0-DIMENSIONAL COMPACTIFICATIONS AND BOOLEAN RINGS

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1. Introduction

A subset of a topological space which is both closed and open is referred to as a clopen subset. Here, a 0-dimensional space is a Hausdorff space which has a basis of clopen sets. By a compactification αX of a completely regular Hausdorff space X, we mean any compact space which contains X as a dense subspace. Two compactifications αX and γX are regarded as being equivalent if there exists a homeomorphism from αX onto γX which keeps X pointwise fixed. We will not distinguish between equivalent compactifications. With this convention, we can partially order any family of compactifications of X by defining $\alpha X \leqq \gamma X$ if there exists a continuous mapping from γX onto αX which leaves X pointwise fixed. This paper is concerned with the study of the partially ordered family $\mathfrak{Z}[X]$ of all 0-dimensional compactifications of a 0-dimensional space X.

In section 4, we have the result that $\mathfrak{Z}[X]$ is always a complete upper semi-lattice and it is a complete lattice if and only if X is locally compact. The analogous result for $\mathfrak{K}[X]$, the family of all compactifications of a completely regular Hausdorff space X, was proven in [3] (page 465, Theorem 20.4). The fact that in order for $\mathfrak{K}[X]$ to be a complete lattice, it is sufficient that X be locally compact, was later rediscovered by Boboc and Siretchi [1] (page 7, Theoreme 2.2).

It will be convenient to make the convention that *from this point on any topological space is assumed to be 0-dimensional unless something is specifically stated to the contrary.* In view of the fact that $\mathfrak{Z}[X]$ is a complete lattice if and only if X is locally compact, several questions immediately present themselves. First, even though X is not locally compact, is it possible that $\mathfrak{Z}[X]$ might still be a lattice? The answer to this question is yes. An example due to Visliseni and Flaksmaier serves to verify this. Secondly, do there exist spaces X for which $\mathfrak{Z}[X]$ is not a lattice? The answer to this question is also yes. If X contains a countable, metrizable dense-in-itself clopen subset, then $\mathfrak{Z}[X]$ fails quite badly at being a lattice. Indeed, $\mathfrak{Z}[X]$ contains a family $\mathscr{F}$ consisting of c (the cardinality of the

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continuum) compactifications with the property that no two-element subset of $\mathscr{F}$ has a lower bound much less a greatest lower bound.

Perhaps a few words about the techniques used to obtain the previously mentioned results are in order. The problem of investigating $\mathfrak{Z}[X]$ is reduced to what seems to be the easier problem of investigating a partially ordered family of Boolean rings. Specifically, let $\mathfrak{B}[X]$ denote the family of all Boolean rings of clopen subsets of X which form a basis for the open subsets of X and partially order $\mathfrak{B}[X]$ by set inclusion. It is shown that $\mathfrak{Z}[X]$ is order isomorphic to $\mathfrak{B}[X]$. Consequently, an investigation of the order structure of $\mathfrak{Z}[X]$ may be made via a study of the order structure of $\mathfrak{B}[X]$.

2. Boolean rings of clopen sets

Let $\mathscr{D}(X)$ be any family of clopen subsets of X which is closed under finite unions, complementation and contains X as an element. Note that $\mathscr{D}(X)$ is then closed under finite intersections since for any $A, B \in \mathscr{D}(X)$, $A \cap B = \complement(\complement A \cup \complement B)$ ($\complement A$ denotes the complement of A in X). It is known that $\mathscr{D}(X)$ is a ring if we define sums and products by $A+B = (A \cup B)-(A \cap B)$ and $AB = A \cap B$. We will refer to such a ring as a Boolean ring of clopen sets or sometimes more simply as a ring of clopen sets. We note that the empty set $\emptyset$ is the additive identity of this ring and X is the multiplicative identity.

The following lemma, together with the fact that $A+X = \complement A$ for any clopen subset A implies that in order to prove that a family $\mathscr{F}$ of clopen subsets is a Boolean ring as defined here, it is sufficient to show that $X \in \mathscr{F}$ and $A+B$ and $AB \in \mathscr{F}$ when $A, B \in \mathscr{F}$.

LEMMA (2.1). *For any finite family $\{A_i\}_{i=1}^N$ of clopen subsets of X,*

$$\bigcup \{A_i\}_{i=1}^N = \sum_{1 \leqq i \leqq N} A_i + \sum_{1 \leqq i < j \leqq N} A_i A_j + \cdots + \sum_{1 \leqq i < j < k \leqq N} A_i A_j A_k + \cdots + A_1 A_2 A_3 \cdots A_N.$$

The proof of this result follows easily using finite induction and will not be given. The proof of the following result will also be omitted for the same reason.

