European Journal of Applied Mathematics (2025), **36**, pp. 27–39
doi:10.1017/S0956792524000184

CAMBRIDGE UNIVERSITY PRESS

PAPER

Uniform propagation of chaos for a dollar exchange econophysics model

Fei Cao[1] and Roberto Cortez[2]

[1]Department of Mathematics and Statistics, University of Massachusetts Amherst, Amherst, MA, 01003, USA
[2]Departamento de Matemáticas, Universidad Andrés Bello, Santiago, Chile
Corresponding author: Fei Cao; Email: fcao@umass.edu

Received: 17 July 2023; **Revised:** 09 March 2024; **Accepted:** 24 March 2024; **First published online** 22 April 2024

Keywords: Econophysics; agent-based model; uniform propagation of chaos; coupling; Wasserstein distance

2020 Mathematics Subject Classification: 82C22 (Primary); 82C31, 91B80, 60J28 (Secondary)

Abstract
We study the poor-biased model for money exchange introduced in Cao & Motsch ((2023) *Kinet. Relat. Models* 16(5), 764–794.): agents are being randomly picked at a rate proportional to their current wealth, and then the selected agent gives a dollar to another agent picked uniformly at random. Simulations of a stochastic system of finitely many agents as well as a rigorous analysis carried out in Cao & Motsch ((2023) *Kinet. Relat. Models* 16(5), 764–794.), Lanchier ((2017) *J. Stat. Phys.* 167(1), 160–172.) suggest that, when both the number of agents and time become large enough, the distribution of money among the agents converges to a Poisson distribution. In this manuscript, we establish a uniform-in-time propagation of chaos result as the number of agents goes to infinity, which justifies the validity of the mean-field deterministic infinite system of ordinary differential equations as an approximation of the underlying stochastic agent-based dynamics.

1. Introduction

In this manuscript, we study a simple mechanism for money exchange in a closed economical system, meaning that there are a fixed number of agents, denoted by N, with an (fixed) average number of dollars $\mu \in \mathbb{N}_+$. We denote by $S_i(t) \in \mathbb{N}$ the amount of dollars the agent i has at time t. Since it is a closed economical system, we have

$$S_1(t) + \cdots + S_N(t) = N\mu = \text{Constant} \qquad \text{for all } t \geq 0. \tag{1.1}$$

Specifically, we consider the so-called poor-biased dollar exchange model investigated in [6]: at random times (generated by an exponential law), an agent i is picked at a rate which is proportional to its current wealth, and he or she will give one dollar to another agent j picked uniformly at random. In particular, if agent i does not have at least one dollar, then he/she will never be picked to give. Mathematically, the update rule of this N-agents system can be represented by

$$\textbf{poor-biased exchange:} \qquad (S_i, S_j) \overset{\lambda\, S_i/N}{\rightsquigarrow} (S_i - 1, S_j + 1). \tag{1.2}$$

We emphasise that in order to ensure that the rate of a typical agent giving a dollar per unit time is of order 1 (so that the correct mean-field analysis as $N \to +\infty$ can be carried out), the rate appearing in (1.2) is set to be $\lambda\, S_i/N$ instead of $\lambda\, S_i$.

We illustrate the dynamics in Figure 1-left. The main task is to identify the limiting distribution of money when both the number of agents N and time t become large enough. We illustrate numerically in Figure 1-right the simulation result using $N = 1000$ agents. Notice that the wealth distribution is well approximated by a Poisson distribution with mean value $\mu = 10$.

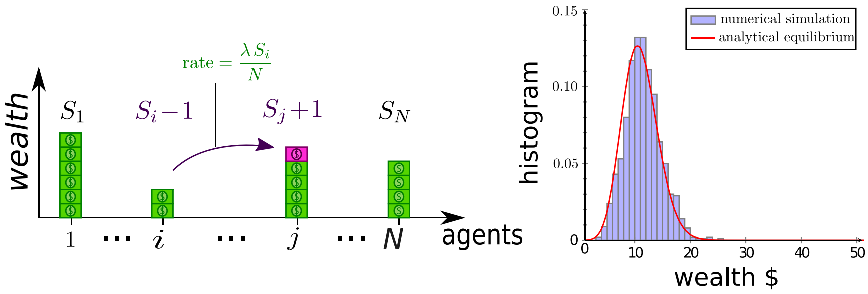

Figure 1. ***Left:*** *illustration of the poor-biased dollar exchange model: at random time, one dollar is passed from a 'giver'* i *to a 'receiver'* j *at a rate proportional to the amount of dollars the 'giver'* i *has.* ***right:*** *the distribution of wealth for the poor-biased dynamics after* 2000 *unit of time with the average amount of dollar per agent* $\mu = 10$*, this distribution is well-approximated by a Poisson distribution with mean value* $\mu = 10$*.*

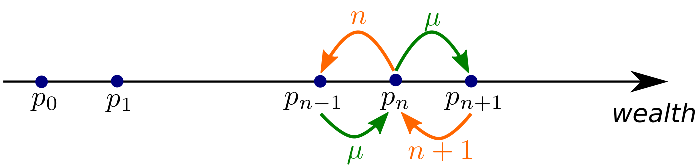

Figure 2. *Schematic illustration of the limiting ODE system (1.3).*

If we denote by $\mathbf{p}(t) = (p_0(t), p_1(t), \ldots)$ the law of the process $S_1(t)$ as $N \to \infty$, i.e. $p_n(t) = \lim_{N\to\infty} \mathbb{P}\left(S_1(t) = n\right)$, it has been shown recently in [6] that the time evolution of $\mathbf{p}(t)$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}(t) = \lambda\, \mathscr{L}[\mathbf{p}(t)] \tag{1.3}$$

with:

$$\mathscr{L}[\mathbf{p}]_n := \begin{cases} p_1 - \mu\, p_0 & \text{for } n = 0, \\ (n+1)\, p_{n+1} + \mu\, p_{n-1} - (n+\mu)\;p_n & \text{for } n \geq 1. \end{cases} \tag{1.4}$$

A harmless normalisation allows us to put $\lambda = 1$ without any loss of generality, which will be implicitly assumed throughout the manuscript. The linear ODE system (1.3) is of Fokker–Planck type and hence admits an interpretation in terms of a 'gain' and 'loss' process, shown in Figure 2.

The transition from the stochastic N-agent dynamics (1.2) to the infinite system of ODEs (1.3) as $N \to \infty$ is accomplished thanks to the notion of *propagation of chaos* [32] and has been rigorously justified in [6]. Unfortunately, only a finite time propagation of chaos result is obtained in [6], meaning that the evolution equation (1.3) is only guaranteed to be a good approximation of the N-agents system over the time span $t \in [0, T]$ with $T > 0$ being arbitrary but fixed. In this work, we aim to establish a uniform-in-time propagation of chaos for this particular dynamics, which refines the previous short time result.

