COMODULES AND CONTRAMODULES

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Abstract. Algebras A and coalgebras C over a commutative ring R are defined by properties of the (endo)functors $A \otimes_R -$ and $C \otimes_R -$ on the category of R-modules $_R\mathbb{M}$. Generalising these notions, monads and comonads were introduced on arbitrary categories, and it turned out that some of their basic relations do not depend on the specific properties of the tensor product. In particular, the adjoint of any comonad is a monad (and vice versa), and hence, for any coalgebra C, $\mathrm{Hom}_R(C, -)$, the right adjoint of $C \otimes_R -$, is a monad on $_R\mathbb{M}$. The modules for the monad $\mathrm{Hom}_R(C, -)$ were called *contramodules* by Eilenberg–Moore and the purpose of this talk is to outline the related constructions and explain the relationship between C-comodules and C-contramodules. The results presented grew out from cooperation with G. Böhm, T. Brzeziński and B. Mesablishvili.

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1. Preliminaries. In this section we recall some formalisms for module categories and their transfer to arbitrary categories. Throughout R will denote a commutative ring with unit and $_R\mathbb{M}$ the category of (left) R-modules. We recall (once again) the basic definitions explicitly since they provide the prototype of the constructions to follow.

For any R-modules N, X, $N \otimes_R X$ is again an R-module and thus we have a functor (endofunctor)
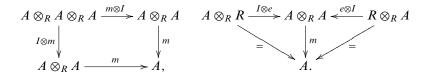
$$N \otimes_R - : {}_R\mathbb{M} \to {}_R\mathbb{M}, \qquad X \mapsto N \otimes_R X,$$
$$f : X \to Y \mapsto I_N \otimes f : N \otimes_R X \to N \otimes_R Y.$$

1.1. Algebras. An R-module A is called an R-*algebra* provided there is an associative R-bilinear form $\tilde{m} : A \times A \to A$, $(a, b) \mapsto ab$, and there exists a unit element 1_A. Forming the tensor product, the map $\tilde{m}$ can be replaced by an R-linear map $m : A \otimes_R A \to A$, $a \otimes b \mapsto ab$, and the unit leads to an R-linear map $e : R \to A$, $r \mapsto r \cdot 1_A$. Thus, an R-algebra A is characterised by two *morphisms in* $_R\mathbb{M}$—multiplication and unit
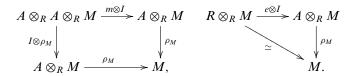
$$m : A \otimes_R A \to A, \quad e : R \to A,$$

and associativity and unitality are expressed by the commutative diagrams

$$
\begin{array}{ccc}
A \otimes_R A \otimes_R A & \xrightarrow{m \otimes I} & A \otimes_R A \\
{\scriptstyle I \otimes m} \downarrow & & \downarrow {\scriptstyle m} \\
A \otimes_R A & \xrightarrow{\quad m \quad} & A,
\end{array}
\qquad
\begin{array}{ccccc}
A \otimes_R R & \xrightarrow{I \otimes e} & A \otimes_R A & \xleftarrow{e \otimes I} & R \otimes_R A \\
& {\scriptstyle =} \searrow & \downarrow {\scriptstyle m} & \swarrow {\scriptstyle =} & \\
& & A. & &
\end{array}
$$

1.2. **Category of A-modules.** Let (A, m, e) be an R-algebra. An R-module M is said to be an *A-module* if there is an R-linear map

$$\rho_M : A \otimes_R M \to M, \quad a \otimes m \mapsto am,$$

inducing commutative diagrams

$$
\begin{array}{ccc}
A \otimes_R A \otimes_R M & \xrightarrow{m \otimes I} & A \otimes_R M \\
{\scriptstyle I \otimes \rho_M} \downarrow & & \downarrow {\scriptstyle \rho_M} \\
A \otimes_R M & \xrightarrow{\rho_M} & M,
\end{array}
\qquad
\begin{array}{ccc}
R \otimes_R M & \xrightarrow{e \otimes I} & A \otimes_R M \\
& {\scriptstyle \simeq} \searrow & \downarrow {\scriptstyle \rho_M} \\
& & M.
\end{array}
$$

A-module morphisms $f : M \to N$ are R-linear maps inducing commutative diagrams

$$
\begin{array}{ccc}
A \otimes_R M & \xrightarrow{I \otimes f} & A \otimes_R N \\
{\scriptstyle \rho_M} \downarrow & & \downarrow {\scriptstyle \rho_N} \\
M & \xrightarrow{f} & N.
\end{array}
$$