LEMMA (2.2). *If $\{A_i\}_{i=1}^N$ are elements of a Boolean ring and $A_i \subseteqq B_i$ for each i, then*

$$\sum_{i=1}^N A_i \subseteqq \bigcup_{i=1}^N B_i.$$

DEFINITION (2.3). The symbol $\mathscr{A}(X)$ denotes the Boolean ring of all clopen subsets of X. The symbol $\mathscr{B}(X)$ denotes the Boolean ring consisting of all compact clopen subsets together with their complements.

We recall that $\mathfrak{B}[X]$ denotes the partially ordered family of all Boolean rings of clopen sets whose elements form a basis for the open subsets of X. $\mathscr{A}(X)$ belongs to $\mathfrak{B}[X]$, in fact, $\mathscr{A}(X)$ is the largest element of $\mathfrak{B}[X]$. However, $\mathscr{B}(X)$ need not form a basis for the open subsets of X. The next result shows precisely when $\mathscr{B}(X)$ belongs to $\mathfrak{B}[X]$.

LEMMA (2.4). *$\mathscr{B}(X)$ belongs to $\mathfrak{B}[X]$ if and only if X is locally compact.*

PROOF. It follows easily that if X is locally compact, then $\mathscr{B}(X) \in \mathfrak{B}[X]$ so we prove only the converse. Suppose then $\mathscr{B}(X) \in \mathfrak{B}[X]$ and let x be any element of X. We must show that x belongs to a compact neighborhood. This is obviously the case if X is compact. Let us consider the case where X is not compact. Then there exists an open set G containing x whose complement is not compact. Since the sets in $\mathscr{B}(X)$ form a basis for the open sets of $X, x \in H \subseteqq G$ for some $H \in \mathscr{B}(X)$. Furthermore, since $\complement G \subseteqq \complement H$ and $\complement G$ is not compact, $\complement H$ cannot be compact. Thus H must be compact.

DEFINITION (2.5). An ideal I of a Boolean ring $\mathscr{D}(X)$ is said to be fixed if there exists an element $x \in X$ such that $x \notin A$ for each $A \in I$. Otherwise, I is said to be free.

THEOREM (2.6). *The following statements are equivalent for any 0-dimensional space X:*

(2.6.1). *X is compact.*

(2.6.2) *Every ideal of $\mathscr{A}(X)$ is fixed.*

(2.6.3) *Every maximal ideal of $\mathscr{A}(X)$ is fixed.*

PROOF. Suppose X is compact and I is any ideal of $\mathscr{A}(X)$. Then $X \notin I$ (since X is the identity of $\mathscr{A}(X)$) and it follows from Lemma (2.1) that no finite union of sets from I is equal to X. Thus $\cup \{A : A \in I\} \neq X$ since X is compact, i.e., there exists an element $x \in X$ such that $x \notin A$ for each $A \in I$. Thus I is a fixed ideal.

It is evident that (2.6.2) implies (2.6.3) so we show (2.6.3) implies (2.6.1). Let $\mathscr{F}$ be any family of clopen sets with property that no finite union of members of $\mathscr{F}$ is equal to X. Let

$$I = \{\sum_{i=1}^{N} A_i B_i : A_i \in \mathscr{A}(X), B_i \in \mathscr{F}, n \in N\}$$

(N denotes the set of positive integers). In view of Lemma (2.2), the assumption that $X \in I$ results in $X = \sum_{i=1}^{N} A_i B_i \subseteqq \bigcup_{i=1}^{N} B_i$ which is a contradiction. Thus $X \notin I$ and it follows that I is an ideal of $\mathscr{A}(X)$. Then I is contained in a maximal ideal M of $\mathscr{A}(X)$ which, by (2.6.3), is fixed. This implies that $X \neq \cup \{B : B \in \mathscr{F}\}$ and it follows that X is compact.

LEMMA (2.7). *For each maximal ideal M of the Boolean ring $\mathscr{D}(X)$, there exists a unique epimorphism φ_M which maps $\mathscr{D}(X)$ onto the two element field $\mathfrak{F} = \{0, 1\}$ such that M is the kernel of φ_M.*

PROOF. Since $\mathscr{D}(X)/M$ is a Boolean field, there exists an isomorphism φ_1 mapping $\mathscr{D}(X)/M$ onto $\mathfrak{F}$. Let φ_2 be the canonical epimorphism from $\mathscr{D}(X)$ onto $\mathscr{D}(X)/M$ and define $\varphi_M = \varphi_1 \circ \varphi_2$. Then φ_M is an epimorphism from $\mathscr{D}(X)$ onto $\mathfrak{F}$. The uniqueness of the epimorphism follows easily.