We first summarise the main result of [6] regarding the large time behaviour of the linear ODE system in the following proposition and refer interested readers to [6] for detailed proofs and discussion.

Lemma 1. *Let* $\mathbf{p}(t) = \{p_n(t)\}_{n\geq 0}$ *be the unique solution of (1.3) with* $p(0) \in V_\mu$*, where*

$$V_\mu := \left\{ \mathbf{p} \,\middle|\, p_n \geq 0,\ \sum_{n=0}^{\infty} p_n = 1,\ \sum_{n=0}^{\infty} n\, p_n = \mu \right\}$$

https://doi.org/10.1017/S0956792524000184 Published online by Cambridge University Press

is the space of probability mass functions on $\mathbb{N}$ *with the pre-fixed mean value* μ. *Then*

$$\sum_{n=0}^{\infty} \mathscr{L}[\mathbf{p}]_n = 0, \text{ and } \sum_{n=0}^{\infty} n\, \mathscr{L}[\mathbf{p}]_n = 0. \tag{1.5}$$

In particular, we have $\mathbf{p}(t) \in V_\mu$ *for all* $t \geq 0$. *Moreover, the unique equilibrium distribution* $\mathbf{p}^* = \{p_n^*\}_n$ *in* V_μ *associated with (1.3) is given by the following Poisson distribution:*

$$p_n^* = \frac{\mu^n\, \mathrm{e}^{-\mu}}{n!}, \quad n \geq 0. \tag{1.6}$$

Moreover, if we introduce the following energy functional for each $\mathbf{p} \in V_\mu$*:*

$$\mathrm{E}[\mathbf{p}] = \sum_{n=0}^{\infty} \frac{p_n^2}{p_n^*} \tag{1.7}$$

and

$$\mathrm{D}[\mathbf{p}] = \sum_{n=0}^{\infty} p_n^* \left(\frac{p_{n+1}}{p_{n+1}^*} - \frac{p_n}{p_n^*}\right)^2, \tag{1.8}$$

then

$$\frac{\mathrm{d}\mathrm{E}[\mathbf{p}(t) - \mathbf{p}^*]}{\mathrm{d}t} = -2\,\mu\, \mathrm{D}[\mathbf{p}(t)] \tag{1.9a}$$

$$\frac{\mathrm{d}^2\mathrm{E}[\mathbf{p}(t) - \mathbf{p}^*]}{\mathrm{d}t^2} \geq -2\, \frac{\mathrm{d}\mathrm{E}[\mathbf{p}(t) - \mathbf{p}^*]}{\mathrm{d}t}. \tag{1.9b}$$

Consequently, $\mathbf{p}(t)$ *decays exponentially fast towards* $\mathbf{p}^*$ *in the sense that*

$$\mathrm{E}[\mathbf{p}(t) - \mathbf{p}^*] \leq \mathrm{E}[\mathbf{p}(0) - \mathbf{p}^*]\, \mathrm{e}^{-t}. \tag{1.10}$$

Remark. The exponential decay result (1.10) is obtained via the celebrated Bakry–Emery approach [1] and the rate appearing on the right-hand side of (1.10) seems to be half of the sharp rate based on numerical simulations carried out in [6]. In other words, numerical experiments suggest that we can strengthen (1.10) to

$$\mathrm{E}[\mathbf{p}(t) - \mathbf{p}^*] \leq \mathrm{E}[\mathbf{p}(0) - \mathbf{p}^*]\, \mathrm{e}^{-2t}. \tag{1.11}$$

Lastly, we emphasise that the convergence to the Poisson distribution can be also studied from a different point of view. For instance, Lanchier [21] investigated a discrete-time analog of the model by sending $t \to \infty$ first prior to sending $N \to \infty$. To give a brief account of this approach, we denote by $\mathbf{S}(t) = (S_1(t), \ldots, S_N(t))$ and recall that the vector $\mathbf{S}(t)$ is a Markov pure-jump process on the following configuration space

$$\mathscr{A}_{N,\mu} := \left\{\mathbf{S} \in \mathbb{N}^N \mid \sum_{n=1}^{N} S_i = N\mu\right\}. \tag{1.12}$$

The key insight behind this approach lies in the fact for any fixed $N \in \mathbb{N}_+$, the dynamics starting from any initial configuration converges (as $t \to \infty$) to the multinomial distribution on $\mathscr{A}_{N,\mu}$ corresponding to taking $N\mu$ independent samples uniformly at random from the set $\{1, \ldots, N\}$, with replacement. More specifically, this distribution, which we denote $\mathscr{M}_N$, is given by

$$\mathscr{M}_N\,(\mathbf{S}) := \binom{N\mu}{S_1, S_2, \ldots, S_N} \prod_{i\in[N]} \frac{1}{N^{S_i}}. \tag{1.13}$$

This roughly means that each dollar will be equally likely to be in any agents' pocket when time becomes sufficiently large. Then the large population limit $N \to \infty$ can be performed with the help of some basic algebra and combinatorial counting techniques.

https://doi.org/10.1017/S0956792524000184 Published online by Cambridge University Press

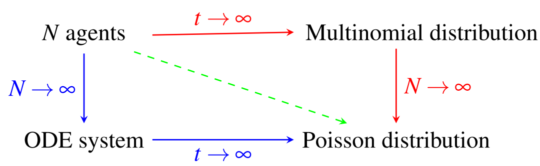

Figure 3. *Roadmap for proving convergence results. The approach of taking the large time limit $t \to \infty$ before taking the large population limit $N \to \infty$ is adapted in Lanchier's recent work [21]. An alternative approach is to send $N \to \infty$ first before investigating the large time asymptotic.*

We encapsulate the various approaches introduced so far in Figure 3.

Although we will only investigate a specific binary exchange models in the present work, the literature on other types of econophysics models based on different exchange rules is vast. To name a few, the so-called immediate exchange model studied in [20] assumes that pairs of agents are randomly and uniformly picked at each random time, and each of the agents transfer a random fraction of its money to the other agents, where these fractions are independent and uniformly distributed in [0, 1]. The so-called uniform reshuffling model investigated in [4] and [15] requires that the total amount of money of two randomly and uniformly picked agents possess before interaction is uniformly redistributed among the two agents after interaction. The so-called unbiased exchange model and the rich-biased exchange proposed in [6] are closely related variants of the poor-biased exchange model investigated in this work, where the variations of these models differ in the rate of an agent (say agent i) being picked to give out a dollar. Indeed, for the unbiased exchange dynamics and the rich-biased exchange dynamics, one modify the corresponding update rules (1.2) to (recall that if agent i has no dollars to give, then nothing will happen)

$$\textbf{unbiased exchange:} \qquad (S_i, S_j) \overset{\lambda/N}{\rightsquigarrow} (S_i - 1, S_j + 1) \tag{1.14}$$

and

$$\textbf{rich-biased exchange:} \qquad (S_i, S_j) \overset{\lambda/(N\,S_i)}{\rightsquigarrow} (S_i - 1, S_j + 1), \tag{1.15}$$

respectively. For other models arising from econophysics, we refer to [2, 5, 9, 10, 17, 22, 26], and the references therein.

