphi_1 \in C(\mathbf{R}; H^{\delta}_{x_1})$, $\partial_{x_2}\varphi_2 \in C(\mathbf{R}; H^{\delta}_{x_2})$ and $\|\phi\|_{H^{\delta_0}} + \|\phi\|_{H^{0,\delta_0}}$ is sufficiently small. Then there exists a positive constant $T > 0$ and a unique solution u of* (1.4) *such that $u \in C([0, T]; Z_\delta)$.*

Our purpose in this paper is to prove the local existence of small solution to (1.4) in usual Sobolev spaces. We now state our result in this paper.

Theorem 1.1. *We assume that $\phi \in H^s$, where $s \geq 5/2$, $\partial_{x_1}\varphi_1 \in C(\mathbf{R}; H^s_{x_1})$, $\partial_{x_2}\varphi_2 \in C(\mathbf{R}; H^s_{x_2})$, and $\|\phi\|_{L^2} < 1/\sqrt{\max\{|d_2|, |d_3|\}}$. Then there exists a positive constant $T > 0$ and a unique solution u of* (1.4) *such that $u \in C([0, T]; H^s)$.*

Theorem 1.1 is considered as an improvement of the previous papers by Chihara [**4**] and Linares and Ponce [**16**]. We only prove Theorem 1.1 in the case of $s = 5/2$ since in the case of $s \geq 5/2$, Theorem 1.1 can be proved in the same way. To obtain our result we introduce the function space.

$$X_T = \{f \in C([0, T]; L^2); \|f\|_{X_T} < \infty\}, \quad Y_T = \{f \in C([0, T]; L^2); \|f\|_{Y_T} < \infty\},$$

where

$$\|f\|_{X_T} = \|f\|_{Y_T} + \|\partial^3_{x_1} f\|_{L^\infty_{x_1} L^2_T L^2_{x_2}} + \|\partial^3_{x_2} f\|_{L^\infty_{x_2} L^2_T L^2_{x_1}},$$

$$\|f\|_{Y_T} = \left\{ \sum_{|\alpha| \leq 2} \|\partial^\alpha f\|^2_{L^\infty_T L^2} + \sum_{|\alpha|=2} \left(\|D^{1/2}_{x_1} \partial^\alpha f\|^2_{L^\infty_T L^2} + \|D^{1/2}_{x_2} \partial^\alpha f\|^2_{L^\infty_T L^2}\right) \right\}^{1/2},$$

$D^\alpha_{x_j} = \mathcal{F}^{-1}|\xi_j|^\alpha \mathcal{F}$, $\partial^\alpha = \partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}$, and $|\alpha| = \alpha_1 + \alpha_2$.

The function space Y_T is the natural Sobolev space when we use the classical energy method with the data $\phi \in H^{5/2}$. The use of the function space X_T suggests that we make use of smoothing properties of solutions to the linear Schrödinger equation (see Section 2). As mentioned in [**16**], it seems that the classical energy method is not sufficient to yield an existence result. In this paper we use the two-dimensional version of the smoothing effect of Kenig–Ponce–Vega type (see, e.g., [**13**]). We note that the method used in this paper does not work to remove the decay condition on the data in the hyperbolic-hyperbolic case which was assumed in [**16, 9**] to obtain local existence results. A smallness assumption on the data can be removed in real analytic data [**10**], however we do not know whether it can be removed or not in the usual Sobolev space.

We conclude this section by giving our strategy of the proof of Theorem 1.1. We apply the contraction mapping principle to the linear Schrödinger equation

$$u(t)=U(t)\phi-iS\left(d_1|v|^2v+d_2v\int_{x_2}^{\infty}\partial_{x_1}|v(x_1,x_2',t)|^2\,dx_2'\right.$$

$$\left.+d_3v\int_{x_1}^{\infty}\partial_{x_2}|v(x_1',x_2,t)|^2\,dx_1'+d_4v\partial_{x_1}\varphi_1+d_5v\partial_{x_2}\varphi_2\right)(t)$$

where $U(t)=\exp(it\Delta)$ and $(Sf)(t)=\int_0^t U(t-s)f(s)\,ds$. In order to do it, we organize the paper as follows. In Section 2 we give some estimates of $U(t)\phi$ and $(Sf)(t)$ (Lemma 2.1) which imply the smoothing properties of solutions and Section 3 is devoted to estimates of the nonlinear term (the right hand side of the above equation). Roughly speaking, we will obtain, by making use of these estimates obtained in Section 3 (Lemma 3.5 – Lemma 3.8),

$$\|u\|_{X_T}\le C\|\phi\|_{H^{5/2}}+CT\|v\|_{X_T}^3+C(d_1,d_2)\|\phi\|^2\|v\|_{X_T},$$

where $C(d_1,d_2)$ is a constant depending only on d_1,d_2 and will be determined in Section 4. This inequality shows that the mapping M defined by $u=Mv$ is the mapping from $X_{T,\rho}=\{f\in X_T;\ \|f\|_{X_T}\le\rho\}$ into itself provided that $\|\phi\|$ is sufficiently small. The constant $C(d_1,d_2)$ gives a condition on the size of $\|\phi\|$ which yields our result Theorem 1.1.

2. Linear Schrödinger equations

In this section we state smoothing properties of the inhomogeneous Schrödinger equations

$$\left.\begin{aligned} i\partial_t u+\Delta u&=f,\quad (x,t)\in\mathbf{R}^2\times\mathbf{R},\\ u(0,x)&=\phi(x).\end{aligned}\right\}\tag{2.1}$$

We let U and S be $U(t)=\exp(it\Delta)$ and $(Sf)(t)=\int_0^t U(t-s)f(s)\,ds$ as defined in Section 1.

Following estimates were obtained by Strichartz **[18]**, Kenig–Ponce–Vega **[13, 14]**, Bekiranov–Ogawa–Ponce **[3]** and Hirata **[12]** etc.

Lemma 2.1. *For the linear operator U and S, we have following estimates.*

$$\|U\phi\|_{L_T^\infty L^2}+\|D_{x_1}^{1/2}U\phi\|_{L_{x_1}^\infty L_T^2L_{x_2}^2}+\|D_{x_2}^{1/2}U\phi\|_{L_{x_2}^\infty L_T^2L_{x_1}^2}\le C_0\|\phi\|_2,\tag{2.2}$$

$$\|\partial_{x_1} Sf\|_{L^\infty_{x_1} L^2_T L^2_{x_2}} \le \begin{cases} \frac{1}{2}\|f\|_{L^1_{x_1} L^2_T L^2_{x_2}}, \\ C_1 \|D^{1/2}_{x_1} f\|_{L^1_T L^2}, \end{cases} \tag{2.3}$$

$$\|\partial_{x_2} Sf\|_{L^\infty_{x_2} L^2_T L^2_{x_1}} \le \begin{cases} \frac{1}{2}\|f\|_{L^1_{x_2} L^2_T L^2_{x_1}}, \\ C_1 \|D^{1/2}_{x_2} f\|_{L^1_T L^2}, \end{cases} \tag{2.4}$$

$$\|Sf\|_{L^\infty_T L^2} \le \|f\|_{L^1_T L^2}. \tag{2.5}$$

Proof. We only prove the first inequality of (2.3), because the factor 1/2 is very important in our theorem. For the proof of the other inequalities, see e.g., [3], [13].