The *category of left A-modules* is denoted by $_A\mathbb{M}$. For any R-module X, $A \otimes_R X$ is an A-module with multiplication induced by the algebra product m. This yields a functor (denoted by the same symbol as the endofunctor)

$$A \otimes_R - : {}_R\mathbb{M} \to {}_A\mathbb{M}, \quad X \mapsto (A \otimes_R X, m \otimes I_X),$$

which is left adjoint to the forgetful functor $U_A : {}_A\mathbb{M} \to {}_R\mathbb{M}$, that is, there is a functorial isomorphism

$$\mathrm{Hom}_A(A \otimes_R M, X) \to \mathrm{Hom}_R(M, X), \quad f \mapsto f(1_A \otimes -).$$

1.3. **Coalgebras.** An R-module C is called an *R-coalgebra* provided there are R-linear maps—comultiplication and counit

$$\Delta : C \to C \otimes_R C, \quad \varepsilon : C \to R,$$

with coassociativity and counitality conditions expressed by inverting the arrows in the diagrams in 1.1.

1.4. **Category of comodules.** Let (C, Δ, ε) be an R-coalgebra. A *left C-comodule* is an R-module M with an R-linear map

$$\varrho^M : M \to C \otimes_R M,$$

with compatibility conditions obtained from those in 1.2 by reversing arrows.

C-comodule morphisms $f : M \to N$ are R-linear maps inducing commutative diagrams

$$
\begin{array}{ccc}
M & \xrightarrow{\ \ f\ \ } & N \\
{\scriptstyle \varrho^M}\big\downarrow & & \big\downarrow{\scriptstyle \varrho^N} \\
C \otimes_R M & \xrightarrow{\ I \otimes f\ } & C \otimes_R N.
\end{array}
$$

The category of left C-comodules is denoted by $^C\mathbb{M}$. For any R-module X, $C \otimes_R X$ is a left C-comodule with coaction induced by Δ. This yields a functor

$$
C \otimes_R - : {}_R\mathbb{M} \to {}^C\mathbb{M}, \quad X \mapsto (C \otimes_R X, \Delta_X),
$$

which is right adjoint to the forgetful functor $U^C : {}^C\mathbb{M} \to {}_R\mathbb{M}$, that is, there is an isomorphism

$$
\mathrm{Hom}^C(M, C \otimes_R X) \to \mathrm{Hom}_R(M, X), \quad f \mapsto (\varepsilon \otimes I_X) \circ f.
$$

Transferring the construction just revisited to arbitrary categories leads to the following notions. By $\mathbb{A}$ we denote any category.

First we express the properties of an algebra in functor terms. Note that $A \otimes_R A$ may be seen as composition of the functor $A \otimes_R -$ with $A \otimes_R -$.

1.5. **Monad on** $\mathbb{A}$. An endofunctor $F : \mathbb{A} \to \mathbb{A}$ is called a *monad on* $\mathbb{A}$ provided there are natural transformations

$$
m : FF \to F, \quad e : I_{\mathbb{A}} \to F,
$$

satisfying associativity and unitality conditions (as for algebras, see 1.1).

1.6. *F***-modules.** Let (F, m, e) be a monad on $\mathbb{A}$. An object A in $\mathbb{A}$ is an *F-module* provided there is a morphism

$$
\varrho_A : F(A) \to A \ \text{ in } \mathbb{A},
$$

satisfying associativity and unitality conditions (as for A-modules, see 1.2).

An *F-module morphism* $f : A \to B$ between F-modules is a morphism in $\mathbb{A}$ subject to commutativity of the diagram

$$
\begin{array}{ccc}
F(A) & \xrightarrow{\ F(f)\ } & F(B) \\
{\scriptstyle \varrho_A}\big\downarrow & & \big\downarrow{\scriptstyle \varrho_B} \\
A & \xrightarrow{\ \ f\ \ } & B.
\end{array}
$$

The category of F-modules, also called the *Eilenberg–Moore category of F*, is denoted by $\mathbb{A}_F$. For any object X in $\mathbb{A}$, $F(X)$ is an F-module with multiplication induced by m. This leads to a functor

$$
\phi_F : \mathbb{A} \to \mathbb{A}_F, \quad X \mapsto (F(X), m_X),
$$

which is left adjoint to the forgetful functor $U_F : \mathbb{A}_F \to \mathbb{A}$, by the functorial isomorphism, for $X \in \mathbb{A}$, $N \in \mathbb{A}_F$,

$$\mathrm{Mor}_F(F(X), N) \to \mathrm{Mor}_\mathbb{A}(X, N), \quad h \mapsto h \circ e_X.$$

Similar to algebras, the notion of a coalgebra also transfers to general categories. Again $\mathbb{A}$ denotes any category.