To the best of our knowledge, uniform-in-time propagation of chaos for other models coming from econophysics has only been studied in [11–13], which include the uniform reshuffling model and the immediate exchange model as special cases. The approach taken in [11–13] is an 'optimal-coupling' type argument, which can be dated back to 1970s [28, 33] and which relies on a stochastic differential equation (SDE) representation of the agent-based stochastic dynamics in terms of Poisson random measures. Unfortunately, the 'optimal-coupling' type framework developed in [11–13] seems inapplicable to the unbiased/poor-biased/rich-biased dynamics mentioned above, as the rate of giving a dollar depends on the wealth of the agent. On the other hand, a very recent work [3] has established a uniform propagation of chaos result for the unbiased exchange model, based on a careful study of the entropy–entropy dissipation relation, at the level of the N-agent system as well as its associated mean-field system of non-linear ODEs. Our approach to the uniform propagation of chaos for the poor-biased exchange model at hand will be built upon probabilistic coupling methods, the explicit knowledge of the invariant measure for the N-agents system, as well as the non-uniform propagation of chaos shown in [6].

To conclude the introduction, we also emphasise that all the aforementioned models fall into the realm of kinetic theory and we refer the interested readers to [27, 34]. As a historical note, many of the earliest econophysics models have been first proposed by the sociologist John Angle (see for instance the review [24] by Thomas Lux). The seminal work [15] is viewed as a standard reference in research on econophysics models, and some of the minor issues in [15] have been identified and addressed in a series of works by Enrico Scalas and his colleagues [18, 31].

https://doi.org/10.1017/S0956792524000184 Published online by Cambridge University Press

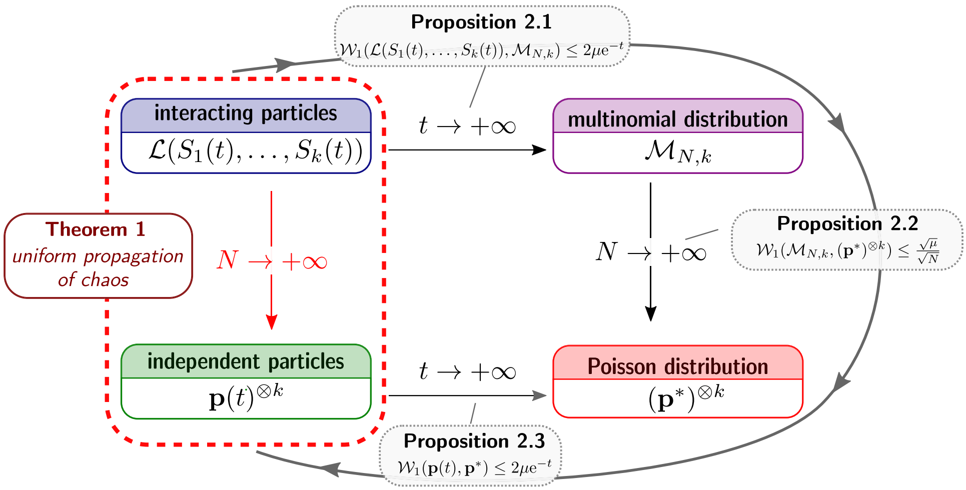

Figure 4. *Schematic illustration of the strategy behind the proof of the uniform-in-time propagation of chaos for the poor-biased dollar exchange model.*

2. Uniform propagation of chaos

Throughout this section, we will employ the notation $\mathscr{L}(\mathbf{X})$ to represent the law of a generic random variable or vector $\mathbf{X}$. We set $[N] := \{1, 2 \ldots, N\}$ for notational simplicity, and we will also quantify convergence of probability measures via the Wasserstein distance (of order 1): for probability measures μ, ν on $\mathbb{N}^d$ with finite first moment, it is defined by

$$\mathscr{W}_1(\mu,\nu) := \inf_{\mathbf{X},\mathbf{Y}} \mathbb{E}\left[\frac{1}{d}\sum_{j=1}^d |X_j - Y_j|\right], \tag{2.1}$$

in which the infimum is taken over all possible couplings of μ and ν, or equivalently over all pair of random vectors $\mathbf{X} = (X_1, \ldots, X_d)$ and $\mathbf{Y} = (Y_1, \ldots, Y_d)$ such that $\mathbf{X}$ and $\mathbf{Y}$ are distributed according to μ and ν, respectively. Before we dive into the details of the proof of Theorem 1 on the uniform propagation of chaos, we summarise the spirit behind the proof in Figure 4 below.

To control $\mathscr{W}_1\left(\mathscr{L}(S_1(t), S_2(t), \ldots, S_k(t)), \mathbf{p}(t)^{\otimes k}\right)$ (where $1 \le k \le N$ is fixed and does not grow as $N \to \infty$), we apply the triangle inequality twice as follows:

$$\begin{aligned}
&\mathscr{W}_1\left(\mathscr{L}(S_1(t), S_2(t), \ldots, S_k(t)), \mathbf{p}(t)^{\otimes k}\right) \\
&\le \mathscr{W}_1\left(\mathscr{L}(S_1(t), S_2(t), \ldots, S_k(t)), \mathscr{M}_{N,k}\right) + \mathscr{W}_1\left(\mathscr{M}_{N,k}, (\mathbf{p}^*)^{\otimes k}\right) \\
&+ \mathscr{W}_1\left(\mathbf{p}(t)^{\otimes k}, (\mathbf{p}^*)^{\otimes k}\right),
\end{aligned} \tag{2.2}$$

where $\mathscr{M}_{N,k}$ is the k-particle marginal of $\mathscr{M}_N$ and $\mathbf{p}^*$ is the Poisson distribution with mean μ. Section 2.1 is devoted to the bound on $\mathscr{W}_1\left(\mathscr{L}(S_1(t), S_2(t), \ldots, S_k(t)), \mathscr{M}_{N,k}\right)$, Sections 2.2 and 2.3 treat the bound on $\mathscr{W}_1\left(\mathscr{M}_{N,k}, (\mathbf{p}^*)^{\otimes k}\right)$ and $\mathscr{W}_1\left(\mathbf{p}(t)^{\otimes k}, (\mathbf{p}^*)^{\otimes k}\right)$, respectively. Finally, the statement and the proof of the uniform propagation of chaos are presented in Section 2.4.

