A SHORTER PROOF OF GOLDIE'S THEOREM

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In this note we present an extremely short proof of Goldie's theorem on the structure of semiprime Noetherian rings [1]. The outline of the proof was given by Procesi and Small in [4]. By utilizing the concept of the singular ideal of a ring we have been able to weaken the hypotheses of many of the steps in [4]. Most significantly, we are able to avoid a reduction to the case of prime rings, and in Lemma 5 we give an informative list of the relationship between regular elements and essential ideals of semiprime rings.

Let S be a subset of a ring R. $\ell(S) = \{x \in R : xS = 0\}$ is called the left annihilator of S; similarly $r(S) = \{x \in R : Sx = 0\}$ is called the right annihilator of S. Note that $r\ell r(S) = r(S)$. It follows that a ring satisfying the ascending chain condition on left annihilators satisfies the descending chain condition on right annihilators.

Let R be any ring and I and J left ideals of R with $I \subseteq J$. I is said to be essential in J if I intersects every non-zero left ideal contained in J non-trivially. If I is essential in R we will call I an essential left ideal. We define $Z(R) = 0$ to mean $r(I) = 0$ for every essential left ideal I.

Let I be a left ideal of a ring R. For $x \in R$, set $(I:x) = \{r \in R : rx \in I\}$. Note that $(I:x)x = I \cap Rx$.

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LEMMA 1. *Let* R *be any ring,* I *and* J *left ideals of* R.

(i) *If* I *is essential in* J, *then* (I:x) *is an essential left ideal of* R *for any* $x \in J$.

(ii) *Conversely, if* $Z(R) = 0$, $x \in R$ *and* (I:x) *is an essential left ideal, then* I *is essential in* I+Rx.

This lemma is due to Johnson [3], and is in fact true for any R-module. For the sake of completeness we repeat the proof.

Proof. Let K be a nonzero left ideal of R. $Kx = 0$ implies $0 \neq K \subseteq (I:x) \cap K$. On the other hand, if $Kx \neq 0$ then $I \cap Kx \neq 0$ since $Kx \subseteq J$. So choosing $0 \neq kx \in Kx \cap I$, $k \in K$, we have $0 \neq k \in K \cap (I:x)$. This proves (i).

Now suppose $Z(R) = 0$ and (I:x) is an essential left ideal of R. Let $0 \neq i + ax \in I + Rx$ with $i \in I$, $a \in R$; we have to show that $R(i + ax) \cap I \neq 0$. From (i), $(I:i + ax) = (I:ax) = ((I:x):a)$ is an essential left ideal of R. Since $Z(R) = 0$, $0 \neq (I:i + ax)(i + ax) = I \cap R(i + ax)$.

LEMMA 2. *Let* R *be a ring with* $Z(R) = 0$, *and* I *a left ideal of* R.

(i) *If* $\ell(B)$ *is essential in* I, *then* $\ell(B) = I$.

(ii) *If* Rx *and* Ry *are essential left ideals, so is* Rxy.

Proof. Suppose that $\ell(B)$ is essential in I and let $x \in I$. Then $(\ell(B):x)$ is an essential left ideal and $(\ell(B):x)xB = 0$, which implies that $xB = 0$, i.e., $x \in \ell(B)$. This proves (i).

For (ii) it suffices to prove that Rxy is essential in Ry. Now $Rx \subseteq (Rxy:y)$, and so $(Rxy:y)$ is essential in R. Hence by Lemma 1(ii), Rxy is an essential submodule of $Rxy + Ry = Ry$.

A ring R is said to be semiprime provided it has no nonzero nilpotent left ideals. Note that for left ideals J and K of a semiprime ring, $JK = 0$ implies $KJ = 0$.

We will constantly refer to the conditions

ℓ(acc): R has the ascending chain condition on left annihilators.

$\oplus$(acc): R contains no infinite direct sums of left ideals.

The following lemma appears in [4]. We repeat the proof.

LEMMA 3. *If* R *is a semiprime ring satisfying* ℓ(acc) *then* $Z(R) = 0$. *Conversely, if* R *is any ring with* $Z(R) = 0$ *and satisfying* $\oplus$(acc), *then* R *has both the ascending and the descending chain conditions on left annihilators*.

Proof. Suppose that I is an essential ideal with $r(I) \neq 0$. Choose $U \neq 0$, a minimal right annihilator $\subseteq r(I)$. $U^2 \neq 0$ since R is semiprime, so there exists $u \in U$ such that $uU \neq 0$. We complete the proof of the first half of the lemma by showing that $Ru \cap I = 0$.

If not, there exists $0 \neq xu \in Ru \cap I$ with $x \in R$. Since $xu \in I$ and $r(I) \supseteq U$, $xuU = 0$. Now $r(x) \cap U$ is a right annihilator contained in U, hence $r(x) \cap U = 0$ or $r(x) \cap U = U$. But $xuU = 0$, so $0 \neq uU \subseteq r(x) \cap U$. Hence we have $r(x) \cap U = U$, which implies that $U \subseteq r(x)$. But then $xu = 0$, a contradiction.