1.7. **Comonad on** $\mathbb{A}$. An endofunctor $G : \mathbb{A} \to \mathbb{A}$ is called a *comonad* if there are natural transformations

$$\delta : G \to GG, \quad \varepsilon : G \to I_\mathbb{A},$$

satisfying coassociativity and counitality conditions (as in 1.3).

1.8. *G-comodules.* Let (G, δ, ε) be a comonad on $\mathbb{A}$. An object A in $\mathbb{A}$ is a *G-comodule* provided there exists a morphism

$$\varrho^A : A \to G(A) \text{ in } \mathbb{A},$$

subject to coassociativity and counitality conditions (as in 1.4).

G-comodule morphisms $g : A \to B$ between G-comodules are morphisms in $\mathbb{A}$ inducing commutative diagrams

$$
\begin{array}{ccc}
A & \xrightarrow{\ g\ } & B \\
{\scriptstyle \varrho^A}\downarrow & & \downarrow{\scriptstyle \varrho^B} \\
G(A) & \xrightarrow{G(g)} & G(B).
\end{array}
$$

The category of G-comodules, also called the *Eilenberg–Moore category of G*, is denoted by $\mathbb{A}^G$. For any X in $\mathbb{A}$, $G(X)$ is a G-comodule with coaction induced by δ. This yields a functor

$$\phi^G : \mathbb{A} \to \mathbb{A}^G, X \mapsto (G(X), \delta_X),$$

which is right adjoint to the forgetful functor $U^G : \mathbb{A}^G \to \mathbb{A}$ by the functorial isomorphism

$$\mathrm{Mor}^G(M, G(X)) \to \mathrm{Mor}_\mathbb{A}(M, X), \quad f \mapsto \varepsilon_X \circ f.$$

Note that for modules of a monad (1.6) as well as for comodules of a comonad (1.7), we have obtained the same basic relations as for A-modules (1.2) and for C-comodules (1.4), respectively. While the latter are derived from the tensor functor (hence a functor with specific properties), we did not impose any condition on the monads or comonads. Bringing these abstract notions back to module categories, the question arises if there are other endofunctors on $_R\mathbb{M}$ which allow for a monad or comonad structure. Some answers to this will be given in the next section.

2. Adjoint endofunctors. We first consider the general case. So let $\mathbb{A}$ be any category. Recall that an endofunctor $F : \mathbb{A} \to \mathbb{A}$ is left adjoint to an endofunctor

$G : \mathbb{A} \to \mathbb{A}$ provided there is a functorial isomorphism

$$\varphi_{X,Y} : \ \mathrm{Mor}_{\mathbb{A}}(F(X), Y) \longrightarrow \mathrm{Mor}_{\mathbb{A}}(X, G(Y)).$$

2.1. Adjoints of monads. Let (F, G) be an adjoint pair of endofunctors on $\mathbb{A}$. Assume F to be a monad with $m : FF \to F$, $e : I_{\mathbb{A}} \to F$. Then we obtain the diagram

$$
\begin{array}{ccc}
\mathrm{Mor}_{\mathbb{A}}(F(X), Y) & \xrightarrow{\ \varphi_{X,Y}\ } & \mathrm{Mor}_{\mathbb{A}}(X, G(Y)) \\
{\scriptstyle m_Y^*}\downarrow & & \downarrow{\scriptstyle ?} \\
\mathrm{Mor}_{\mathbb{A}}(FF(X), Y) & \xrightarrow{\ \varphi_{F(X),Y}\ } & \mathrm{Mor}_{\mathbb{A}}(X, GG(Y)),
\end{array}
$$

where $m^* = \mathrm{Mor}_{\mathbb{A}}(m, -)$ and the question mark can be replaced by the natural transformation

$$\varphi_{F(X),Y} \circ m_Y^* \circ \varphi_{X,Y}^{-1}.$$

Together with a corresponding diagram for the unit we conclude by the Yoneda lemma that there is a comonad structure on G:

$$\delta : G \to GG, \quad \varepsilon : G \to I_{\mathbb{A}}.$$

It may come as a surprise that, in the situation considered in 2.1, the category of F-modules is isomorphic to the category of G-comodules.