Before we dive into the rigorous mathematical treatment of the poor-biased exchange model in the upcoming sections, we would like to mention a few reasons as to why we limit our scope to this particular model. Although the strategy outlined via Figure 4 seems quite general, quite applicable, and there are many dollar/wealth exchange models that seem to fit the approach, the implementation of the paradigm depends on specific modelling details as well as the nature of the mean-field limit equation. For instance, the mean-field system of ODEs corresponding to the unbiased exchange model (1.14) and the rich-biased exchange model (1.15) are both (highly) nonlinear, and for both models it is a very delicate task

https://doi.org/10.1017/S0956792524000184 Published online by Cambridge University Press

to establish an analog of Proposition 2.1 and/or Proposition 2.2 below (in fact, it is conjectured in [6] that the rich-biased dynamics does not propagate chaos uniformly in time). The poor-biased exchange dynamics also enjoys a particular nice feature mentioned at the beginning of section 2.1 which is absent in its sibling models (1.14) or (1.15).

2.1. Equilibration of the N-agents dynamics

One nice feature of the poor-biased exchange model is that, from the point of view of each individual dollar, the dynamics is very simple: each dollar jumps randomly and independently from pocket to pocket. More specifically, we denote $M := N\mu$ and introduce

$$\mathbf{a}(t) := (a_1(t), \ldots, a_M(t)) \in \{1, \ldots, N\}^M, \tag{2.3}$$

in which $a_k(t)$ represents the (label of the) agent assigned to dollar k at time t. The dollar-wise poor-biased dynamics is specified as follows: at each random time generated by an exponential clock with rate $\frac{NM}{N-1}$, we pick a dollar $k \in \{1, \ldots, M\}$ and an agent $i \in \{1, \ldots, N\}$ independently and uniformly at random, then we update the label (or value) of a_k to i. Notice that when $i = a_k$, nothing happens; this is why we set the rate to be $\frac{NM}{N-1}$, so that the *effective* jump rate of each dollar is 1. Consequently, when going back to the agent-wise dynamics (see (2.6) below), one recovers exactly the same rates specified in (1.2).

A natural way to couple two such processes $\mathbf{a}(t) \in \{1, \ldots, N\}^M$ and $\mathbf{b}(t) \in \{1, \ldots, N\}^M$ (with possibly different initial conditions) is to employ the same jump times, and pick the same dollar $k \in \{1, \ldots, M\}$ and agent $i \in \{1, \ldots, N\}$ in each update. To compare these processes, we will use the distance on $\{1, \ldots, N\}^M$ given by

$$\rho(\mathbf{a}, \mathbf{b}) := \sum_{k=1}^{M} \mathbb{1}\{a_k \neq b_k\} \tag{2.4}$$

for all $\mathbf{a}, \mathbf{b} \in \{1, \ldots, N\}^M$. Now, if we let τ_k be the first time dollar k is picked (notice that once a dollar is chosen, its position in both systems $\mathbf{a}$ and $\mathbf{b}$ will be the same, indefinitely), then $\{\tau_k\}_{k=1}^M$ will be i.i.d with $\tau_k \sim \text{Exponential}\left(\frac{N}{N-1}\right)$, whence

$$\begin{aligned} \mathbb{E}\,\rho\,(\mathbf{a}(t), \mathbf{b}(t)) &= \sum_{k=1}^{M} \mathbb{P}\,(a_k(t) \neq b_k(t)) \\ &\leq \sum_{k=1}^{M} \mathbb{P}\,(\tau_k > t) \\ &= M\,\mathrm{e}^{-t\frac{N}{N-1}} \\ &\leq N\,\mu\,\mathrm{e}^{-t}. \end{aligned} \tag{2.5}$$

This upper bound grows linearly with N, which in principle is unsatisfactory. However, we will now re-write this coupling in the setting of the agent-wise dynamics, and we will use an appropriate distance to compare these processes, then the estimate (2.5) will become useful.

Specifically, given $\mathbf{a}, \mathbf{b} \in \{1, \ldots, N\}^M$, we denote $\mathbf{Q}^{\mathbf{a}} = (Q_1^{\mathbf{a}}, \ldots, Q_N^{\mathbf{a}}) \in \mathscr{A}_{N,\mu}$ the vector of the number of dollars that each agent has according to $\mathbf{a}$, that is

$$Q_i^{\mathbf{a}} := \sum_{k=1}^{M} \mathbb{1}\{a_k = i\}, \tag{2.6}$$

for all $1 \leq i \leq N$, and similarly for $\mathbf{Q}^{\mathbf{b}} = (Q_1^{\mathbf{b}}, \ldots, Q_N^{\mathbf{b}})$. We introduce the distance on $\mathscr{A}_{N,\mu}$ via

$$\mathrm{d}\,(\mathbf{S}, \mathbf{R}) := \frac{1}{N} \sum_{i=1}^{N} |S_i - R_i| \tag{2.7}$$

https://doi.org/10.1017/S0956792524000184 Published online by Cambridge University Press

for all $\mathbf{S}, \mathbf{R} \in \mathscr{A}_{N,\mu}$. Then:

$$\begin{aligned}
\mathrm{d}\left(\mathbf{Q}^{\mathbf{a}}, \mathbf{Q}^{\mathbf{b}}\right) &= \frac{1}{N} \sum_{i=1}^{N} \left| \sum_{k=1}^{M} \mathbb{1}\{a_k = i\} - \sum_{k=1}^{M} \mathbb{1}\{b_k = i\} \right| \\
&\leq \frac{1}{N} \sum_{i=1}^{N} \sum_{k=1}^{M} |\mathbb{1}\{a_k = i\} - \mathbb{1}\{b_k = i\}| \\
&= \frac{1}{N} \sum_{i=1}^{N} \sum_{k=1}^{M} (\mathbb{1}\{a_k = i, b_k \neq a_k\} + \mathbb{1}\{b_k = i, a_k \neq b_k\}) \\
&= \frac{1}{N} \sum_{k=1}^{M} \mathbb{1}\{a_k \neq b_k\} \left(\sum_{i=1}^{N} \mathbb{1}\{a_k = i\} + \sum_{i=1}^{N} \mathbb{1}\{b_k = i\} \right) \\
&= \frac{2}{N} \rho(\mathbf{a}, \mathbf{b}).
\end{aligned} \tag{2.8}$$

Now, define the processes $\mathbf{S}(t)$ and $\mathbf{R}(t)$ by

$$\mathbf{S}(t) = \mathbf{Q}^{\mathbf{a}(t)} \qquad \text{and} \qquad \mathbf{R}(t) = \mathbf{Q}^{\mathbf{b}(t)},$$

where $(\mathbf{a}(t), \mathbf{b}(t))$ is the coupling defined above. It is straightforward to verify that $\mathbf{S}(t)$ and $\mathbf{R}(t)$ are indeed realisations of the poor-biased exchange model (1.2). Consequently, using the estimates (2.5) and (2.8), we arrive at

$$\begin{aligned}
\mathscr{W}_1\left(\mathscr{L}(\mathbf{S}(t)), \mathscr{L}(\mathbf{R}(t))\right) &\leq \mathbb{E}\, \mathrm{d}\left(\mathbf{S}(t), \mathbf{R}(t)\right) \\
&\leq \frac{2}{N}\, \mathbb{E}\, \rho\left(\mathbf{a}(t), \mathbf{b}(t)\right) \\
&\leq 2\,\mu\, \mathrm{e}^{-t}.
\end{aligned} \tag{2.9}$$

We summarise the previous discussions into the following proposition.