We remark that

$$Sf = -i\mathcal{F}^{-1}_{x,t}(\tau + \xi_1^2 + \xi_2^2 - i0)^{-1}\mathcal{F}_{x,t} f \tag{2.6}$$

if $f(x,t) \equiv 0$ for $t < 0$. Here, $\mathcal{F}_{x,t}$ is the Fourier transform with respect to whole space-time variables (x,t). In fact, a simple calculation shows

$$\mathcal{F}^{-1}_{x,t}(\tau + \xi_1^2 + \xi_2^2 - i0)^{-1}\mathcal{F}_{x,t} f|_{t=0} = -i\int_{-\infty}^{0} U(-s)f(s)\,ds,$$

and then,

$$-i\mathcal{F}^{-1}_{x,t}(\tau + \xi_1^2 + \xi_2^2 - i0)^{-1}\mathcal{F}_{x,t} f = \int_0^t U(t-s)f(\cdot,s)\,ds - \int_{-\infty}^{0} U(-s)f(s)\,ds.$$

This shows our claim (2.6). We apply Plancherel's equality to (2.6) to obtain

$$\begin{aligned} \sup_{x_1\in\mathbf{R}} \|Sf\|_{L^2_t L^2_{x_2}} &= \sup_{x_1\in\mathbf{R}} \|\mathcal{F}^{-1}_{x,t}\xi_1(\tau + \xi_1^2 + \xi_2^2 - i0)^{-1}\mathcal{F}_{x,t} f\|_{L^2_t L^2_{x_2}} \\ &= \sup_{x_1\in\mathbf{R}} \|\mathcal{F}^{-1}_{x_1}\xi_1(\tau + \xi_1^2 + \xi_2^2 - i0)^{-1}\mathcal{F}_{x_1}(\mathcal{F}_{x_2,t} f)\|_{L^2_\tau L^2_{\xi_2}} \\ &= \sup_{x_1\in\mathbf{R}} \Big\| \int_{\mathbf{R}} G(x_1 - \tilde{x}_1; \xi_2, \tau)(\mathcal{F}_{x_2,t} f)(\tilde{x}_1, \xi_2, \tau)\,d\tilde{x}_1 \Big\|_{L^2_\tau L^2_{\xi_2}}. \end{aligned}$$

Here

$$G(z; \xi_2, \tau) := \frac{1}{2\pi}\int_{\mathbf{R}} \frac{\xi_1 \exp(iz\xi_1)}{\xi_1^2 + (\tau + \xi_2^2) - i0}\,d\xi_1.$$

The residue calculus shows

$$G(z;\xi_2,\tau)=\begin{cases}\frac{i}{2}\exp(iz(-\tau-\xi_2^2)^{1/2}), & \text{if } z>0, \tau+\xi_2^2<0,\\ \frac{i}{2}\exp(-z(\tau+\xi_2^2)^{1/2}), & \text{if } z>0, \tau+\xi_2^2>0,\\ -\frac{i}{2}\exp(-iz(-\tau-\xi_2^2)^{1/2}), & \text{if } z<0, \tau+\xi_2^2<0,\\ -\frac{i}{2}\exp(z(\tau+\xi_2^2)^{1/2}), & \text{if } z<0, \tau+\xi_2^2>0,\end{cases}$$

that is, G is uniformly bounded with respect to z, τ, ξ_2 by $1/2$. Then we get

$$\begin{aligned}&\sup_{x_1\in\mathbf{R}}\|\mathcal{F}^{-1}_{x,t}\xi_1(\tau+\xi_1^2+\xi_2^2-i0)^{-1}\mathcal{F}_{x,t}f\|_{L^2_tL^2_{x_2}}\\ &\le\sup_{x_1\in\mathbf{R}}\int_{\mathbf{R}}\|G(x_1-\tilde{x}_1;\xi_2,\tau)(\mathcal{F}_{x_2,t}f)(\tilde{x}_1,x_2,\tau)\|_{L^2_\tau L^2_{\xi_2}}d\tilde{x}_1\\ &\le\frac{1}{2}\int_{\mathbf{R}}\|(\mathcal{F}_{x_2,t}f)(\cdot,x_1,\cdot)\|_{L^2_\tau L^2_{\xi_2}}dx_1\\ &=\frac{1}{2}\int_{\mathbf{R}}\|f(\cdot,x_1,\cdot)\|_{L^2_tL^2_{x_2}}dx_1\\ &=\frac{1}{2}\|f\|_{L^1_{x_1}L^2_tL^2_{x_2}}.\end{aligned}$$

This shows the desired result. □

The next lemma is a Hölder type estimate of Leibniz rule for a fractional order derivative.

Lemma 2.2. *Let* $0<\alpha<1$ *and* $1<p<\infty$. *Then*

$$\|D^\alpha_x(fg)-fD^\alpha_xg-gD^\alpha_xf\|_p\le C\|g\|_\infty\|D^\alpha_xf\|_p.$$

Let $p, p_1, p_2\in(1,\infty)$ *such that* $1/p=1/p_1+1/p_2$. *Then*

$$\|D^\alpha_x(fg)-fD^\alpha_xg-gD^\alpha_xf\|_p\le C\|g\|_{p_1}\|D^\alpha_xf\|_{p_2}.$$

For the proof of this lemma, see Appendix of [15, Theorem A.1].

3. The estimates for the nonlinear terms

In what follows, we use the following notation.

$$F(v)=\sum_{j=1}^3 f_j(v),$$

where

$$f_1(v)=d_1|v|^2v,\quad f_2(v)=d_2v\int_{x_2}^{\infty}\partial_{x_1}|v(x_1,x_2')|^2\,dx_2',$$
$$\text{and}\quad f_3(v)=d_3v\int_{x_1}^{\infty}\partial_{x_2}|v(x_1',x_2)|^2\,dx_1'.$$

By a direct calculation we have

$$\begin{cases}\partial_{x_1}^2 f_2(v)=d_2\left(g_1(v)+2v\displaystyle\int_{x_2}^{\infty}\operatorname{Re}(v\partial_{x_1}^3\bar v)\,dx_2'\right),\\ \partial_{x_2}^2 f_2(v)=2d_2\left(\partial_{x_2}^2 v\displaystyle\int_{x_2}^{\infty}\operatorname{Re}(v\partial_{x_1}\bar v)\,dx_2'-2\partial_{x_2}v\operatorname{Re}(v\partial_{x_1}\bar v)\right.\\ \qquad\left.-v\operatorname{Re}(\partial_{x_2}v\cdot\partial_{x_1}\bar v+v\partial_{x_1}\partial_{x_2}\bar v)\right),\\ \partial_{x_1}\partial_{x_2}f_2(v)=2d_2\left(\partial_{x_1}\partial_{x_2}v\displaystyle\int_{x_2}^{\infty}\operatorname{Re}(v\partial_{x_1}\bar v)\,dx_2'+\partial_{x_2}v\int_{x_2}^{\infty}|\partial_{x_1}v|^2+\operatorname{Re}(v\partial_{x_1}^2\bar v)\,dx_2'\right.\\ \qquad\left.-\partial_{x_1}v\operatorname{Re}(v\partial_{x_1}\bar v)-v(|\partial_{x_1}v|^2+\operatorname{Re}(v\partial_{x_1}^2\bar v))\right),\end{cases}\tag{I1}$$

where

$$g_1(v)=4\partial_{x_1}v\int_{x_2}^{\infty}|\partial_{x_1}v|^2+\operatorname{Re}(v\partial_{x_1}^2\bar v)\,dx_2'+2\partial_{x_1}^2v\int_{x_2}^{\infty}\operatorname{Re}(v\partial_{x_1}\bar v)\,dx_2'$$
$$+6v\int_{x_2}^{\infty}\operatorname{Re}(\partial_{x_1}v\cdot\partial_{x_1}^2\bar v)\,dx_2',$$

and

$$\begin{cases}\partial_{x_1}^2 f_3(v)=2d_3\left(\partial_{x_1}^2 v\displaystyle\int_{x_1}^{\infty}\operatorname{Re}(v\partial_{x_2}\bar v)\,dx_1'-2\partial_{x_1}v\operatorname{Re}(v\partial_{x_2}\bar v)\right.\\ \qquad\left.-v\operatorname{Re}(\partial_{x_1}v\cdot\partial_{x_2}\bar v+v\partial_{x_1}\partial_{x_2}\bar v)\right),\\ \partial_{x_2}^2 f_3(v)=2d_3\left(g_2(v)+v\displaystyle\int_{x_1}^{\infty}\operatorname{Re}(v\partial_{x_2}^3\bar v)\,dx_1'\right),\\ \partial_{x_1}\partial_{x_2}f_3(v)=2d_3\left(\partial_{x_1}\partial_{x_2}v\displaystyle\int_{x_1}^{\infty}\operatorname{Re}(v\partial_{x_2}\bar v)\,dx_1'+\partial_{x_1}v\int_{x_1}^{\infty}|\partial_{x_2}v|^2+\operatorname{Re}(v\partial_{x_2}^2\bar v)\,dx_1'\right.\\ \qquad\left.-\partial_{x_2}v\operatorname{Re}(v\partial_{x_2}\bar v)-v(|\partial_{x_2}v|^2+\operatorname{Re}(v\partial_{x_2}^2\bar v))\right),\end{cases}\tag{I2}$$