2.2. Modules and comodules for an adjoint pair. Let (F, G) be an adjoint pair of functors with unit $\eta : I_{\mathbb{A}} \to GF$ and counit $\varepsilon : FG \to I_{\mathbb{A}}$. Let F be a monad and consider G with the induced comonad structure. Then the (Eilenberg–Moore) categories $\mathbb{A}_F$ and $\mathbb{A}^G$ are isomorphic to each other.

Proof. The isomorphism is given by a functor leaving objects and morphisms unchanged and turning F-module structure maps to G-comodule structure maps and vice versa. An F-module $\varrho_A : F(A) \to A$ induces a morphism

$$A \xrightarrow{\ \eta A\ } GF(A) \xrightarrow{\ G\varrho_A\ } G(A),$$

making A a G-comodule. Similarly, a comodule $\varrho^A : A \to G(A)$ induces

$$F(A) \xrightarrow{\ F\varrho^A\ } FG(A) \xrightarrow{\ \varepsilon_A\ } A,$$

defining an F-module structure on A.

It remains to show that module morphisms are also comodule morphisms. An F-module morphism $f : A' \to A$ yields a commutative diagram

$$
\begin{array}{ccc}
F(A') & \xrightarrow{\ F(f)\ } & F(A) \\
{\scriptstyle \varrho_{A'}}\downarrow & & \downarrow{\scriptstyle \varrho_A} \\
A' & \xrightarrow{\ f\ } & A \,,
\end{array}
$$

from which we obtain the commutative diagram

$$
\begin{array}{ccc}
A' & \xrightarrow{\ \ f\ \ } & A \\
{\scriptstyle \eta A'}\big\downarrow & & \big\downarrow{\scriptstyle \eta A} \\
GF(A') & \xrightarrow{\ GF(f)\ } & GF(A) \\
{\scriptstyle G\varrho_{A'}}\big\downarrow & & \big\downarrow{\scriptstyle G\varrho_A} \\
G(A') & \xrightarrow{\ G(f)\ } & G(A).
\end{array}
$$

Commutativity of the outer rectangle shows that f is also a G-comodule morphism. Similarly one proves that G-comodule morphisms are also F-module morphisms. $\quad\square$

2.3. **Adjoints of comonads.** Let (F, G) be an adjoint pair of endofunctors on $\mathbb{A}$. Now assume F to be a comonad. Then—with similar arguments as in 2.1—one can see that G has a monad structure.

Related to such a pair we have the categories $\mathbb{A}^F$ and $\mathbb{A}_G$. Unlike the monad–comonad case considered in 2.2, here we do not get an equivalence between $\mathbb{A}^F$ and $\mathbb{A}_G$. There is, however, an equivalence between certain subcategories (Kleisli subcategories, see [**1**, Section 2]). We will come back to this problem for $_R\mathbb{M}$ (in 3.9).

Having the general setting developed above, one may ask about the meaning of this in the category of R-modules. For this, recall the following.

2.4. **Adjoint endofunctors on $_R\mathbb{M}$.** For any R-module M, the endofunctors $M \otimes_R -$ and $\mathrm{Hom}_R(M, -)$ form an adjoint pair by the isomorphism

$$
\psi^M_{X,Y} : \mathrm{Hom}_R(M \otimes_R X,\, Y) \to \mathrm{Hom}_R(X,\, \mathrm{Hom}_R(M, Y)), \quad \gamma \mapsto [x \mapsto \gamma(- \otimes x)],
$$

with unit $\nu : I \to \mathrm{Hom}_R(M, M \otimes_R -)$ and counit $\varepsilon : M \otimes_R \mathrm{Hom}(M, -) \to I$.

First we apply 2.1 and 2.2 to algebras.

PROPOSITION 2.5. *For an R-module A the following are equivalent:*

(a) *A is an R-algebra.*
(b) *$A \otimes_R - : {}_R\mathbb{M} \to {}_R\mathbb{M}$ is a monad.*
(c) *$\mathrm{Hom}_R(A, -) : {}_R\mathbb{M} \to {}_R\mathbb{M}$ is a comonad.*

In this case the categories $_A\mathbb{M} = \mathbb{M}_{A\otimes_R-}$ and $\mathbb{M}^{\mathrm{Hom}(A,-)}$ are isomorphic.