Proposition 2.1. *Let $\mathbf{S}(t)$ and $\mathbf{R}(t)$ be two realisations of the poor-biased exchange model given by (1.2), starting from any given pair of initial configurations $\mathbf{S}(0), \mathbf{R}(0) \in \mathscr{A}_{N,\mu}$. Then, we have*

$$\mathscr{W}_1\left(\mathscr{L}(\mathbf{S}(t)), \mathscr{L}(\mathbf{R}(t))\right) \leq 2\,\mu\, \mathrm{e}^{-t}.$$

In particular, since the multinomial distribution $\mathscr{M}_N$ is the unique stationary distribution for the poor-biased exchange dynamics as $t \to \infty$ while N is kept frozen (see for instance [21]), we also deduce that

$$\mathscr{W}_1\left(\mathscr{L}(\mathbf{S}(t)), \mathscr{M}_N\right) \leq 2\,\mu\, \mathrm{e}^{-t}. \tag{2.10}$$

Remark. As any coupling of the full vector gives rise to a coupling of the marginals, we immediately deduce from Proposition 2.1 that

$$\mathscr{W}_1\left(\mathscr{L}(S_1(t)), \mathrm{Binomial}\left(\mu N, \frac{1}{N}\right)\right) \leq 2\,\mu\, \mathrm{e}^{-t}. \tag{2.11}$$

Denoting by $\mathscr{M}_{N,k}$ the k-particle marginal distribution of $\mathscr{M}_N$ for each fixed $k \geq 1$, one also has

$$\mathscr{W}_1\left(\mathscr{L}(S_1(t), S_2(t), \ldots, S_k(t)), \mathscr{M}_{N,k}\right) \leq 2\,\mu\, \mathrm{e}^{-t}. \tag{2.12}$$

2.2. Chaos of the multinomial to Poisson

In this subsection, we will establish a quantitative bound on the Wasserstein distance between the multinomial distribution $\mathscr{M}_N$ and the tensorised Poisson distribution $\mathrm{Poisson}(\mu)^{\otimes N}$, which might be of independent interest. We recall that $\mathscr{M}_N$, rigorously defined by (1.13), is the distribution corresponding to the experiment of tossing $N\mu$ balls independently in N equally likely urns.

https://doi.org/10.1017/S0956792524000184 Published online by Cambridge University Press

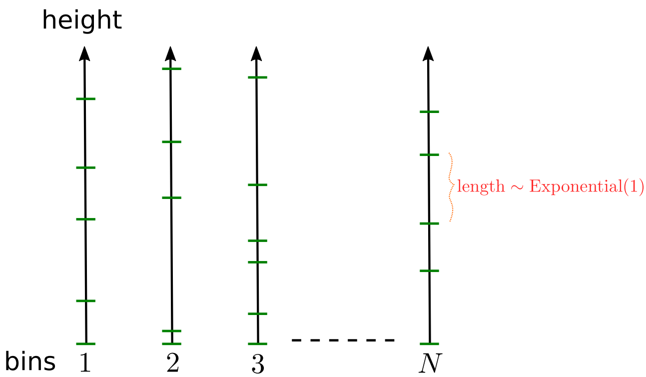

Figure 5. *Construction of N independent copies of a one-dimensional Poisson process via an infinite collection of i.i.d. exponentially distributed random variables.*

Our goal is to show the following bound:

Proposition 2.2. *For each $\mu \in \mathbb{N}_+$ and each $N \geq 2$, we have*

$$\mathscr{W}_1\left(\mathscr{M}_N, \text{Poisson}(\mu)^{\otimes N}\right) \leq \frac{\sqrt{2\mu/\pi}}{\sqrt{N}} \leq \frac{\sqrt{\mu}}{\sqrt{N}}. \tag{2.13}$$

Proof. We proceed by a coupling argument: the idea is to define random vectors

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{pmatrix} \sim \mathscr{M}_N, \qquad \mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_N \end{pmatrix} \sim \text{Poisson}(\mu)^{\otimes N}$$

in a convenient way. Specifically, for each bin $i = 1, 2, \ldots, N$, we consider an infinite collection of Exponential(1) distributed independent random variables, stacked together. That is, we have N independent copies of a one-dimensional Poisson process (see Figure 5 for an illustration). Every interval represents a ball that can potentially be tossed in the bin underneath.

Now, consider a horizontal 'bar' that covers all N bins, initially at height 0. Start rising this bar, and stop when exactly μN full intervals (among all N stacks) lie below it. In other words: if we consider the union of all the N Poisson point processes and arrange its atoms in increasing order, the final height of the bar is exactly the location of the (μN)-th atom. Call

$$X_i := \text{ number of full intervals below the bar at bin } i$$

for each $1 \leq i \leq N$. It is readily seen that $\mathbf{X} \sim \mathscr{M}_N$ because 'rising the bar' is the same as adding new balls at random among the N bins, thanks to the loss of memory property. Also, let

$$Y_i := \text{ number of full intervals below height } \mu \text{ at bin } i$$

for each $1 \leq i \leq N$. Clearly, $Y_1, Y_2, \ldots, Y_N$ are independent Poisson(μ) distributed random variables. See Figure 6 for a concrete illustration of the constructed coupling.