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where

$$g_2(v) = 4\partial_{x_2} v \int_{x_1}^{\infty} |\partial_{x_2} v|^2 + \mathrm{Re}(v\partial_{x_2}^2 \bar{v})\, dx_1' + 2\partial_{x_2}^2 v \int_{x_1}^{\infty} \mathrm{Re}(v\partial_{x_2}\bar{v})\, dx_1'$$
$$+ 6v \int_{x_1}^{\infty} \mathrm{Re}(\partial_{x_2} v \cdot \partial_{x_2}^2 \bar{v})\, dx_1'.$$

By applying Lemma 2.1 to these identities, we obtain the following estimates.

Lemma 3.1. *We have*

$$\|\partial_{x_1}^3 SF(v)\|_{L_{x_1}^\infty L_T^2 L_{x_2}^2} \le |d_2| \|v \int_{x_2}^{\infty} \mathrm{Re}(v\partial_{x_1}^3 \bar{v})\, dx_2' \|_{L_{x_1}^1 L_T^2 L_{x_2}^2}$$
$$+ C_1(\|D_{x_1}^{1/2} g_1(v)\|_{L_T^1 L^2} + \|D_{x_1}^{1/2} \partial_{x_1}^2 f_1(v)\|_{L_T^1 L^2} + \|D_{x_1}^{1/2} \partial_{x_1}^2 f_3(v)\|_{L_T^1 L^2}),$$

and

$$\|\partial_{x_2}^3 SF(v)\|_{L_{x_2}^\infty L_T^2 L_{x_1}^2} \le |d_3| \|v \int_{x_1}^{\infty} \mathrm{Re}(v\partial_{x_2}^3 \bar{v})\, dx_1' \|_{L_{x_2}^1 L_T^2 L_{x_1}^2}$$
$$+ C_1(\|D_{x_2}^{1/2} g_2(v)\|_{L_T^1 L^2} + \|D_{x_2}^{1/2} \partial_{x_2}^2 f_1(v)\|_{L_T^1 L^2} + \|D_{x_2}^{1/2} \partial_{x_2}^2 f_2(v)\|_{L_T^1 L^2}).$$

Lemma 3.2. *Let f, g, h be complex valued functions on $\mathbf{R}^2$, and $p_1, \ldots, p_6 \in (1, \infty)$ such that $1/2 = 1/p_1 + 1/p_2 + 1/p_3$, $1/2 = 1/p_4 + 1/p_5 + 1/p_6$. Then, we have*

$$\|D_{x_1}^{1/2}\left(f \int_{x_2}^{\infty} gh\, dx_2'\right)\|$$
$$\le \begin{cases} C\{\|D_{x_1}^{1/2} f\|_{L_{x_2}^2 L_{x_1}^{p_1}} \|g\|_{L_{x_2}^2 L_{x_1}^{p_2}} \|h\|_{L_{x_2}^2 L_{x_1}^{p_3}} \\ \quad + \|f\|_{L_{x_2}^2 L_{x_1}^\infty}(\|g\|_{L_{x_2}^2 L_{x_1}^\infty} \|D_{x_1}^{1/2} h\| + \|h\|_{L_{x_2}^2 L_{x_1}^\infty} \|D_{x_1}^{1/2} g\|)\}, \\ C\{\|D_{x_1}^{1/2} f\| \|g\|_{L_{x_2}^2 L_{x_1}^\infty} \|h\|_{L_{x_2}^2 L_{x_1}^\infty} \\ \quad + \|f\|_{L_{x_2}^2 L_{x_1}^{p_1}}(\|g\|_{L_{x_2}^2 L_{x_1}^{p_2}} \|D_{x_1}^{1/2} h\|_{L_{x_2}^2 L_{x_1}^{p_3}} + \|h\|_{L_{x_2}^2 L_{x_1}^{p_2}} \|D_{x_1}^{1/2} g\|_{L_{x_2}^2 L_{x_1}^{p_3}})\} \end{cases} \quad (3.1)$$

and

$$\|D_{x_2}^{1/2}\left(f \int_{x_2}^{\infty} gh\, dx_2'\right)\|$$
$$\le \begin{cases} C\{\|\langle D_{x_2}\rangle^{1/2} f\| \|g\|_{L_{x_1}^\infty L_{x_2}^2} \|h\|_{L_{x_1}^\infty L_{x_2}^2} + \|f\|_{L_{x_1}^{p_4} L_{x_2}^{p_1}} \|g\|_{L_{x_1}^{p_5} L_{x_2}^{p_2}} \|h\|_{L_{x_1}^{p_6} L_{x_2}^{p_3}}\}, \\ C\{\|\langle D_{x_2}\rangle^{1/2} f\|_{L_{x_1}^{p_1} L_{x_2}^2} \|g\|_{L_{x_1}^{p_2} L_{x_2}^2} \|h\|_{L_{x_1}^{p_3} L_{x_2}^2} \\ \quad + \|f\|_{L_{x_1}^{p_4} L_{x_2}^{p_1}} \|g\|_{L_{x_1}^{p_5} L_{x_2}^{p_2}} \|h\|_{L_{x_1}^{p_6} L_{x_2}^{p_3}}\}, \end{cases} \quad (3.2)$$

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where

$$\langle D_{x_j}\rangle = (1-\partial_{x_j}^2)^{1/2}.$$

Proof. Lemma 3.2 is obtained by Lemma 2.2. □

Lemma 3.3.

$$\begin{aligned} &\|D_{x_1}^{1/2} g_1(v)\|_{L_T^1 L^2} + \|D_{x_1}^{1/2}\partial_{x_1}^2 f_1(v)\|_{L_T^1 L^2} \\ &+ \|D_{x_1}^{1/2}\partial_{x_1}^2 f_3(v)\|_{L_T^1 L^2} \le CT\|v\|_{Y_T}^3, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &\|D_{x_2}^{1/2} g_2(v)\|_{L_T^1 L^2} + \|D_{x_2}^{1/2}\partial_{x_2}^2 f_1(v)\|_{L_T^1 L^2} \\ &+ \|D_{x_2}^{1/2}\partial_{x_2}^2 f_2(v)\|_{L_T^1 L^2} \le CT\|v\|_{Y_T}^3, \end{aligned} \tag{3.4}$$

and

$$\|D_{x_1}^{1/2}\partial_{x_1}\partial_{x_2} F(v)\|_{L_T^1 L^2} + \|D_{x_1}^{1/2}\partial_{x_1}\partial_{x_2} F(v)\|_{L_T^1 L^2} \le CT\|v\|_{Y_T}^3. \tag{3.5}$$

Proof. Lemma 3.3 is obtained by Lemma 3.2. □

Lemma 3.4.