LEMMA (2.8). *Suppose A is a clopen subset of X which is not compact. Then $\varphi_M(A) = 1$ for some free maximal ideal M of $\mathscr{A}(X)$.*

PROOF. Since A is not compact, there exists a family $\{H_\alpha : \alpha \in \Lambda\}$ of clopen subsets of A with the finite intersection property such that $\cap \{H_\alpha : \alpha \in \Lambda\} = \emptyset$. Let I denote the ideal of $\mathscr{A}(X)$ generated by $\complement A$ together with the family $\{\complement H_\alpha : \alpha \in \Lambda\}$. Now $X \in I$ implies that

$$X = B(\complement A) + \sum_{i=1}^{N} B_i(\complement H_{\alpha_i}) \subseteqq \complement A \cup [\bigcup_{i=1}^{N} \complement H_{\alpha_i}]$$

which, in turn, implies $\bigcap_{i=1}^{N} H_{\alpha_i} = \emptyset$. This is a contradiction and we must conclude that $X \notin I$, i.e., I is a proper ideal of $\mathscr{A}(X)$. Then I is contained in a maximal ideal M of $\mathscr{A}(X)$. The assumption that M is fixed leads to the conclusion that for some $x \in X$ and each $B \in M$, $x \notin B$. This implies that $x \in \cap \{H_\alpha : \alpha \in \Lambda\}$ which is a contradiction. Thus, M is free. Since $\complement A \in M$, $\varphi_M(\complement A) = 0$ which implies that $\varphi_M(A) = 1$.

LEMMA (2.9). *Let $\mathscr{D}(X) \in \mathfrak{B}[X]$ and let M and N be two free maximal ideals of $\mathscr{D}(X)$. Then*

$$\mathscr{D}(X; M, N) = \{A \in \mathscr{D}(X) : \varphi_M(A) = \varphi_N(A)\}$$

belongs to $\mathfrak{B}[X]$.

PROOF. For any $A, B \in \mathscr{D}(X; M, N)$, $\varphi_M(A+B) = \varphi_N(A+B)$ and $\varphi_M(AB) = \varphi_N(AB)$. Furthermore, $\varphi_M(X) = 1 = \varphi_N(X)$. Thus, $\mathscr{D}(X; M, N)$ is a Boolean ring of clopen sets. To complete the proof, we must show that $\mathscr{D}(X; M, N)$ is a basis for the open subsets of X. Suppose G is any open subset of X and $x \in G$. Since M and N are free ideals, there exist $A_1 \in M$ and $A_2 \in N$ such that $x \in A_1 \cap A_2$. Moreover, since $\mathscr{D}(X)$ is a basis for the open subsets of X, $x \in A_3 \subseteqq G$ for some $A_3 \in \mathscr{D}(X)$. Hence, $x \in A_1A_2A_3 \subseteqq G$ which completes the proof since $\varphi_M(A_1A_2A_3) = 0 = \varphi_N(A_1A_2A_3)$.

LEMMA (2.10). $\mathscr{B}(X) = \cap\, \mathfrak{B}[X]$.

PROOF. Suppose A is a compact clopen set and $\mathscr{D}(X)$ is any element of $\mathfrak{B}[X]$. Then $A = A_1 \cup A_2 \cup A_3 \cdots \cup A_N$ for a finite number of clopen subsets $A_i \in \mathscr{D}(X)$. This implies $A \in \mathscr{D}(X)$ since $\mathscr{D}(X)$ is closed under finite

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unions. Since, in addition, $\mathscr{D}(X)$ is closed under complementation, $\complement A \in \mathscr{D}(X)$. This implies that $\mathscr{B}(X) \subseteqq \mathscr{D}(X)$ and hence that $\mathscr{B}(X) \subseteqq \cap \mathfrak{B}[X]$.

Now suppose A is a clopen subset of X which does not belong to $\mathscr{B}(X)$. Then neither A nor $\complement A$ is compact. By Lemma (2.8), there exist free maximal ideals M and N of $\mathscr{A}(X)$ such that $\varphi_M(A) = 1 = \varphi_N(\complement A)$. Thus, $\varphi_N(A) = 0 \neq \varphi_M(A)$ which implies that $A \notin \mathscr{A}(X; M, N)$. It follows from Lemma (2.9) that $A \notin \cap \mathfrak{B}[X]$.