Observe that to go from $\mathbf{X}$ to $\mathbf{Y}$ (or vice versa), we either add balls (i.e., full intervals) to some bins or remove balls from some bins, but we never add balls to some bins and remove balls from other bins simultaneously. This means that $X_i - Y_i$ has the same sign for all $i = 1, 2, \ldots, N$. Also, recall that for a random variable $Z \sim \text{Poisson}(m)$ with m integer, we have the following exact formula for the expected mean deviation:

$$\mathbb{E}[|Z - m|] = 2e^{-m}\frac{m^{m+1}}{m!},$$

https://doi.org/10.1017/S0956792524000184 Published online by Cambridge University Press

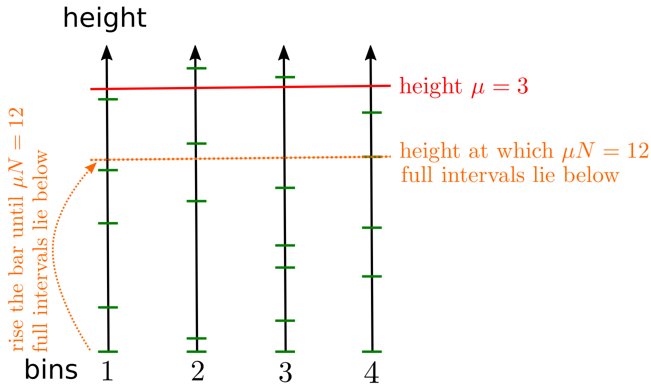

Figure 6. *Coupling of the two random vectors $\boldsymbol{X} \sim \mathscr{M}_N$ and $\boldsymbol{Y} \sim$ Poisson$(\mu)^{\otimes N}$ with $N = 4$ and $\mu = 3$. In this example, $\boldsymbol{X} = (X_1 = 3, X_2 = 2, X_3 = 4, X_4 = 3)$ and $\boldsymbol{Y} = (Y_1 = 4, Y_2 = 3, Y_3 = 4, Y_4 = 4)$.*

see for instance [14, 30]. Consequently, defining $Z := \sum_{i=1}^N Y_i \sim \text{Poisson}(\mu N)$, we deduce that

$$\begin{aligned}\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N |X_i - Y_i|\right] &= \frac{1}{N}\,\mathbb{E}\left[\left|\sum_{i=1}^N X_i - \sum_{i=1}^N Y_i\right|\right]\\ &= \frac{1}{N}\,\mathbb{E}[|\mu N - Z|]\\ &= \frac{1}{N} 2\mathrm{e}^{-\mu N}\frac{(\mu N)^{\mu N+1}}{(\mu N)!}\\ &\leq \frac{\sqrt{2\mu/\pi}}{\sqrt{N}},\end{aligned} \tag{2.14}$$

where we have used the bound $n! \geq \sqrt{2\pi n}\,(n/\mathrm{e})^n$, coming from Stirling's approximation. The desired bound now follows. □

Remark. Since a coupling of the marginals can be induced from a coupling of the full vector, Proposition 2.2 implies that

$$\mathscr{W}_1\left(\text{Binomial}\left(\mu N, \frac{1}{N}\right), \text{Poisson}(\mu)\right) \leq \frac{\sqrt{2\mu/\pi}}{\sqrt{N}} \leq \frac{\sqrt{\mu}}{\sqrt{N}}, \tag{2.15}$$

as well as

$$\mathscr{W}_1\left(\mathscr{M}_{N,k}, \text{Poisson}(\mu)^{\otimes k}\right) \leq \frac{\sqrt{2\mu/\pi}}{\sqrt{N}} \leq \frac{\sqrt{\mu}}{\sqrt{N}}, \tag{2.16}$$

for each fixed $1 \leq k \leq N$.

Remark. There are a number of technical papers on the Poisson approximation of the multinomial distribution using a variety of distances. We refer the interested readers to [23] and [25].

2.3. Equilibration of the mean-field ODE towards Poisson

We are now ready to establish the last piece of result before we prove the desired uniform-in- time propagation of chaos. We first recall that a non-uniform propagation of chaos result for the poor-biased

https://doi.org/10.1017/S0956792524000184 Published online by Cambridge University Press

exchange model has already been obtained in [6] (see their Theorem 4.3): assuming $\mathscr{L}(S_1(0)) = \mathbf{p}(0)$, then for each $t \geq 0$ and each $N \geq 2$, we have

$$\mathscr{W}_1\left(\mathscr{L}(S_1(t)), \mathbf{p}(t)\right) \leq \frac{4\,\mu\,\mathrm{e}^t}{N}, \tag{2.17}$$

where $\mathbf{p}(t)$ is the unique solution to the mean-field system of linear ODEs (1.3). Combining this result with Propositions 2.1 and 2.2 leads us to the following estimate:

Proposition 2.3. *Suppose that* $\mathbf{p}(t) = \{p_n(t)\}_{n\geq 0}$ *is the unique solution of (1.3) and* $\mathbf{p}^*$ *is the equilibrium Poisson distribution (1.6) to (1.3). Then for each* $t \geq 0$,

$$\mathscr{W}_1\left(\mathbf{p}(t), \mathbf{p}^*\right) \leq 2\,\mu\,\mathrm{e}^{-t}. \tag{2.18}$$

Proof. An elementary application of the triangle inequality for the Wasserstein distance W_1, together with the estimates (2.11), (2.15) and (2.17), yields

$$\begin{aligned}\mathscr{W}_1\left(\mathbf{p}(t), \mathbf{p}^*\right) &\leq \mathscr{W}_1\left(\mathbf{p}(t), \mathscr{L}(S_1(t))\right) + \mathscr{W}_1\left(\mathscr{L}(S_1(t)), \mathrm{Binomial}\left(\mu N, \frac{1}{N}\right)\right) \\ &\quad + \mathscr{W}_1\left(\mathrm{Binomial}\left(\mu N, \frac{1}{N}\right), \mathbf{p}^*\right) \\ &\leq \frac{4\,\mu\,\mathrm{e}^t}{N} + 2\,\mu\,\mathrm{e}^{-t} + \frac{\sqrt{\mu}}{\sqrt{N}}.\end{aligned}$$

As this estimate is valid for any N, sending $N \to \infty$ gives rise to the desired bound (2.18). □

Remark. It is very natural to expect that a result of the type (2.18), which concerns only the large-time behaviour of the mean-field ODE system (1.3), can be obtained in a purely analytic way without resorting to any probabilistic argument of the underlying stochastic agent-based dynamics. Indeed, it has already been shown in [6] that the so-called χ^2 'distance' from $\mathbf{p}(t)$ to $\mathbf{p}^*$, defined by

$$\chi^2\left(\mathbf{p}(t), \mathbf{p}^*\right) := \sum_{n=0}^{\infty} \frac{|p_n(t) - p_n^*|^2}{p_n^*},$$

satisfies

$$\chi^2\left(\mathbf{p}(t), \mathbf{p}^*\right) \leq \chi^2\left(\mathbf{p}(0), \mathbf{p}^*\right)\,\mathrm{e}^{-t}. \tag{2.19}$$

Taking into account the possibility to bound the Wasserstein distance $\mathscr{W}_1\left(\mathbf{p}(t), \mathbf{p}^*\right)$ by the χ^2 "distance" $\chi^2\left(\mathbf{p}(t), \mathbf{p}^*\right)$, such as (see for instance [29] for its proof)

$$\mathscr{W}_1\left(\mathbf{p}(t), \mathbf{p}^*\right) \leq \sqrt{\mu^2 + \mu}\,\sqrt{\chi^2\left(\mathbf{p}(t), \mathbf{p}^*\right)}, \tag{2.20}$$

we obtain

$$\mathscr{W}_1\left(\mathbf{p}(t), \mathbf{p}^*\right) \leq \sqrt{\mu^2 + \mu}\,\sqrt{\chi^2\left(\mathbf{p}(0), \mathbf{p}^*\right)}\,\mathrm{e}^{-\frac{t}{2}}. \tag{2.21}$$

Notice that (2.18) is sharper than (2.21). However, it is conjectured in [6] that $\chi^2\left(\mathbf{p}(t), \mathbf{p}^*\right)$ decays like e^{-2t} (based on some heuristic reasoning and numerical experiments), which would then lead to an estimate similar to (2.18).