$$\begin{aligned} &\Big\|v\int_{x_1}^{\infty} \mathrm{Re}(v\partial_{x_2}^3 v)\,dx_1'\Big\|_{L_{x_2}^1 L_T^2 L_{x_1}^2} \\ &\le (2T\|v\|_{L_T^\infty L^2}\|\partial_t v\|_{L_T^\infty L^2} + \|v(0)\|^2)\|\partial_{x_2}^3 v\|_{L_{x_2}^\infty L_T^2 L_{x_1}^2}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\Big\|v\int_{x_2}^{\infty} \mathrm{Re}(v\partial_{x_2}^3 v)\,dx_2'\Big\|_{L_{x_1}^1 L_T^2 L_{x_2}^2} \\ &\le (2T\|v\|_{L_T^\infty L^2}\|\partial_t v\|_{L_T^\infty L^2} + \|v(0)\|^2)\|\partial_{x_1}^3 v\|_{L_{x_1}^\infty L_T^2 L_{x_2}^2}. \end{aligned} \tag{3.7}$$

Proof. The proof of (3.7) is obtained in the same way as in (3.6), so we only consider (3.6). First, we note that

$$\Big\|v\int_{x_1}^{\infty} \mathrm{Re}(v\partial_{x_2}^3 v)\,dx_1'\Big\|_{L_{x_2}^1 L_T^2 L_{x_1}^2} \le \|v\|_{L_{x_2}^1 L_T^\infty L_{x_1}^2}^2 \|\partial_{x_2}^3 v\|_{L_{x_2}^\infty L_T^2 L_{x_1}^2} \tag{3.8}$$

by Hölder's inequality. Since

$$\|v(t)\|_{L_{x_1}^2}^2 = \int_0^t \frac{d}{d\tau}\|v(\tau)\|_{L_{x_1}^2}^2\,d\tau + \|v(0)\|_{L_{x_1}^2}^2,$$

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taking L^∞ norm on $[0, T]$ of the both hand sides, we have

$$\|v\|^2_{L^\infty_T L^2_{x_1}} \le 2\|\partial_t v\|_{L^2_T L^2_{x_1}} \|v\|_{L^2_T L^2_{x_1}} + \|v(0)\|^2_{L^2_{x_1}}. \tag{3.9}$$

Taking L^1 norm of (3.9) with respect to x_2-variable, we have

$$\begin{aligned}\|v\|^2_{L^1_{x_2} L^\infty_T L^2_{x_1}} &\le 2\|\partial_t v\|_{L^2_T L^2} \|v\|_{L^2_T L^2} + \|v(0)\|^2 \\ &\le 2T\|\partial_t v\|_{L^\infty_T L^2} \|v\|_{L^\infty_T L^2} + \|v(0)\|^2.\end{aligned} \tag{3.10}$$

Combining (3.8) and (3.10), we obtain (3.6). □

By using Lemma 3.3 and Lemma 3.4 in the right hand side of Lemma 3.1, we obtain the following lemma.

Lemma 3.5. *We have*

$$\begin{aligned}&\|\partial^3_{x_1} SF(v)\|_{L^\infty_{x_1} L^2_T L^2_{x_2}} \\ &\le CT\|v\|^3_{Y_T} + (2|d_2|T\|v\|_{L^\infty_T L^2}\|\partial_t v\|_{L^\infty_T L^2} + |d_2|\|v(0)\|^2)\|\partial^3_{x_1} v\|_{L^\infty_{x_1} L^2_T L^2_{x_2}},\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}&\|\partial^3_{x_2} SF(v)\|_{L^\infty_{x_2} L^2_T L^2_{x_1}} \\ &\le CT\|v\|^3_{Y_T} + (2|d_3|T\|v\|_{L^\infty_T L^2}\|\partial_t v\|_{L^\infty_T L^2} + |d_3|\|v(0)\|^2_2)\|\partial^3_{x_2} v\|_{L^\infty_{x_2} L^2_T L^2_{x_1}}.\end{aligned} \tag{3.12}$$

Lemma 3.6. *We have*

$$\begin{aligned}&\int_0^T |\mathrm{Im}(D^{1/2}_{x_1}\partial^2_{x_1} F(v), D^{1/2}_{x_1}\partial^2_{x_1} u)|\,dt \le CT\|v\|^3_{Y_T}\|u\|_{Y_T} \\ &+ (4T\|v\|_{L^\infty_T L^2}\|\partial_t v\|_{L^\infty_T L^2} + 2|d_2|\|v(0)\|^2\|\partial^3_{x_1} v\|_{L^\infty_{x_1} L^2_T L^2_{x_2}})\|\partial^3_{x_1} u\|_{L^\infty_{x_1} L^2_T L^2_{x_2}},\end{aligned} \tag{3.13}$$

$$\begin{aligned}&\int_0^T |\mathrm{Im}(D^{1/2}_{x_2}\partial^2_{x_2} F(v), D^{1/2}_{x_2}\partial^2_{x_2} u)|\,dt \le CT\|v\|^3_{Y_T}\|u\|_{Y_T} \\ &+ (4T\|v\|_{L^\infty_T L^2}\|\partial_t v\|_{L^\infty_T L^2} + 2|d_3|\|v(0)\|^2\|\partial^3_{x_2} v\|_{L^\infty_{x_2} L^2_T L^2_{x_1}})\|\partial^3_{x_2} u\|_{L^\infty_{x_2} L^2_T L^2_{x_1}},\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}&\int_0^T |\mathrm{Im}(D^{1/2}_{x_2}\partial_{x_1}\partial_{x_2} F(v), D^{1/2}_{x_2}\partial_{x_1}\partial_{x_2} u)|\,dt \\ &+ \int_0^T |\mathrm{Im}(D^{1/2}_{x_1}\partial_{x_1}\partial_{x_2} F(v), D^{1/2}_{x_1}\partial_{x_1}\partial_{x_2} u)|\,dt \\ &\le CT\|v\|^3_{Y_T}\|u\|_{Y_T}.\end{aligned} \tag{3.15}$$

Proof. We only consider (3.13), since the other estimates are very similar. From integration by parts and the Schwartz inequality we easily see that the left hand side of (3.13) is bounded from above by

$$
\begin{aligned}
&2(\|D_{x_1}^{1/2}\partial_{x_1}^2 f_1(v)\|_{L_T^1L^2} + \|D_{x_1}^{1/2}\partial_{x_1}^2 f_3(v)\|_{L_T^1L^2} + \|D_{x_1}^{1/2} g_1(v)\|_{L_T^1L^2})\|D_{x_1}^{1/2}\partial_{x_1}^2 u\|_{L_T^\infty L^2}\\
&+ 2|d_2|\|v\int_{x_2}^\infty 2\,\mathrm{Re}(v\partial_{x_1}^3\bar{v})\,dx_2' \cdot D_{x_1}\partial_{x_1}^2 u\|_{L_T^1L^1}\\
\le{}& 2(\|D_{x_1}^{1/2}\partial_{x_1}^2 f_1(v)\|_{L_T^1L^2} + \|D_{x_1}^{1/2}\partial_{x_1}^2 f_3(v)\|_{L_T^1L^2} + \|D_{x_1}^{1/2} g_1(v)\|_{L_T^1L^2})\|D_{x_1}^{1/2}\partial_{x_1}^2 u\|_{L_T^\infty L^2}\\
&+ 2|d_2|\|v\int_{x_2}^\infty 2\,\mathrm{Re}(v\partial_{x_1}^3\bar{v})\,dx_2'\|_{L_{x_1}^1L_T^2L_{x_2}^2}\|D_{x_1}\partial_{x_1}^2 u\|_{L_{x_1}^\infty L_T^2L_{x_2}^2}.
\end{aligned}
$$

We apply Lemma 3.3 and Lemma 3.4 to the above to obtain (3.13). □

In a similar way as the proof of Lemma 3.6 (3.13) we have the following lemma.