Proof. The assertions are special cases of 2.1 and 2.2. The algebra structure on A, $m : A \otimes_R A \to A$ and $e : R \to A$ correspond to the comonad structure on $\mathrm{Hom}_R(A, -)$:

$$
\mathrm{Hom}_R(A, -) \xrightarrow{\ m^*\ } \mathrm{Hom}_R(A \otimes_R A, -) \xrightarrow{\ \psi^A_{A,-}\ } \mathrm{Hom}_R(A, \mathrm{Hom}_R(A, -)),
$$
$$
\mathrm{Hom}_R(A, -) \xrightarrow{\ e^*\ } \mathrm{Hom}_R(R, -),
$$

where $m^* = \mathrm{Hom}_R(m, -)$ and $e^* = \mathrm{Hom}_R(e, -)$.

The A-module structure $\rho_N : A \otimes_R N \to N$ induces a $\mathrm{Hom}_R(A, -)$-comodule structure on N:

$$\widehat{\rho_N} : N \xrightarrow{\;\nu_N\;} \mathrm{Hom}_R(A, A \otimes_R N) \xrightarrow{\;\mathrm{Hom}(A, \rho_N)\;} \mathrm{Hom}_R(A, N).$$

On the other hand, a comodule structure $\rho^N : N \to \mathrm{Hom}_R(A, N)$ induces an A-module structure on N:

$$\widetilde{\rho^N} : A \otimes_R N \xrightarrow{\;I \otimes \rho^N\;} A \otimes_R \mathrm{Hom}_R(A, N) \xrightarrow{\;\varepsilon_N\;} N.$$

The equivalence between the module and comodule categories is given by

$$\mathbb{M}_{A \otimes_R -} \to \mathbb{M}^{\mathrm{Hom}(A, -)}, \quad (M, \rho_N) \mapsto (M, \widehat{\rho_N}),$$

keeping the morphisms unchanged. □

2.6. **Value at R.** Since R is a generator in $_R\mathbb{M}$ and $A \otimes_R -$ preserves direct sums and epimorphisms, the functor $A \otimes_R - : {}_R\mathbb{M} \to {}_R\mathbb{M}$ is fully determined by the value at R, that is, by $A \otimes_R R \simeq A$. Similarly, a natural transformation $\varphi : A \otimes_R - \to B \otimes_R -$ between tensor functors is of the form $\varphi_R \otimes_R -$, where $\varphi_R : A \to B$ is an R-linear map.

In general, $\mathrm{Hom}_R(A, -)$ is not determined by $A^* = \mathrm{Hom}_R(A, R)$, unless it preserves direct sums and epimorphisms, that is, unless A is a finitely generated and projective R-module. However, $\mathrm{Hom}_R(A, -)$ is determined by $\mathrm{Hom}_R(A, Q)$ for any cogenerator $Q \in {}_R\mathbb{M}$ since it is left exact and preserves direct products. For a natural transformation $\psi : \mathrm{Hom}_R(A, -) \to \mathrm{Hom}_R(B, -)$ between Hom functors, it follows by the Yoneda lemma that $\psi = \mathrm{Hom}_R(\psi_R, -)$, where $\psi_R : B \to A$ is an R-linear map.

3. Comodules and contramodules. In this section, let (C, Δ, ε) be an R-coalgebra (see 1.3).

The category $^C\mathbb{M}$ of left C-comodules is well investigated. Besides the basic facts mentioned in 1.4, there are more relations which partly result from special properties of the functor $C \otimes_R -$ as being right exact and preserving coproducts (see [**2**]).

3.1. **The category $^C\mathbb{M}$.**

(1) $^C\mathbb{M}$ is an additive category with coproducts and cokernels.
(2) For any generator $P \in {}_R\mathbb{M}$, $C \otimes_R P$ is a subgenerator in $^C\mathbb{M}$, in particular, C is a subgenerator in $^C\mathbb{M}$.
(3) For any monomorphism $f : X \to Y$ in $_R\mathbb{M}$, $f \otimes_R I_C : C \otimes_R X \to C \otimes_R Y$ is a monomorphism in $^C\mathbb{M}$.
(4) For any family X_λ of R-modules, $C \otimes_R \prod_\Lambda X_\lambda$ is the product of the C-comodules $C \otimes_R X_\lambda$.