2.4. Proof of the uniform propagation of chaos

We now assemble all the previous estimates together to prove a uniform-in-time propagation of chaos result for the poor-biased exchange dynamics. The key idea behind the proof is to carefully choose the time t as a function of the number of agents N, inspired from a very recent work [3] on a closely related model.

https://doi.org/10.1017/S0956792524000184 Published online by Cambridge University Press

We will need the following general version of (2.17), see [6]: assuming that $S_1(t), \ldots, S_N(t)$ are i.i.d. with law $\mathbf{p}(0)$, then for any fixed number of marginals k, we have

$$\mathscr{W}_1\left(\mathscr{L}(S_1(t), \ldots, S_k(t)), \mathbf{p}(t)^{\otimes k}\right) \leq \frac{4\,k\,\mu\,\mathrm{e}^t}{N}. \tag{2.22}$$

Theorem 1. *Assume that $S_1(t), \ldots, S_N(t)$ are i.i.d. with law $\mathbf{p}(0)$. Then, for all fixed $k \geq 1$ and for all $N \geq 2$ and $t \geq 0$, we have*

$$\mathscr{W}_1\left(\mathscr{L}(S_1(t), \ldots, S_k(t)), \mathbf{p}(t)^{\otimes k}\right) \leq \frac{4\,k\,\mu + \sqrt{\mu}}{\sqrt{N}}. \tag{2.23}$$

Proof. Notice that the appropriate scaling in the definition of Wasserstein distance (2.1) ensures that a tensorised version of (2.18) remains valid as well, i.e.,

$$\mathscr{W}_1\left(\mathbf{p}(t)^{\otimes k}, (\mathbf{p}^*)^{\otimes k}\right) \leq 2\,\mu\,\mathrm{e}^{-t}. \tag{2.24}$$

Thus, from (2.12), (2.16) and (2.24), we have

$$\begin{aligned}
&\mathscr{W}_1\left(\mathscr{L}(S_1(t), \ldots, S_k(t)), \mathbf{p}(t)^{\otimes k}\right)\\
&\leq \mathscr{W}_1\left(\mathscr{L}(S_1(t), \ldots, S_k(t)), \mathscr{M}_{N,k}\right) + \mathscr{W}_1\left(\mathscr{M}_{N,k}, (\mathbf{p}^*)^{\otimes k}\right) + \mathscr{W}_1\left((\mathbf{p}^*)^{\otimes k}, \mathbf{p}(t)^{\otimes k}\right)\\
&\leq 4\,\mu\,\mathrm{e}^{-t} + \frac{\sqrt{\mu}}{\sqrt{N}}.
\end{aligned}$$

Now, set $T = \frac{\log N}{2}$. Using the last estimate when $t \geq T$, and (2.22) when $t \leq T$, gives

$$\mathscr{W}_1\left(\mathscr{L}(S_1(t), \ldots, S_k(t)), \mathbf{p}(t)^{\otimes k}\right) \leq \begin{cases} \dfrac{4\,k\,\mu}{\sqrt{N}} & \text{if } t \leq T,\\[2ex] \dfrac{4\,\mu + \sqrt{\mu}}{\sqrt{N}} & \text{if } t \geq T, \end{cases} \tag{2.25}$$

from which (2.23) follows. □

Remark. At this point, it would be interesting to quantify the expected error between the (random) empirical distribution $\bar{S}(t) := \frac{1}{N}\sum_i \delta_{S_i(t)}$ and the limiting probability mass function $\mathbf{p}(t)$. If the error is measured using $\mathscr{W}_1$, this amounts to estimate the quantity $\mathbb{E}[\mathscr{W}_1(\bar{S}(t), \mathbf{p}(t))]$. In [19, Theorem 2.4], under some assumptions, it is proven that this notion of chaos is, in fact, equivalent to the convergence of the law of k marginals in the $\mathscr{W}_1$ distance, with some explicit estimates. Thus, from Theorem 1, one can deduce the convergence of the empirical measure, although possibly with sub-optimal rates. In order to obtain better rates, one could try to prove a non-uniform-in-time propagation of chaos result in terms of $\mathbb{E}[\mathscr{W}_1(\bar{S}(t), \mathbf{p}(t))]$, similar to the one in [6]; then, an argument analogous to the proof of Theorem 1 would yield uniform propagation of chaos for the empirical measure. Also, one might be interested in proving some large deviation type results (for instance, if we run the agent-based model once and then compute its histogram, can we control the expected error compared to its mean-field limit?). We leave such problems as challenging open tasks which deserve further research activities.

3. Conclusion

In this manuscript, an agent-based dollar exchange model (called the poor-biased exchange model in [6]) for wealth (re-)distribution is studied. We rigorously proved a uniform-in-time propagation of chaos result for this model which, to the best of our knowledge, is not available in the literature prior to the present work. Our proof is based on several probabilistic coupling approaches and the non-uniform propagation of chaos established in [6], as well as some ideas from the recent work [3] for a closely related model. We emphasise that the poor-biased exchange model investigated in this paper has at least two 'siblings', known as the unbiased exchange model and the rich-biased exchange model [6], where

https://doi.org/10.1017/S0956792524000184 Published online by Cambridge University Press

the rate that a typical agent will be picked to give are modified according to (1.14), (1.15), respectively. One possible follow-up work would be to have a rigorous proof of the sharp estimate (recall (1.11) or equivalently (2.19)) for the solution of the mean-field ODE system (1.3) in the χ^2 'distance' conjectured in [6].

As of now, one exciting direction of research for econophysics models involves the inclusion of a central bank or even several banks and hence the possibility of agents being in debt (see for instance the recent work [7, 8, 22]). We plan to extend the framework and analysis of the present work to this more realistic setting.