Lemma 3.7. *We have*

$$
\sum_{|\alpha|\le 2}\int_0^T |\mathrm{Im}(\partial^\alpha F(v), \partial^\alpha u)|\,dt \le CT\|v\|_{Y_T}^3\|u\|_{Y_T}. \tag{3.16}
$$

Proof. By a simple calculation we have

$$
\begin{aligned}
\mathrm{Im}(\partial_{x_1}^2 F(v), \partial_{x_1}^2 u) ={}&\mathrm{Im}(\partial_{x_1}^2 f_1(v), \partial_{x_1}^2 u) + d_2\,\mathrm{Im}(g_1(v), \partial_{x_1}^2 u) + \mathrm{Im}(\partial_{x_1}^2 f_3(v), \partial_{x_1}^2 u)\\
&+ 2d_2\,\mathrm{Im}\left(v\int_{x_2}^\infty \mathrm{Re}(v\partial_{x_1}^3\bar{v})\,dx_2', \partial_{x_1}^2 u\right).
\end{aligned} \tag{3.17}
$$

Similarly, we have

$$
\begin{aligned}
\mathrm{Im}(\partial_{x_2}^2 F(v), \partial_{x_2}^2 u) ={}&\mathrm{Im}(\partial_{x_2}^2 f_1(v), \partial_{x_2}^2 u) + d_3\,\mathrm{Im}(g_2(v), \partial_{x_2}^2 u) + \mathrm{Im}(\partial_{x_2}^2 f_2(v), \partial_{x_2}^2 u)\\
&+ 2d_3\,\mathrm{Im}\left(v\int_{x_1}^\infty \mathrm{Re}(v\partial_{x_2}^3\bar{v})\,dx_1', \partial_{x_2}^2 u\right).
\end{aligned} \tag{3.18}
$$

We consider the last terms of the right hand sides of (3.17) and (3.18). By virtue of the self adjointness of $D_x^{1/2}$, we have

$$
\begin{aligned}
&\mathrm{Im}\left(v\int_{x_2}^\infty \mathrm{Re}(v\partial_{x_1}^3\bar{v})\,dx_2', \partial_{x_1}^2 u\right)\\
={}& \mathrm{Im}\left(\frac{\partial_{x_1}}{D_{x_1}^{1/2}} v\int_{x_2}^\infty \mathrm{Re}(v\partial_{x_1}^2\bar{v})\,dx_2', D_{x_1}^{1/2}\partial_{x_1}^2 u\right) - \mathrm{Im}\left(\partial_{x_1} v\int_{x_2}^\infty \mathrm{Re}(v\partial_{x_1}^2\bar{v})\,dx_2', \partial_{x_1}^2 u\right)\\
&- \mathrm{Im}\left(v\int_{x_2}^\infty \mathrm{Re}(\partial_{x_1} v\partial_{x_1}^2\bar{v})\,dx_2', \partial_{x_1}^2 u\right),
\end{aligned}
$$

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and

$$\operatorname{Im}\left(v\int_{x_1}^{\infty}\operatorname{Re}(v\partial_{x_2}^3\bar{v})\,dx_1',\ \partial_{x_2}^2 u\right)$$
$$=\operatorname{Im}\left(\frac{\partial_{x_2}}{D_{x_2}^{1/2}}v\int_{x_1}^{\infty}\operatorname{Re}(v\partial_{x_2}^2\bar{v})\,dx_1',\ D_{x_2}^{1/2}\partial_{x_2}^2 u\right)-\operatorname{Im}\left(\partial_{x_2}v\int_{x_1}^{\infty}\operatorname{Re}(v\partial_{x_2}^2\bar{v})\,dx_1',\ \partial_{x_2}^2 u\right)$$
$$-\operatorname{Im}\left(v\int_{x_1}^{\infty}\operatorname{Re}(\partial_{x_2}v\partial_{x_2}^2\bar{v})\,dx_2',\ \partial_{x_2}^2 u\right).$$

Hence we have by the Schwartz inequality

$$\begin{aligned}&\left|\operatorname{Im}\left(v\int_{x_2}^{\infty}\operatorname{Re}(v\partial_{x_1}^3\bar{v})\,dx_2',\ \partial_{x_1}^2 u\right)\right|\\ &\le \|D_{x_1}^{1/2}\partial_{x_1}^2 u\|\,\|D_{x_1}^{1/2}\left(v\int_{x_2}^{\infty}\operatorname{Re}(v\partial_{x_1}^2\bar{v})\,dx_2'\right)\|\\ &\quad+\|\partial_{x_1}^2 u\|\left(\|(\partial_{x_1}v)\int_{x_2}^{\infty}\operatorname{Re}(v\partial_{x_1}^2\bar{v})\,dx_2'\|+\|v\int_{x_2}^{\infty}\operatorname{Re}(\partial_{x_1}v\partial_{x_1}^2\bar{v})\,dx_2'\|\right),\end{aligned}\tag{3.19}$$

and

$$\begin{aligned}&\left|\operatorname{Im}\left(v\int_{x_1}^{\infty}\operatorname{Re}(v\partial_{x_2}^3\bar{v})\,dx_1',\ \partial_{x_2}^2 u\right)\right|\\ &\le \|D_{x_2}^{1/2}\partial_{x_2}^2 u\|\,\|D_{x_2}^{1/2}\left(v\int_{x_1}^{\infty}\operatorname{Re}(v\partial_{x_2}^2\bar{v})\,dx_1'\right)\|\\ &\quad+\|\partial_{x_2}^2 u\|\left(\|(\partial_{x_2}v)\int_{x_1}^{\infty}\operatorname{Re}(v\partial_{x_2}^2\bar{v})\,dx_1'\|+\|v\int_{x_1}^{\infty}\operatorname{Re}(\partial_{x_2}v\partial_{x_2}^2\bar{v})\,dx_1'\|\right).\end{aligned}\tag{3.20}$$

Integrating (3.19) and (3.20) in time variable t and using Lemma 3.3, we obtain

$$\begin{aligned}&\|\operatorname{Im}\left(v\int_{x_2}^{\infty}\operatorname{Re}(v\partial_{x_1}^3\bar{v})\,dx_2',\ \partial_{x_1}^2 u\right)\|_{L_T^1}+\|\operatorname{Im}\left(v\int_{x_1}^{\infty}\operatorname{Re}(v\partial_{x_2}^3\bar{v})\,dx_1',\ \partial_{x_2}^2 u\right)\|_{L_T^1}\\ &\le CT\|u\|_{Y_T}\|v\|_{Y_T}^3.\end{aligned}\tag{3.21}$$

We have by (3.15), (3.16), (3.21) and Lemma 3.2

$$\|\operatorname{Im}(\partial_{x_1}^2F(v),\ \partial_{x_1}^2 u)\|_{L_T^1}+\|\operatorname{Im}(\partial_{x_2}^2F(v),\ \partial_{x_2}^2 u)\|_{L_T^1}\le CT\|u\|_{Y_T}\|v\|_{Y_T}^3.\tag{3.22}$$

A similar calculation shows

$$\|\operatorname{Im}(\partial_{x_1}\partial_{x_2}F(v),\ \partial_{x_1}\partial_{x_2}u)\|_{L_T^1}\le CT\|u\|_{Y_T}\|v\|_{Y_T}^3,\tag{3.23}$$

and the estimate

$$\sum_{|\alpha|\le 1} \|\mathrm{Im}(\partial^\alpha F(v), \partial^\alpha u)\|_{L^1_T} \le CT\|u\|_{Y_T}\|v\|^3_{Y_T} \tag{3.24}$$

is obtained by Sobolev's and Hölder's inequalities. From (3.23)–(3.24) the lemma follows. □

We next consider the term

$$G(v;\varphi) = d_4 v\partial_{x_1}\varphi_1 + d_5 v\partial_{x_2}\varphi_2.$$

By a simple calculation we have

$$\partial^2_{x_1}G(v;\varphi) = d_4(\partial^2_{x_1}v\cdot\partial_{x_1}\varphi_1 + v\partial^3_{x_1}\varphi_1 + 2\partial_{x_1}v\cdot\partial^2_{x_1}\varphi_1) + d_5\partial^2_{x_1}v\cdot\partial_{x_2}\varphi_2,$$
$$\partial^2_{x_2}G(v;\varphi) = d_4\partial^2_{x_2}v\cdot\partial_{x_1}\varphi_1 + d_5(\partial^2_{x_2}v\cdot\partial_{x_2}\varphi_2 + v\partial^3_{x_2}\varphi_2 + 2\partial_{x_2}v\cdot\partial^2_{x_2}\varphi_2),$$

and

$$\partial_{x_1}\partial_{x_2}G(v;\varphi) = d_4(\partial_{x_1}\partial_{x_2}v\cdot\partial_{x_1}\varphi_1 + v\partial^2_{x_1}\varphi_1) + d_5(\partial_{x_1}\partial_{x_2}v\cdot\partial_{x_2}\varphi_2 + v\partial^2_{x_2}\varphi_2).$$

By these identities we obtain the following lemma.