Now we are in a position to prove the main result of this section.

THEOREM (2.11). *The partially ordered family $\mathfrak{B}[X]$ is always a complete upper semi-lattice. It is a complete lattice if and only if X is locally compact.*

PROOF. Let $\mathscr{F}$ be any subfamily of $\mathfrak{B}[X]$ and let $\mathscr{D}(X)$ be the intersection of all rings in $\mathfrak{B}[X]$ which contain each ring in $\mathscr{F}$. One easily verifies that $\mathscr{D}(X)$ is the least upper bound for $\mathscr{F}$. Thus, $\mathfrak{B}[X]$ is a complete upper semi-lattice.

Now suppose X is locally compact. Then by Lemma (2.4), $\mathscr{B}(X) \in \mathfrak{B}[X]$. This, together with Lemma (2.10), implies that $\mathscr{B}(X)$ is the smallest element of $\mathfrak{B}[X]$. Since any complete upper semi-lattice containing a smallest element is a complete lattice, the desired conclusion follows.

On the other hand, if $\mathfrak{B}[X]$ is a complete lattice, it contains a smallest element which must necessarily be of the form $\cap \mathfrak{B}[X]$. This, together with Lemmas (2.4) and (2.10), implies that X is locally compact.

3. $\mathfrak{Z}[X]$ and $\mathfrak{B}[X]$

Our main goal in this section is to show that $\mathfrak{Z}[X]$ is order isomorphic to $\mathfrak{B}[X]$. Before proving the first result of this section, it will be convenient to introduce some notation. One easily shows that for any $x \in X$, $\{A \in \mathscr{A}(X) : x \notin A\}$ is a maximal ideal of $\mathscr{A}(X)$. We will denote this ideal by M_x. It follows from Theorem (2.6) that X is compact if and only if every maximal ideal of $\mathscr{A}(X)$ is of the form M_x for some $x \in X$.

THEOREM (3.1). *Suppose X and Y are compact. Then a mapping φ from $\mathscr{A}(X)$ into $\mathscr{A}(Y)$ is a monomorphism with the property that $\varphi(X) = Y$ if and only if there exists a continuous function h mapping Y onto X such that for any clopen set A in $\mathscr{A}(X)$, $\varphi(A) = h^{-1}[A]$.*

PROOF. First suppose φ is a monomorphism from $\mathscr{A}(X)$ into $\mathscr{A}(Y)$ with the property that $\varphi(X) = Y$ and for any point $y \in Y$, let $M = \{A \in \mathscr{A}(X) : y \notin \varphi(A)\}$. Suppose A is any element of $\mathscr{A}(X)$. Then

$$Y = \varphi(X) = \varphi(A + \complement A) = \varphi(A) + \varphi(\complement A).$$

Moreover,

$$\emptyset = \varphi(\emptyset) = \varphi(A \cdot \complement A) = \varphi(A) \cdot \varphi(\complement A).$$

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Thus, $\varphi(\complement A) = \complement\varphi(A)$. If $A \notin M$, then $y \in \varphi(A)$ which implies that $y \notin \complement\varphi(A) = \varphi(\complement A)$. Thus, $\complement A \in M$. Since $X = A + \complement A$, there is no ideal of $\mathscr{A}(X)$ $(\neq \mathscr{A}(X))$ which properly contains M, i.e., M is a maximal ideal of $\mathscr{A}(X)$. As we observed previously, it follows from Theorem (2.6) that there exists a point $x \in X$ such that $M = M_x$. This point x is unique and we define $h(y) = x$. Thus, $M_{h(y)} = \{A \in \mathscr{A}(X) : y \notin \varphi(A)\}$. Using this fact, we see that for any $A \in \mathscr{A}(X)$, the following statements are successively equivalent:

$$y \in h^{-1}[A],\ h(y) \in A,\ A \notin M_{h(y)},\ y \in \varphi(A).$$

Therefore, $\varphi(A) = h^{-1}[A]$. The continuity of h is a consequence of the previous statement and the fact that $\mathscr{A}(X)$ is a base for the open subsets of X.

Suppose $h[Y]$ is not dense in X, then there exists a nonempty clopen subset A of X which does not intersect $h[Y]$. But then $\varphi(A) = h^{-1}[A] = \emptyset$ which contradicts the fact that φ is injective. Hence $h[Y]$ is dense in X. This implies that $h[Y] = X$ since Y is compact and X is Hausdorff.