Acknowledgements. We would like to thank Sebastien Motsch for his help on generating Figure 4 in this manuscript. We also want to express our gratitude towards two anonymous reviewers for providing helpful suggestions and comments that greatly improved the readability of this manuscript.

Competing interests. There is no competing interests to declare for this manuscript.

References

[1] Bakry, D. & Émery, M. (1985) Diffusions hypercontractives. In: *Séminaire de Probabilités XIX 1983/84*, Springer, pp. 177–206.
[2] Cao, F. (2023) Explicit decay rate for the Gini index in the repeated averaging model. *Math. Methods Appl. Sci.* **46**(4), 3583–3596.
[3] Cao, F. & Jabin, P.-E. (2022) From interacting agents to Boltzmann-Gibbs distribution of money. Preprint, arXiv: 2208.05629.
[4] Cao, F., Jabin, P.-E. & Motsch, S. (2023) Entropy dissipation and propagation of chaos for the uniform reshuffling model. *Math. Models Methods Appl. Sci.* **33**(4), 829–875.
[5] Cao, F. & Marshall, N. F. (2024) From the binomial reshuffling model to Poisson distribution of money. *Netw. Heterog. Media* **19**(1), 24–43.
[6] Cao, F. & Motsch, S. (2023) Derivation of wealth distributions from biased exchange of money. *Kinet. Relat. Models* **16**(5), 764–794.
[7] Cao, F. & Motsch, S. (2023) Uncovering a two-phase dynamics from a dollar exchange model with bank and debt. *SIAM J. Appl. Math.* **83**(5), 1872–1891.
[8] Cao, F. & Reed, S. (2023) A biased dollar exchange model involving bank and debt with discontinuous equilibrium. Preprint, arXiv: 2311.07851.
[9] Chakraborti, A. & Chakrabarti, B. K. (2000) Statistical mechanics of money: how saving propensity affects its distribution. *Eur. Phys. J. B-Condens. Matter Complex Syst.* **17**(1), 167–170.
[10] Chatterjee, A., Chakrabarti, B. K. & Manna, S. S. (2004) Pareto law in a kinetic model of market with random saving propensity. *Phys. A Stat. Mech. Appl.* **335**(1-2), 155–163.
[11] Cortez, R. (2016) Uniform propagation of chaos for Kac's 1D particle system. *J. Stat. Phys.* **165**(6), 1102–1113.
[12] Cortez, R. (2018) Particle system approach to wealth redistribution. Preprint:1809.05372.
[13] Cortez, R. & Fontbona, J. (2016) Quantitative propagation of chaos for generalized Kac particle systems. *Ann. Appl. Prob.* **26**(2), 892–916.
[14] Crow, E. L. (1958) The mean deviation of the Poisson distribution. *Biometrika* **45**(3/4), 556–559.
[15] Dragulescu, A. & Yakovenko, V. M. (2000) Statistical mechanics of money. *Eur. Phys. J. B-Condens. Matter Complex Syst.* **17**(4), 723–729.
[16] Düring, B., Georgiou, N., Merino-Aceituno, S. & Scalas, E. (2022) Continuum and thermodynamic limits for a simple random-exchange model. *Stoch. Process. Appl.* **149**, 248–277.
[17] Düring, B., Matthes, D. & Toscani, G. (2008) Kinetic equations modelling wealth redistribution: a comparison of approaches. *Phys. Rev. E* **78**(5), 056103.
[18] Garibaldi, U., Scalas, E. & Viarengo, P. (2007) Statistical equilibrium in simple exchange games II. The redistribution game. *Eur. Phys. J. B* **60**(2), 241–246.
[19] Hauray, M. & Mischler, S. (2014) On Kac's chaos and related problems. *J. Funct. Anal.* **266**(10), 6055–6157.
[20] Heinsalu, E. & Marco, P. (2014) Kinetic models of immediate exchange. *Eur. Phys. J. B* **87**(8), 1–10.
[21] Lanchier, N. (2017) Rigorous proof of the Boltzmann-Gibbs distribution of money on connected graphs. *J. Stat. Phys.* **167**(1), 160–172.
[22] Lanchier, N. & Reed, S. (2019) Rigorous results for the distribution of money on connected graphs (models with debts). *J. Stat. Phys.* **176**(5), 1115–1137.
[23] Loh, W.-L. (1992) Stein's method and multinomial approximation. *Ann. Appl. Prob.* **2**(3), 536–554.
[24] Lux, T. (2005) Emergent statistical wealth distributions in simple monetary exchange models: a critical review. *Econophys. Wealth Distrib. Econophys.-Kolkata* **I**, 51–60.
[25] McDonald, D. R. (1980) On the Poisson approximation to the multinomial distribution. *Canad. J. Stat.* **8**(1), 115–118.

https://doi.org/10.1017/S0956792524000184 Published online by Cambridge University Press

[26] Merle, M. & Salez, J. (2019) Cutoff for the mean-field zero-range process. *Ann. Prob.* **47**(5), 3170–3201. Publisher: Institute of Mathematical Statistics.
[27] Mischler, S. & Mouhot, C. (2013) Kac's program in kinetic theory. *Invent. Math.* **193**(1), 1–147.
[28] Murata, H. (1977) Propagation of chaos for Boltzmann-like equation of noncutoff type in the plane. *Hiroshima Math. J.* **7**(2), 479–515. Publisher: Hiroshima University, Mathematics Program.
[29] Pinelis, I. An inequality involving the Wasserstein distance and chi-squared distance (URL version: 2022-10-27). *Available at:* https://mathoverflow.net/q/433344.
[30] Ramasubban, T. A. (1958) The mean difference and the mean deviation of some discontinuous distributions. *Biometrika* **45**(3/4), 549–556.
[31] Scalas, E., Garibaldi, U. & Donadio, S. (2006) Statistical equilibrium in simple exchange games I: methods of solution and application to the Bennati-Dragulescu-Yakovenko (BDY) game. *Eur. Phys. J. B-Condens. Matter Complex Syst.* **53**(2), 267–272.
[32] Sznitman, A.-S. (1991) Topics in propagation of chaos. In: *Ecole d'été de probabilités de Saint-Flour XIX–1989*, Springer, pp. 165–251.
[33] Tanaka, H. (1978) Probabilistic treatment of the Boltzmann equation of Maxwellian molecules. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* **46**(1), 67–105. Publisher: Springer.
[34] Villani, C. (2002). A review of mathematical topics in collisional kinetic theory. In: *Handbook of Mathematical Fluid Dynamics*, Vol. **I**, North-Holland, Amsterdam, pp. 71–305.

Cite this article: Cao F. and Cortez R. (2025). Uniform propagation of chaos for a dollar exchange econophysics model. *European Journal of Applied Mathematics*, **36**, 27–39. https://doi.org/10.1017/S0956792524000184