Lemma 3.8. *We have*

$$\|\partial^3_{x_1}SG(v;\varphi)\|_{L^\infty_{x_1}L^2_TL^2_{x_2}} \le C_\varphi T\|v\|_{Y_T},$$
$$\|\partial^3_{x_2}SG(v;\varphi)\|_{L^\infty_{x_2}L^2_TL^2_{x_1}} \le C_\varphi T\|v\|_{Y_T},$$

$$\sum_{|\alpha|\le 2}\int_0^T |\mathrm{Im}(D^{1/2}_{x_1}\partial^\alpha G(v;\varphi), D^{1/2}_{x_1}\partial^\alpha u)|dt \le C_\varphi T\|v\|_{Y_T}\|u\|_{Y_T},$$

and $$\sum_{|\alpha|\le 2}\int_0^T |\mathrm{Im}(D^{1/2}_{x_2}\partial^\alpha G(v;\varphi), D^{1/2}_{x_2}\partial^\alpha u)|dt \le C_\varphi T\|v\|_{Y_T}\|u\|_{Y_T},$$

where

$$C_\varphi = C(\|\partial_{x_1}\varphi_1\|_{H^{5/2}_{x_1}} + \|\partial_{x_2}\varphi_2\|_{H^{5/2}_{x_2}}).$$

Proof. By Lemma 2.1 we find that the left hand sides of the first two inequalities are estimated from above by

$$C_1\|D^{1/2}_{x_1}\partial^2_{x_1}G(v;\varphi)\|_{L^1_TL^2}, \quad \text{and} \quad C_1\|D^{1/2}_{x_2}\partial^2_{x_2}G(v;\varphi)\|_{L^1_TL^2}.$$

The Schwartz inequality shows that the left hand sides of the last two inequalities are estimated from above by

$$C\sum_{|\alpha|\le 2}\|D_{x_1}^{1/2}\partial^\alpha G(v;\varphi)\|_{L_T^1L^2}\|u\|_{Y_T},\quad \text{and}\quad C\sum_{|\alpha|\le 2}\|D_{x_2}^{1/2}\partial^\alpha G(v;\varphi)\|_{L_T^1L^2}\|u\|_{Y_T}.$$

Therefore we have the lemma by Lemma 2.2 and Sobolev's inequality. □

In a similar way as in the proofs of Lemma 3.5–Lemma 3.8, we have the following lemma.

Lemma 3.9. *We assume that* $v(x,0)=w(x,0)$. *Then we have*

$$\begin{aligned}
&\|\partial_{x_1}^3 S(F(v)+G(v;\varphi)-F(w)-G(w;\varphi))\|_{L_{x_1}^\infty L_T^2L_{x_2}^2}\\
&\quad+\|\partial_{x_2}^3 S(F(v)+G(v;\varphi)-F(w)-G(w;\varphi))\|_{L_{x_2}^\infty L_T^2L_{x_1}^2}\\
&\le CT(\|v-w\|_{X_T}+\|\partial_t(v-w)\|_{L_T^\infty L^2})(C_\varphi+\|v\|_{X_T}^2+\|w\|_{X_T}^2+\|\partial_t v\|_{L_T^\infty L^2}+\|\partial_t w\|_{L_T^\infty L^2}),\\
&\sum_{|\alpha|=2}\left(\int_0^T|\mathrm{Im}(D_{x_1}^{1/2}\partial^\alpha(F(v)-F(w)),D_{x_1}^{1/2}\partial^\alpha u)|\,dt+\int_0^T|\mathrm{Im}(D_{x_2}^{1/2}\partial^\alpha(F(v)-F(w)),D_{x_2}^{1/2}\partial^\alpha u)|\,dt\right)\\
&\le CT(\|v-w\|_{Y_T}+\|\partial_t(v-w)\|_{L_T^\infty L^2})(\|v\|_{Y_T}^2+\|w\|_{Y_T}^2+\|\partial_t v\|_{L_T^\infty L^2}+\|\partial_t w\|_{L_T^\infty L^2})\|u\|_{X_T},\\
&\sum_{|\alpha|\le 2}\int_0^T|2\,\mathrm{Im}(\partial^\alpha(F(v)-F(w)),\partial^\alpha u)|\,dt\le CT\|v-w\|_{Y_T}(\|v\|_{Y_T}^2+\|w\|_{Y_T}^2)\|u\|_{Y_T},\\
&\sum_{|\alpha|\le 2}\left(\int_0^T|\mathrm{Im}(D_{x_1}^{1/2}\partial^\alpha(G(v;\varphi)-G(w;\varphi)),D_{x_1}^{1/2}\partial^\alpha u)|\,dt\right.\\
&\quad\left.+\int_0^T|\mathrm{Im}(D_{x_2}^{1/2}\partial^\alpha(G(v;\varphi)-G(w;\varphi)),D_{x_2}^{1/2}\partial^\alpha u)|\,dt\right)\\
&\le C_\varphi T\|v-w\|_{Y_T}\|u\|_{Y_T},
\end{aligned}$$

and

$$\sum_{|\alpha|\le 2}\int_0^T|\mathrm{Im}(\partial^\alpha(G(v;\varphi)-G(w;\varphi)),\partial^\alpha u)|\,dt\le C_\varphi T\|v-w\|_{Y_T}\|u\|_{Y_T}.$$

4. Proof of Theorem 1

We define the sequence $\{u_n\}_{n\in\mathbb{N}\cup\{0\}}$ as follows:

$$\begin{cases} u_0=U\phi,\\ u_n=u_0-iS(F(u_{n-1})+G(u_{n-1};\varphi)), \end{cases}\tag{4.1}$$

where F and G are the same as defined in Section 3. We first remark $u_0 \in X_T$ for some $\rho > 0$ by virtue of the first estimate in Lemma 2.1. From now on we will prove that $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $X_{T,\rho}$ for some time T, where

$$X_{T,\rho} = \{f \in X_T;\ \|f\|_{Y_T} \le \rho/2,\ \|\partial_{x_1}^3 f\|_{L^\infty_{x_1}L^2_TL^2_{x_2}} \le \rho/4,\ \|\partial_{x_2}^3 f\|_{L^\infty_{x_2}L^2_TL^2_{x_1}} \le \rho/4\}.$$

We assume that $u_j(t) \in X_{T,\rho}$ for all $0 \le j \le n-1$. By Lemma 3.5 and Lemma 3.8, we have

$$\begin{aligned}\|\partial_{x_1}^3 u_n\|_{L^\infty_{x_1}L^2_TL^2_{x_2}} \le{}& C_0\|D_{x_1}^{1/2}\partial_{x_1}^2\phi\| + CT\|u_{n-1}\|_{Y_T}^3\\ &+ |d_2|(2T\|u_{n-1}\|_{L^\infty_TL^2}\|\partial_t u_{n-1}\|_{L^\infty_TL^2} + \|\phi\|^2)\|\partial_{x_1}^3 u_{n-1}\|_{L^\infty_{x_2}L^2_TL^2_{x_1}}\\ &+ C_\varphi T\|u_{n-1}\|_{Y_T}^3\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}\|\partial_{x_2}^3 u_n\|_{L^\infty_{x_2}L^2_TL^2_{x_1}} \le{}& C_0\|D_{x_2}^{1/2}\partial_{x_2}^2\phi\| + CT\|u_{n-1}\|_{Y_T}^3\\ &+ |d_3|(2T\|u_{n-1}\|_{L^\infty_TL^2}\|\partial_t u_{n-1}\|_{L^\infty_TL^2} + \|\phi\|^2)\|\partial_{x_2}^3 u_{n-1}\|_{L^\infty_{x_2}L^2_TL^2_{x_1}}\\ &+ C_\varphi T\|u_{n-1}\|_{Y_T}^3.\end{aligned} \tag{4.3}$$