For the converse note that from any infinite proper chain of left annihilators we can extract an infinite direct sum by Lemma 2(i).

LEMMA 4. *Suppose that* R *is a semiprime ring satisfying* $\oplus$(acc). *Let* I *be any left ideal of* R, *and let* $a \in I$ *with* $\ell(a)$ *minimal among all* $\ell(x)$ *with* $x \in I$. *Then* Ra *is essential in* I.

Proof. Let J be any left ideal $\subseteq I$ with $Ra \cap J = 0$. For any $x \in J$, $\ell(a+x) \supseteq \ell(a) \cap \ell(x)$; and in fact $\ell(a+x) = \ell(a) \cap \ell(x) \subseteq \ell(a)$ since $Ra \cap J = 0$. By the minimality of $\ell(a)$ we must have $\ell(a+x) = \ell(a) \cap \ell(x) = \ell(a)$. Hence $\ell(a) \subseteq \ell(x)$. Since x was arbitrary, $\ell(a)J = 0 = J\ell(a)$.

Suppose now that $x \in \ell(a^2)$. Then $xa \in \ell(a)$, so $Jxa = 0$. But then $Jx \subseteq \ell(a)$, so $(Jx)^2 = 0$, whence $Jx = 0$. We have thus shown that $J\ell(a^2) = 0$; and similarly we can prove that $J\ell(a^i) = 0$ for all integers $i > 0$.

Either $J = 0$ or else $Ja^i \neq 0$ for all $i > 0$ (for $Ja^i = 0$ implies that $J \subseteq \ell(a^i)$, whence $J^2 = 0$). In the latter case, consider $\sum_{i=1}^{\infty} Ja^i$. This sum cannot be direct, so there exist $x_k, \ldots, x_n \in J$, $n > k$, such that $x_k a^k + \ldots + x_n a^n = 0$ with $x_k a^k \neq 0$. Now $(x_k + \ldots + x_n a^{n-k})a^k = 0$ implies that $R(x_k + \ldots + x_n a^{n-k}) \subseteq \ell(a^k)$, and so $JR(x_k + \ldots + x_n a^{n-k}) = 0$. But then $JRx_k = JR(-x_{k+1}a - \ldots - x_n a^{n-k}) \subseteq J \cap Ra = 0$, which leads to a contradiction since $x_k \in J$. Hence $J = 0$, and it follows that Ra is essential in I.

LEMMA 5. *Let R be a semiprime ring.*

(i) *If R satisfies ℓ(acc), and Ra is an essential left ideal, then a is regular, i.e.,* $r(a) = 0 = \ell(a)$.

(ii) *If R satisfies $\oplus$(acc), and $\ell(a) = 0$, then Ra is an essential left ideal.*

(iii) *If R satisfies both ℓ(acc) and $\oplus$(acc), then every essential left ideal contains a regular element.*

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Proof. (i). Suppose that Ra is essential. By Lemma 3, $r(a) = r(Ra) = 0$. Since R satisfies the ascending chain condition on left annihilators, there exists an integer n such that $\ell(a^n) = \ell(a^{n+1})$. Suppose $ya^n = x \in Ra^n \cap \ell(a)$. Then $0 = xa = ya^{n+1}$, so $y \in \ell(a^{n+1}) = \ell(a^n)$, whence $x = ya^n = 0$. Thus $Ra^n \cap \ell(a) = 0$. But by Lemma 2(ii) Ra^n is essential. Hence $\ell(a) = 0$.

Both (ii) and (iii) are consequences of Lemma 4; (ii) is immediate, while for (iii) we need only invoke part (i).

A ring Q with identity is said to be a *left quotient ring* of a ring R if $R \subseteq Q$, every regular element of R is invertible in Q, and every element of Q is of the form $a^{-1}b$ with $a,b \in R$.

It is known [2; p.262] that the following *common multiple condition* is necessary and sufficient that a ring R containing regular elements have a left quotient ring: given $a,b \in R$ with a regular, there exist $c,d \in R$ with d regular such that $ca = db$.

THEOREM (Goldie). *Let* R *be a semiprime ring satisfying both* ℓ(acc) *and* $\oplus$(acc). *Then* R *has a left quotient ring* Q *which is semisimple with minimum condition*.

Proof. Observe that by a trivial application of Lemma 5(iii) R contains regular elements. Next, let $a,b \in R$ be given with a regular. Ra is essential by Lemma 5(ii) and hence (Ra:b) is essential. By Lemma 5(iii), (Ra:b) must contain a regular element, and this yields the common multiple condition.

To prove that Q is semisimple with minimum condition it suffices to show that every left ideal of Q is a direct summand. If I is a nonzero left ideal of Q, then by Zorn's lemma there

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exists a left ideal K of R such that $(I \cap R) \oplus K$ is essential in R. Then by Lemma 5(iii), $Q = Q((I \cap R) \oplus K)$ which equals $Q(I \cap R) \oplus QK = I \oplus QK$ since Q is the left quotient ring of R. This completes the proof of the theorem.

Remark. In view of Lemma 3 we could replace the condition ℓ(acc) in the theorem with the hypothesis $Z(R) = 0$.

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