The R-module structure of C has a strong influence on the properties of $^C\mathbb{M}$ [**2**, 3.14].

PROPOSITION 3.2. *For an R-coalgebra C, the following are equivalent:*

(a) *C is flat as an R-module.*
(b) *Every monomorphism in $^C\mathbb{M}$ is injective.*

(c) *The forgetful functor $^C\mathbb{M} \to {}_R\mathbb{M}$ respects monomorphisms.*

In this case, $^C\mathbb{M}$ is an abelian category and

 (i) *for any cogenerator $Q \in {}_R\mathbb{M}$, $C \otimes_R Q$ is a cogenerator in $^C\mathbb{M}$;*
 (ii) *for any injective $X \in {}_R\mathbb{M}$, $C \otimes_R X$ is injective in $^C\mathbb{M}$.*

In view of the adjointness of the endofunctors $C \otimes_R -$ and $\mathrm{Hom}_R(C, -)$, the latter has a monad structure by 2.3.

PROPOSITION 3.3. *For an R-module C, the following are equivalent:*

(a) *C is an R-coalgebra.*
(b) *$C \otimes_R - : {}_R\mathbb{M} \to {}_R\mathbb{M}$ is a comonad.*
(c) *$\mathrm{Hom}_R(C, -) : {}_R\mathbb{M} \to {}_R\mathbb{M}$ is a monad.*

Hereby the coalgebra structure maps $\Delta : C \to C \otimes_R C, \varepsilon : C \to R$, correspond to the monad structure

$$\mathrm{Hom}_R(C, \mathrm{Hom}_R(C, -)) \xrightarrow{\simeq} \mathrm{Hom}_R(C \otimes_R C, -) \xrightarrow{\Delta^*} \mathrm{Hom}_R(C, -),$$
$$\mathrm{Hom}_R(R, -) \xrightarrow{\varepsilon^*} \mathrm{Hom}_R(C, -),$$

where $\Delta^* = \mathrm{Hom}_R(\Delta, -)$, $\varepsilon^* = \mathrm{Hom}_R(\varepsilon, -)$, and the isomorphism is from 2.4.

Henceforth we will write $[C, -] = \mathrm{Hom}_R(C, -)$ for short. By applying 1.6 we have the definition of $[C, -]$-modules and the category $\mathbb{M}_{[C,-]}$ with the (free) functor

$$\phi_{[C,-]} : {}_R\mathbb{M} \to \mathbb{M}_{[C,-]}$$

being left adjoint to the forgetful functor $U_{[C,-]} : \mathbb{M}_{[C,-]} \to {}_R\mathbb{M}$. Following [**4**], the $[C, -]$-modules are also called *C-contramodules*.

By left exactness of the functor $[C, -]$ we obtain (with similar arguments as in the comodule case) special properties of $[C, -]$-modules.

3.4. **The category $\mathbb{M}_{[C,-]}$.** Let C be an R-coalgebra.

(1) $\mathbb{M}_{[C,-]}$ is an additive category with products and kernels.
(2) For any $M \in \mathbb{M}_{[C,-]}$, $\mathrm{Hom}_{[C,-]}([C, R], M) \simeq M$.
(3) For any epimorphism $h : X \to Y$ in ${}_R\mathbb{M}$, $[C, h] : [C, X] \to [C, Y]$ is an epimorphism (not necessarily surjective) in $\mathbb{M}_{[C,-]}$.
(4) For any family X_λ of R-modules, $[C, \bigoplus_\Lambda X_\lambda]$ is the coproduct of the $[C, -]$-modules $[C, X_\lambda]$.

For C-comodules, it is not always clear if the kernel of a C-comodule morphism is a subcomodule. Here we need some condition to make the cokernel of a $[C, -]$-module morphism again a $[C, -]$-module.

PROPOSITION 3.5. *Let C be an R-algebra. If C_R is projective, then for any $[C, -]$-submodule $K \subset M$, the R-module M/K is a $[C, -]$-module.*

Proof. By assumption we have the commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & [C, K] & \longrightarrow & [C, M] & \longrightarrow & [C, M/K] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & M/K & \longrightarrow & 0,
\end{array}
$$

which can be completed commutatively by an R-morphism $[C, M/K] \to M/K$. It is routine to check that this provides M/K with a $[C, -]$-module structure. $\square$

As observed above, in $\mathbb{M}_{[C,-]}$ epimorphisms need not be surjective maps. This is the case under the following conditions.