Here u_{n-1} satisfies the differential equality

$$\begin{cases} i\partial_t u_{n-1} = -\Delta u_{n-1} + F(u_{n-2}) + G(u_{n-2};\varphi),\\ u_{n-1}(0) = \phi,\end{cases}$$

where we define $u_{-1} = 0$. So, by virtue of the usual Sobolev inequalities, we have

$$\begin{aligned}\|\partial_t u_{n-1}\|_{L^\infty_TL^2} &\le \|\Delta u_{n-1}\|_{L^\infty_TL^2} + \|F(u_{n-2})\|_{L^\infty_TL^2} + \|G(u_{n-2})\|_{L^\infty_TL^2}\\ &\le \|\Delta u_{n-1}\|_{L^\infty_TL^2} + |d_1|\|u_{n-2}\|_{L^\infty_TL^6}^3\\ &\quad + |d_2|\|u_{n-2}\|_{L^\infty_TL^\infty_{x_1}L^2_{x_2}}\|\partial_{x_1}|u_{n-2}|^2\|_{L^\infty_TL^2_{x_1}L^1_{x_2}}\\ &\quad + |d_3|\|u_{n-2}\|_{L^\infty_TL^\infty_{x_2}L^2_{x_1}}\|\partial_{x_2}|u_{n-2}|^2\|_{L^\infty_TL^2_{x_2}L^1_{x_1}}\\ &\quad + |d_4|\|u_{n-2}\|_{L^\infty_TL^2}\|\varphi_1\|_{L^\infty_TL^\infty_{x_1}} + |d_5|\|u_{n-2}\|_{L^\infty_TL^2}\|\varphi_2\|_{L^\infty_TL^\infty_{x_2}}\\ &\le \|\Delta u_{n-1}\|_{L^\infty_TL^2} + C\|u_{n-2}\|_{L^\infty_TH^1}^3 + C_\varphi\|u_{n-2}\|_{L^\infty_TL^2}.\end{aligned}$$

Applying this estimate to (4.2) and (4.3), we have

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$$
\begin{aligned}
&\|\partial_{x_1}^3 u_n\|_{L_{x_1}^\infty L_T^2 L_{x_2}^2} \\
&\le C_0\|\phi\|_{H^{5/2}} + CT\|u_{n-1}\|_{Y_T}^3 + C_\varphi T\|u_{n-1}\|_{Y_T}^3 \\
&\quad + |d_2|(2T\|u_{n-1}\|_{L_T^\infty L^2}(\|\Delta u_{n-1}\|_{L_T^\infty L^2} + C\|u_{n-2}\|_{L_T^\infty H^1}^3) + \|\phi\|^2)\|\partial_{x_1}^3 u_{n-1}\|_{L_{x_1}^\infty L_T^2 L_{x_2}^2} \\
&\le C_0\|\phi\|_{H^{5/2}} + |d_2|\|\phi\|^2\|\partial_{x_1}^3 u_{n-1}\|_{L_{x_1}^\infty L_T^2 L_{x_2}^2} \\
&\quad + C_\varphi T\|u_{n-1}\|_{Y_T}(\|u_{n-1}\|_{Y_T}^2 + (\|u_{n-1}\|_{Y_T} + \|u_{n-2}\|_{Y_T}^3)\|\partial_{x_1}^3 u_{n-1}\|_{L_{x_1}^\infty L_T^2 L_{x_2}^2}) \\
&\le C_0\|\phi\|_{H^{5/2}} + \tfrac{1}{4}\rho|d_2|\|\phi\|^2 + C_\varphi T\tfrac{1}{2}\rho(\tfrac{1}{4}\rho^2 + \tfrac{1}{4}\rho(\tfrac{1}{2}\rho + \tfrac{1}{8}\rho^3)) \\
&= C_0\|\phi\|_{H^{5/2}} + \tfrac{1}{4}\rho|d_2|\|\phi\|^2 + \tfrac{1}{64}C_\varphi T\rho^3(12+\rho^3)
\end{aligned}
\tag{4.4}
$$

and

$$
\begin{aligned}
&\|\partial_{x_2}^3 u_n\|_{L_{x_2}^\infty L_T^2 L_{x_1}^2} \\
&\le C_0\|\phi\|_{H^{5/2}} + CT\|u_{n-1}\|_{Y_T}^3 + C_\varphi T\|u_{n-1}\|_{Y_T}^3 \\
&\quad + |d_3|(2T\|u_{n-1}\|_{L_T^\infty L^2}(\|\Delta u_{n-1}\|_{L_T^\infty L^2} + C\|u_{n-2}\|_{L_T^\infty H^1}^3) + \|\phi\|^2)\|\partial_{x_2}^3 u_{n-1}\|_{L_{x_2}^\infty L_T^2 L_{x_1}^2} \\
&\le C_0\|\phi\|_{H^{5/2}} + |d_3|\|\phi\|^2\|\partial_{x_2}^3 u_{n-1}\|_{L_{x_2}^\infty L_T^2 L_{x_1}^2} \\
&\quad + C_\varphi T\|u_{n-1}\|_{Y_T}(\|u_{n-1}\|_{Y_T}^2 + (\|u_{n-1}\|_{Y_T} + \|u_{n-2}\|_{Y_T}^3)\|\partial_{x_2}^3 u_{n-1}\|_{L_{x_2}^\infty L_T^2 L_{x_1}^2}) \\
&\le C_0\|\phi\|_{H^{5/2}} + \tfrac{1}{4}\rho|d_3|\|\phi\|^2 + C_\varphi T\tfrac{1}{2}\rho(\tfrac{1}{4}\rho^2 + \tfrac{1}{4}\rho(\tfrac{1}{2}\rho + \tfrac{1}{8}\rho^3)) \\
&= C_0\|\phi\|_{H^{5/2}} + \tfrac{1}{4}\rho|d_3|\|\phi\|^2 + \tfrac{1}{64}C_\varphi T\rho^3(12+\rho^3).
\end{aligned}
\tag{4.5}
$$

Now, by the assumptions on ϕ, we can define a small positive constant δ such that $\max(|d_2|, |d_3|)\|\phi\|^2 \le 1 - 8\delta$. For this δ, we put ρ such that $C_0\|\phi\|_{H^{5/2}} \le \delta\rho$ and T such that $\frac{1}{64}C_\varphi T\rho^2(12+\rho^3) \le \delta$. Under these conditions, we see that

$$\|\partial_{x_1}^3 u_n\|_{L_{x_1}^\infty L_T^2 L_{x_2}^2} \le \rho/4, \quad \text{and} \quad \|\partial_{x_2}^3 u_n\|_{L_{x_2}^\infty L_T^2 L_{x_1}^2} \le \rho/4. \tag{4.6}$$

Next, to estimate $D_x^{1/2}\partial^2 u$, we note that (4.1) is equivalent to

$$i\partial_t u_0 + \Delta u_0 = 0, \quad u_0(0) = \phi, \tag{4.7}$$

and

$$i\partial_t u_n + \Delta u_n = F(u_{n-1}) + G(u_{n-1}), \quad u_n(0) = \phi. \tag{4.8}$$