Now suppose φ is a mapping from $\mathscr{A}(X)$ into $\mathscr{A}(Y)$ and there exists a continuous function h from Y onto X such that $\varphi(A) = h^{-1}[A]$. It is immediate that $\varphi(X) = Y$ and that φ is a homomorphism. Furthermore, if $\emptyset = \varphi(A) = h^{-1}[A]$, then $A = \emptyset$, i.e., φ is injective.

Theorem (3.2) *Suppose X and Y are compact. A mapping φ from $\mathscr{A}(X)$ onto $\mathscr{A}(Y)$ is an isomorphism if and only if there exists a homeomorphism k from X onto Y such that $\varphi(A) = k[A]$ for each $A \in \mathscr{A}(X)$.*

Proof. Sufficiency follows easily. To prove necessity we use the previous theorem. First of all we note that $\varphi(X) = Y$ and therefore there exists a continuous function h mapping Y onto X such that $\varphi(A) = h^{-1}[A]$ for each $A \in \mathscr{A}(X)$. Similarly, there exists a continuous function k mapping X onto Y such that $\varphi^{-1}(B) = k^{-1}[B]$ for each $B \in \mathscr{A}(Y)$. For any clopen subset A of X, $A = \varphi^{-1}(\varphi(A)) = k^{-1}[h^{-1}[A]] = (h \circ k)^{-1}[A]$. We can use this fact to prove that k is a bijection. Suppose a and b are distinct points of X. Then some clopen subset A of X contains a but not b. Then $a \in (h \circ k)^{-1}[A]$ and $b \notin (h \circ k)^{-1}[A]$, i.e., $h(k(a)) \in A$ and $h(k(b)) \notin A$. Thus $k(a) \neq k(b)$. Similarly, h is a bijection from Y onto X. This proves that both h and k are homeomorphisms. To see that $k = h^{-1}$, let any $x \in X$ be given. For any clopen subset A of X containing x, we have $x \in A = (h \circ k)^{-1}[A]$, i.e., $h(k(x)) \in A$. Since the clopen subsets of X form a basis for the open subsets of X, this implies that $h(k(x)) = x$, that is, $k(x) = h^{-1}(x)$. This completes the proof.

Now let αX be a (0-dimensional) compactification of X. We define a mapping φ_α from $\mathscr{A}(\alpha X)$ into $\mathscr{A}(X)$ by $\varphi_\alpha(A) = A \cap X$. We will denote the image of $\mathscr{A}(\alpha X)$ under φ_α by $\Gamma(\alpha X)$. The proof of the following result is routine and is omitted.

LEMMA (3.3). *The mapping φ_α is a monomorphism from $\mathscr{A}(\alpha X)$ into $\mathscr{A}(X)$ and $\Gamma(\alpha X)$ belongs to $\mathfrak{B}[X]$.*

Before stating the next result, we recall that for any two compactifications αX and γX, $\alpha X \leqq \gamma X$ if there exists a continuous function mapping γX onto αX which keeps X pointwise fixed.

THEOREM (3.4). $\alpha X \leqq \gamma X$ *if and only if* $\Gamma(\alpha X) \subseteqq \Gamma(\gamma X)$.

PROOF. First suppose $\alpha X \leqq \gamma X$. Then there exists a continuous function h mapping γX onto αX which keeps X pointwise fixed. Since X is dense in γX and γX is Hausdorff, it follows that h must map $\gamma X - X$ onto $\alpha X - X$. Let A be any element of $\Gamma(\alpha X)$. Then $A = X \cap A^*$ where A^* is a clopen subset of αX and it follows that $X \cap h^{-1}[A^*] \in \Gamma(\gamma X)$. But

$$X \cap h^{-1}[A^*] = h^{-1}[X] \cap h^{-1}[A^*] = h^{-1}[X \cap A^*] = h^{-1}[A] = A.$$

This proves that $\Gamma(\alpha X) \subseteqq \Gamma(\gamma X)$.