PROPOSITION 3.6. *For an R-coalgebra C, the following are equivalent:*

(a) *C_R is projective.*
(b) *Every epimorphism in $\mathbb{M}_{[C,-]}$ is surjective.*
(c) *The forgetful functor $\mathbb{M}_{[C,-]} \to \mathbb{M}_R$ respects epimorphisms.*

In this case, $\mathbb{M}_{[C,-]}$ is an abelian category and

(i) *for any generator $P \in {}_R\mathbb{M}$, $[C, P]$ is a generator in $\mathbb{M}_{[C,-]}$;*
(ii) *for any projective $Y \in {}_R\mathbb{M}$, $[C, Y]$ is projective in $\mathbb{M}_{[C,-]}$.*

Proof. (b)$\Rightarrow$(a) For any epimorphism $f : K \to L$ in ${}_R\mathbb{M}$, $\mathrm{Hom}(C, f) :$ $\mathrm{Hom}_R(C, K) \to \mathrm{Hom}_R(C, K)$ is an epimorphism in $\mathbb{M}_{[C,-]}$ and hence surjective by (b). This means that C_R is projective.

(c)$\Rightarrow$(a) is shown with a similar argument.

(a)$\Rightarrow$(c) Assume C_R to be projective and consider an epimorphism $f : M \to N$ in $\mathbb{M}_{[C,-]}$. Then the cokernel $h : N \to N/f(M)$ is a morphims in $\mathbb{M}_{[C,-]}$ and $0f = hf = 0$. Since f is an epimorphism, this implies $N = f(M)$. $\square$

Recall that for any R-coalgebra C, the dual space $C^* = \mathrm{Hom}_R(C, R)$ has a ring structure by the convolution product for $f, g \in C^*$, $f * g = (f \otimes g) \circ \Delta$. The relation between C-comodules and modules over the dual ring of C is well studied (e.g. [**2**, Section 19]). Now it follows from the general observations in 2.1 and 2.3 that a coalgebra C gives rise to two comonads and two monads on ${}_R\mathbb{M}$:

$$
C \otimes_R - \text{ and } \mathrm{Hom}_R(C^*, -), \qquad C^* \otimes_R - \text{ and } \mathrm{Hom}_R(C, -).
$$

Relations between these are given by morphisms which are well known in module theory—but usually not viewed under this aspect.

3.7. The comonads $C \otimes_R -$ and $[C^*, -]$. The comonad morphism

$$
\alpha : C \otimes_R - \to \mathrm{Hom}_R(C^*, -), \quad c \otimes - \mapsto [f \mapsto f(c)-]
$$

yields a faithful functor

$$
\begin{array}{rcl}
G_\alpha : & {}^C\mathbb{M} \longrightarrow \mathbb{M}^{[C^*,-]} \simeq \mathbb{M}_{C^*}, \\
N \xrightarrow{\varrho^N} C \otimes_R N & \mapsto & N \xrightarrow{\varrho^N} C \otimes_R N \xrightarrow{\alpha_N} \mathrm{Hom}_R(C^*, N),
\end{array}
$$

and the following are equivalent:

  (a) α_N is injective for each $N \in {}_R\mathbb{M}$.
  (b) G_α is a full functor.
  (c) C is a locally projective left R-module.

If these conditions are satisfied, ${}^C\mathbb{M}$ is equal to $\sigma[C_{C^*}]$, the full subcategory of $\mathbb{M}_{C^*}$ subgenerated by C.

Similar to 3.7, C-contramodules can be related to C^*-modules.

3.8. **The monads $[C, -]$ and $- \otimes_R C^*$.** The monad morphism

$$\beta : - \otimes_R C^* \to \mathrm{Hom}_R(C, -), \quad - \otimes f \mapsto [c \mapsto f(c)-]$$

yields a faithful functor

$$F_\beta : \qquad \mathbb{M}_{[C,-]} \longrightarrow \mathbb{M}_{C^*},$$
$$\mathrm{Hom}_R(C, M) \xrightarrow{\varrho_M} M \longmapsto M \otimes_R C^* \xrightarrow{\beta_M} \mathrm{Hom}_R(C, M) \xrightarrow{\varrho_M} M,$$

and the following are equivalent:

  (a) β is surjective for all $M \in {}_R\mathbb{M}$.
  (b) F_β is an isomorphism.
  (c) C is a finitely generated and projective R-module.