Applying $D_{x_1}^{1/2}\partial_{x_1}^2$, to both sides of (4.7) and (4.8), multiplying both sides of the resulting equations by $D_{x_1}^{1/2}\partial_{x_1}^2\bar{u}_0(t)$ and $D_{x_1}^{1/2}\partial_{x_2}^2\bar{u}_n(t)$, respectively, integrating over $\mathbf{R}^2$, and taking the imaginary part, we obtain

$$\frac{d}{dt}\|D_{x_1}^{1/2}\partial_{x_1}^2 u_0(t)\|^2 = 0, \tag{4.9}$$

and

$$\frac{d}{dt}\|D_{x_1}^{1/2}\partial_{x_1}^2 u_n(t)\|^2 = 2\,\mathrm{Im}(D_{x_1}^{1/2}\partial_{x_1}^2(F(u_{n-1}(t)) + G(u_{n-1}(t))), D_{x_1}^{1/2}\partial_{x_1}^2 u_n(t)). \tag{4.10}$$

Integrating (4.9) and (4.10) in and using Lemma 3.6, we find that

$$\|D_{x_1}^{1/2}\partial_{x_1}^2 u_0\|_{L_T^\infty L^2}^2 = \|D_{x_1}^{1/2}\partial_{x_1}^2\phi\|^2, \tag{4.11}$$

and

$$\begin{aligned}\|D_{x_1}^{1/2}\partial_{x_1}^2 u_n\|_{L_T^\infty L^2}^2 &\le \|D_{x_1}^{1/2}\partial_{x_1}^2\phi\|^2 + 2CT\|u_{n-1}\|_{Y_T}^3\|u_n\|_{Y_T}\\ &+ (8T\|u_{n-1}\|_{L_T^\infty L^2}\|\partial_t u_{n-1}\|_{L_T^\infty L^2} + 4|d_2|\|\phi\|^2\|\partial_{x_1}^3 u_{n-1}\|_{L_{x_1}^\infty L_T^2 L_{x_2}^2})\|\partial_{x_1}^3 u_n\|_{L_{x_1}^\infty L_T^2 L_{x_2}^2}\\ &+ C_\varphi T\|u_{n-1}\|_{Y_T}\|u_n\|_{Y_T}.\end{aligned} \tag{4.12}$$

In the same way as in the proofs of (4.11) and (4.12) we have

$$\|D_{x_2}^{1/2}\partial_{x_2}^2 u_0\|_{L_T^\infty L^2}^2 = \|D_{x_2}^{1/2}\partial_{x_2}^2\phi\|^2, \tag{4.13}$$

$$\begin{aligned}\|D_{x_2}^{1/2}\partial_{x_2}^2 u_n\|_{L_T^\infty L^2}^2 &\le \|D_{x_2}^{1/2}\partial_{x_2}^2\phi\|^2 + 2CT\|u_{n-1}\|_{Y_T}^3\|u_n\|_{Y_T}\\ &+ (8T\|u_{n-1}\|_{L_T^\infty L^2}\|\partial_t u_{n-1}\|_{L_T^\infty L^2} + 4|d_3|\|\phi\|^2\|\partial_{x_2}^3 u_{n-1}\|_{L_{x_2}^\infty L_T^2 L_{x_1}^2})\|\partial_{x_2}^3 u_n\|_{L_{x_2}^\infty L_T^2 L_{x_1}^2}\\ &+ C_\varphi T\|u_{n-1}\|_{Y_T}\|u_n\|_{Y_T},\end{aligned} \tag{4.14}$$

$$\|D_{x_1}^{1/2}\partial_{x_1}\partial_{x_2}u_0\|_{L_T^\infty L^2}^2 + \|D_{x_2}^{1/2}\partial_{x_1}\partial_{x_2}u_0\|_{L_T^\infty L^2}^2 = \|D_{x_1}^{1/2}\partial_{x_1}\partial_{x_2}\phi\|^2 + D_{x_2}^{1/2}\partial_{x_1}\partial_{x_2}\phi\|^2, \tag{4.15}$$

and

$$\begin{aligned}&\|D_{x_1}^{1/2}\partial_{x_1}\partial_{x_2}u_n\|_{L_T^\infty L^2}^2 + \|D_{x_2}^{1/2}\partial_{x_1}\partial_{x_2}u_n\|_{L_T^\infty L^2}^2\\ &\le \|D_{x_1}^{1/2}\partial_{x_1}\partial_{x_2}\phi\|^2 + \|D_{x_2}^{1/2}\partial_{x_1}\partial_{x_2}\phi\|^2 + CT\|u_{n-1}\|_{Y_T}^3\|u_n\|_{Y_T} + C_\varphi T\|u_{n-1}\|_{Y_T}\|u_n\|_{Y_T}.\end{aligned} \tag{4.16}$$

Integration by parts shows that

$$\begin{aligned}\|D_{x_1}^{1/2}\partial_{x_2}^2 u_n\|^2 &\le \|D_{x_2}^{1/2}\partial_{x_2}^2 u_n\|\|D_{x_2}^{1/2}\partial_{x_1}\partial_{x_2}u_n\|\\ &\le \varepsilon\|D_{x_2}^{1/2}\partial_{x_2}^2 u_n\|^2 + \tfrac{1}{4\varepsilon}\|D_{x_2}^{1/2}\partial_{x_1}\partial_{x_2}u_n\|^2,\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}\|D_{x_2}^{1/2}\partial_{x_1}^2 u_n\|^2 &\le \|D_{x_1}^{1/2}\partial_{x_1}^2 u_n\|\|D_{x_1}^{1/2}\partial_{x_1}\partial_{x_2}u_n\|\\ &\le \varepsilon\|D_{x_1}^{1/2}\partial_{x_1}^2 u_n\|^2 + \tfrac{1}{4\varepsilon}\|D_{x_1}^{1/2}\partial_{x_1}\partial_{x_2}u_n\|^2,\end{aligned} \tag{4.18}$$

where $\varepsilon > 0$ is determined later. By the usual energy method and Lemma 3.7 we have

$$\sum_{|\alpha|\le 2} \|\partial^\alpha u_n\|^2_{L^\infty_T L^2} \le \sum_{|\alpha|\le 2} \|\partial^\alpha \phi\|^2 + CT(\rho+\rho^3)\|u_n\|_{Y_T}. \tag{4.19}$$

From (4.11)–(4.19) and the Schwartz inequality it follows that

$$\|u_n\|^2_{Y_T} \le C\|\phi\|^2_{H^{5/2}} + \tfrac{1}{16}C_\varphi T\rho^2(4+\rho^2) + \tfrac{1}{32}(8+\varepsilon)T\rho^3(4+\rho^2) + \tfrac{1}{32}(4+\varepsilon)(|d_2|+|d_3|)\|\phi\|^2\rho^2.$$

Hence, if necessary, we retake ρ and T such that

$$\begin{cases} \tfrac{1}{16}(8+\varepsilon)T\rho(4+\rho) \le \delta, \\ C\|\phi\|^2_{H^{5/2}} \le \tfrac{1}{2}\delta\rho^2, \\ \tfrac{1}{8}C_\varphi T(4+\rho^2) \le \delta, \end{cases}$$

we find that

$$\|u_n\|_{Y_T} \le \frac{\rho}{2}. \tag{4.20}$$

Moreover, for $u_0(t) = U(t)\phi$ we have the following estimate by Lemma 2.1

$$\|u_0\|_{X_T} \le C_0\|\phi\|_{H^{5/2}} \le \delta\rho < \rho/4. \tag{4.21}$$

The induction argument and (4.6), (4.20), (4.21) show that $\{u_n\}$ is a well-defined sequence in $X_{T,\rho}$.

Using Lemma 3.9 instead of Lemmas 3.5–3.8, a similar calculation shows $\{u_n\}$ is a Cauchy sequence which implies Theorem 1.1. □

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Department of Applied Mathematics
Science University of Tokyo
1–3, Kagurazaka
Shinjuku-ku
Tokyo 162
Japan
E-mail address: nhayashi@rs.kagu.sut.ac.jp

Department of Mathematics
Sophia University
7–1, Kioicho
Chiyoda-ku
Tokyo 102
Japan
E-mail address: h-hirata@mm.sophia.ac.jp