Now suppose $\Gamma(\alpha X) \subseteqq \Gamma(\gamma X)$. Then $\varphi = \varphi_\gamma^{-1} \circ \varphi_\alpha$ is a monomorphism from $\mathscr{A}(\alpha X)$ into $\mathscr{A}(\gamma X)$ with the property that $\varphi(\alpha X) = \gamma X$. By Theorem (3.1), there exists a continuous function h mapping γX onto αX such that $\varphi(A) = h^{-1}[A]$ for each $A \in \alpha X$. Then

$$h^{-1}[A] = \varphi(A) = \varphi_\gamma^{-1} \circ \varphi_\alpha(A) = \varphi_\gamma^{-1}(X \cap A)$$

which implies that $\varphi_\gamma(h^{-1}[A]) = X \cap A$. Therefore,

$$X \cap h^{-1}[A] = X \cap A. \tag{3.4.1}$$

Now suppose $x \in X$ and A is any clopen subset of αX which contains x. Then $x \in X \cap A$ and it follows from (3.4.1) that $x \in h^{-1}[A]$, i.e., $h(x)$ also belongs to A. This implies that $h(x) = x$ since αX is 0-dimensional. Thus $\alpha X \leqq \gamma X$ and the theorem is proved.

Suppose $\Gamma(\alpha X) = \Gamma(\gamma X)$. Then by Theorem (3.4), $\alpha X \leqq \gamma X$ and $\gamma X \leqq \alpha X$. This means that there exists a continuous function h mapping γX onto αX which keeps X pointwise fixed and a continuous function k mapping αX onto γX which keeps X pointwise fixed. Both $h \circ k$ and $k \circ h$ are the identity maps when restricted to the dense subspace X. Thus, $h \circ k$ is the identity on αX and $k \circ h$ is the identity on γX. This means $k = h^{-1}$ and h is a homeomorphism from γX onto αX. Hence, αX and γX are equivalent. Since we do not distinguish between equivalent compactifications, it follows that the mapping Γ is injective. This fact, together with Theorem (3.4), results in

COROLLARY (3.5). *The mapping Γ is an order isomorphism from $\mathfrak{Z}[X]$ into $\mathfrak{B}[X]$.*

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We are now in a position to state and prove the main result of this section.

THEOREM (3.6). *$\mathfrak{Z}[X]$ and $\mathfrak{B}[X]$ are order isomorphic.*

PROOF. In view of the previous corollary, it will be sufficient to show that for any ring $\mathscr{D}(X) \in \mathfrak{B}[X]$, there exists a compactification αX such that $\Gamma(\alpha X) = \mathscr{D}(X)$. Let $\mathscr{M}$ be the family of all maximal ideals of $\mathscr{D}(X)$. Let αX be an index set for $\mathscr{M}$ with the convention that if a maximal ideal M is fixed (i.e., $M = \{A \in \mathscr{D}(X) : x \notin A\}$ for some $x \in X$), then its index is that unique point x which belongs to the complement of each clopen set in M. Throughout this proof, the index of an ideal will appear as a superscript. Following this convention, we define, for each $A \in \mathscr{D}(X)$,

$$H_A = \{p \in \alpha X : A \in M^p\}.$$

It follows easily that $H_A \cup H_B = H_{AB}$. We topologize αX by taking $\{H_A : A \in \mathscr{D}(X)\}$ as a basis for the closed subsets of αX. This is the structure space of the ring $\mathscr{D}(X)$. It is known [2, p. 111, 7M] that the structure space of any commutative ring with identity is a compact T_1 space.

Suppose $A \in \mathscr{D}(X)$ and M^p is a maximal ideal of $\mathscr{D}(X)$ which does not contain A. Then the ideal generated by M^p together with A is all of $\mathscr{D}(X)$ and we have $X = B+AC$ for suitable $B \in M^p$ and $C \in \mathscr{D}(X)$. Thus

$$\complement(AC) = X+AC = B \in M^p.$$

But then, $\complement A \subseteqq \complement(AC)$ implies that

$$\complement A = \complement A \cdot (\complement(AC)) \in M^p.$$

Consequently, for any maximal ideal M^p of $\mathscr{D}(X)$ and any $A \in \mathscr{D}(X)$, either A or $\complement A$ belongs to M^p. It follows from this that for any $A \in \mathscr{D}(X)$, $H_A \cup H_{\complement A} = \alpha X$. Since, in addition, $H_A \cap H_{\complement A} = \emptyset$, it follows that for every $A \in \mathscr{D}(X)$, H_A is a clopen subset of αX. Since $\{H_A : A \in \mathscr{D}(X)\}$ is a basis for the closed sets of αX, it follows that αX is 0-dimensional. Moreover, it is Hausdorff since any 0-dimensional T_1 space is Hausdorff. One easily shows that $H_A \cap X = X-A$ for each clopen subset A of X. From this it follows that X is indeed a subspace of αX. To show that X is dense in αX, suppose $H_A \neq \emptyset$. Then $p \in H_A$ for some $p \in \alpha X$ which is equivalent to saying that $A \in M^p$ for some maximal ideal M^p of $\mathscr{D}(X)$. Then $A \neq X$ and there exists a point $x \in X-A$. It follows that $M^x = \{B \in \mathscr{D}(X) : x \in X-B\}$ is a maximal ideal of $\mathscr{D}(X)$ which contains A. Thus, $x \in H_A$ which proves that X is dense in αX.