In general, C is not a $[C, -]$-module and $[C, R]$ is not a C-comodule. In fact, $[C, R] \in {}^C\mathbb{M}$ holds provided C is finitely generated and projective as an R-module.

We now consider the relationship between the categories of C-comodules and C-contramodules. As indicated in 2.3, the two categories need not be equivalent. This becomes evident from their properties listed in 3.4 and 3.1, respectively. However, they are related by an adjoint pair of functors.

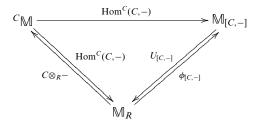3.9. **Correspondence of categories.** Let C be an R-coalgebra. Then

$$\mathrm{Hom}^C(C, -) : {}^C\mathbb{M} \to \mathbb{M}_{[C,-]}, \quad M \mapsto \mathrm{Hom}^C(C, M)$$

is a functor which has a left adjoint given by the *contratensor product* defined for $(N, \rho) \in {}^C\mathbb{M}$ and $(M, \alpha) \in \mathbb{M}_{[C,-]}$ as the coequaliser

$$\mathrm{Hom}_R(C, M) \otimes_R N \underset{\alpha \otimes_R I_N}{\overset{h}{\rightrightarrows}} M \otimes_R N \longrightarrow M \otimes_{[C,-]} N,$$

for the map $h : f \otimes_R n \mapsto (f \otimes_R I_N) \circ \rho(n)$.

This follows by Dubuc's adjoint triangle theorem applied to the diagram

by the existence of the functor in the upper line and the fact that $^C\mathbb{M}$ has coequalisers (see [**1**, Section 4], [**3**, Appendix] and [**6**]).

3.10. **Equivalence of subcategories.** Take any $X \in {}_R\mathbb{M}$. Since $C \otimes_R -$ is right adjoint to the forgetful functor, $\text{Hom}^C(C, -)$ takes

$$C \otimes_R X \mapsto \text{Hom}^C(C, C \otimes_R X) \simeq \text{Hom}_R(C, X).$$

On the other hand, the functor $C \otimes_{[C,-]} -$ transfers

$$\text{Hom}_R(C, X) \mapsto C \otimes_{[C,-]} \text{Hom}_R(C, X) \simeq C \otimes_R X.$$

This shows that the full subcategory of $^C\mathbb{M}$, whose objects are of the form $C \otimes_R X$, is equivalent to the full subcategory of $\mathbb{M}_{[C,-]}$, with objects $\text{Hom}_R(C, X)$, $X \in {}_R\mathbb{M}$ (*Kleisli subcategories*, e.g. [**1**]).

Finally we apply the formalism developed to bialgebras and bimodules.

3.11. **Bialgebras.** An R-module B is called an *R-bialgebra* if it is

 (i) an algebra (B, m, e) and a coalgebra (B, δ, ε),
(ii) the structures are compatible.

Then we know from 2.1 and 2.3 that $\text{Hom}_R(B, -)$ is a monad as well as a comonad and certain compatibility conditions hold.

Thus, a bimonad may be defined by properties of two monads $B \otimes_R -$ and $\text{Hom}_R(B, -)$ or else by the same two functors endowed with comonad structures.

3.12. **Bimodules.** For an R-bialgebra B, (mixed) *left B-bimodules* are defined as R-modules equipped with a left B-module and a left B-comodule structure subject to some compatibility conditions. With morphisms respecting the module as well as the comodule structure, they form a category which we denote by $^B_B\mathbb{M}$. There is a functor

$$B \otimes_R - : {}_R\mathbb{M} \to {}^B_B\mathbb{M}, \quad M \mapsto (B \otimes_R M, m \otimes I, \delta \otimes I),$$

and B is called a *Hopf algebra* provided this is an equivalence of categories. The latter can be characterised by the existence of an *antipode*.

Now, considering the monad and comonad $[B, -] = \text{Hom}(B, -)$, we can also define bimodules leading to the category $M^{[B,-]}_{[B,-]}$ and the functor

$$\text{Hom}_R(B, -) : {}_R\mathbb{M} \to \mathbb{M}^{[B,-]}_{[B,-]}, \quad M \mapsto (\text{Hom}_R(B, M), m^*, \delta^*),$$

which is an equivalence if and only if B is a Hopf algebra.

These relationships are explained in more detail in [**1**]. It is shown in [**5**] and [**7**] how they can be transferred to *bimonads* on arbitrary categories.

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