We have established the fact that αX is a 0-dimensional compactification of X. To complete the proof, we need only show that $\Gamma(\alpha X) = \mathscr{D}(X)$. Let φ_α be the monomorphism (Lemma (3.3)) mapping $\mathscr{A}(\alpha X)$ into $\mathscr{A}(X)$

which is defined by $\varphi_\alpha(A) = X \cap A$. Since $\varphi_\alpha(H_A) = X - A$ for each $A \in \mathscr{D}(X)$, it follows that $\mathscr{D}(X) \subseteqq \varphi_\alpha[\mathscr{A}(\alpha X)] = \Gamma(\alpha X)$. Since, in any Boolean ring, $A \cup B = A + B + AB$, we have $\varphi(A \cup B) = \varphi(A) \cup \varphi(B)$ for any homomorphism φ from one such ring into another. Now let $A \in \Gamma(\alpha X)$. Then $A = X \cap A^*$ for some $A^* \in \mathscr{A}(\alpha X)$ and since A^* is compact, $A^* = \cup \{H_{B_i}\}_{i=1}^N$ for $\{B_i\}_{i=1}^N \subseteqq \mathscr{D}(X)$. Thus,

$$A = \varphi_\alpha(A^*) = \varphi_\alpha(\bigcup\{H_{B_i}\}_{i=1}^N) = \bigcup\{\varphi_\alpha(H_{B_i})_{i=1}^N\} = \bigcup\{X - B_i\}_{i=1}^N \in \mathscr{D}(X).$$

Thus, $\Gamma(\alpha X) = \mathscr{D}(X)$ and the proof is complete.

4. The partially ordered family $\mathfrak{Z}[X]$

Because of Theorem (3.6), the problem of investigating the order structure of $\mathfrak{Z}[X]$ has been reduced to the problem of investigating the order structure of $\mathfrak{B}[X]$. As an immediate consequence of Theorems (3.6) and (2.11), we obtain

THEOREM (4.1). *The partially ordered family $\mathfrak{Z}[X]$ of all 0-dimensional compactifications of X is always a complete upper semi-lattice. It is a complete lattice if and only if X is locally compact.*

Suppose we consider the case where X is not locally compact. In view of Theorem (4.1), $\mathfrak{Z}[X]$ is not a complete lattice. It is reasonable, however, to ask if $\mathfrak{Z}[X]$ might, at least, be a lattice. An example due to J. Visliseni and J. Flaksmaier [6] shows that there are cases where the answer is affirmative. We will discuss briefly this example which they construct in the proof of their Theorem 1. For more detail, one may consult [6].

Let D be an infinite discrete space and, as is customary, let βD denote the Stone-Čech compactification of D. Then $\beta D - D$ contains a copy T of D whose closure in βD is homeomorphic to βD and is contained in $\beta D - D$. Let $X = \beta D - T$. Then $\beta X = \beta D$ and $\beta X - X = T$. The space X is 0-dimensional. Furthermore, every compactification of X is obtained by taking a finite number of finite subsets of $\beta X - X$ and identifying each of these subsets with a point. It follows that $\mathfrak{Z}[X]$ is a lattice. It also follows that every compactification of X is 0-dimensional, i.e., $\mathfrak{Z}[X] = \mathfrak{K}[X]$ (the family of all compactifications of X) in this case. In general, $\mathfrak{Z}[Y]$ is a proper subset of $\mathfrak{K}[Y]$. In fact, Theorem (2.2) of [4] states that if Y is any locally compact normal Hausdorff (not necessarily 0-dimensional) space which contains an infinite discrete closed subset and K is any Peano space, then there exists a compactification αY of Y such that $\alpha Y - Y$ is homeomorphic to K. If K is not the space consisting of one point, then such a compactification will not be 0-dimensional.

At this point we still have not exhibited a space X where $\mathfrak{Z}[X]$ is not a lattice. The next result indicates that such spaces are rather abundant and that some of then fail quite badly at being lattices.

THEOREM (4.2). *Let X be a space which contains a countable dense-in-itself metrizable clopen subset. Then there exists a subfamily $\mathscr{F}$ of $\mathfrak{Z}[X]$ of cardinality c with the property that no two-element subset of $\mathscr{F}$ has a lower bound much less a greatest lower bound.*

PROOF. In view of Theorem (3.6), it is sufficient to prove the analogous statement for $\mathfrak{B}[X]$. By hypothesis, $X = Q \cup Y$ where Q is a countable dense-in-itself metrizable clopen subset of X. According to the statement following the corollary on page 107 of [5], we may take Q to be the space of all rational numbers. Define two real numbers k and t to be equivalent of $k-t$ is rational. Choose precisely one element from each of the equivalence classes and denote the resulting set by K. Then K has cardinality c and the difference of any two elements of K is irrational.

Now for each $k \in K$, we construct a Boolean ring $\mathscr{D}_k(X)$. For rational numbers, r and s, let

$$L_{kr} = \{q \in Q : q < k+r\},$$
$$R_{kr} = \{q \in Q : q > k+r\},$$
$$I_{krs} = \{q \in Q : k+r < q < k+s\}.$$

The family consisting of all L_{kr}, R_{kr} and I_{krs} is a basis for Q consisting of clopen sets. Let $\mathscr{D}_k(X)$ consist of all sets of the form $G \cup H$ where G is a clopen subset of Y and H is either empty, equal to Q or is a finite union of sets of the form L_{kr}, R_{kr} and I_{krs}. Then $\mathscr{D}_k(X)$ is closed under finite unions, complementation and contains X. Moreover, the sets in $\mathscr{D}_k(X)$ form a basis for the open subsets of X. Thus $\mathscr{D}_k(X) \in \mathfrak{B}[X]$.

Suppose k and t are two distinct elements of K. We want to show that $\mathscr{D}_k(X)$ and $\mathscr{D}_t(X)$ have no common lower bound. Suppose $A \in \mathscr{D}_k(X) \cap \mathscr{D}_t(X)$. Then $A = G_k \cup H_k = G_t \cup H_t$ where G_k and G_t are clopen subsets of Y, H_k is either empty, equal to Q or is a finite union of mutually disjoint sets of the form L_{kr}, R_{kr} and I_{krs} and H_t is either empty, equal to Q or is a finite union of mutually disjoint sets of the form L_{tr}, R_{tr} and I_{trs}. Since $H_k \cap G_t = \emptyset = H_t \cap G_k$, it follows that $H_k = H_t$. The assumption that $Q \neq H_k \neq \emptyset$ leads to one of the following conclusions:

(i) There exist rational numbers r_1 and r_2 such that $L_{kr_1} = L_{tr_2}$.

(ii) There exist rational numbers r_1 and r_2 such that $R_{kr_1} = R_{tr_2}$.

(iii) There exist rational numbers r_1, r_2, s_1, s_2 such that $r_1 < s_1$, $r_2 < s_2$ and $I_{kr_1s_1} = I_{tr_2s_2}$.

Each of these leads to the contradiction that $k-t = r_2-r_1$, i.e., that $k-t$ is rational. Therefore, we must conclude that $H_k = H_t$ is either $\emptyset$ or Q.

It follows that $\mathscr{D}_k(X) \cap \mathscr{D}_t(X)$ consists of all clopen subsets A of X such that either $Q \cap A = \emptyset$ or $Q \subseteqq A$. Consequently, no proper nonempty clopen subset of Q is the union of members of $\mathscr{D}_k(X) \cap \mathscr{D}_t(X)$. Since any element $\mathscr{D}(X) \in \mathfrak{B}[X]$ is a basis for X, it follows that there exists no $\mathscr{D}(X)$ which is smaller than both $\mathscr{D}_k(X)$ and $\mathscr{D}_t(X)$. This completes the proof.

References

[1] N. Boboc and Gh. Siretchi, 'Sur la compactification d'un espace topologique', *Bull. Math. Sci. Math. Phys. R. P. Roumaine* (N.S.) 5 (53) (1961), 155–165 (1964).
[2] L. Gillman and M. Jerison, *Rings of continuous functions* (D. Van Nostrand, New York, 1960).
[3] R. G. Lubben, 'Concerning the decomposition and amalgamation of points, upper semi-continuous collections and topological extensions', *Trans. Amer. Math. Soc.* 49 (1961), 410–466
[4] K. D. Magill, Jr.,' A note on compactifications', *Math. Zeit.* 94 (1966), 322–325.
[5] W. Sierpinski, *Introduction to general topology* (The University of Toronto Press, 1934).
[6] J. Visliseni and J. Flaksmaier, 'The power and structure of the lattice of all compact extensions of a completely regular space', *Doklady* 165, (1965), 1423–1425.

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