Compositio Mathematica **117:** 337–359, 1999.

Nonvanishing of Central Hecke L-Values and Rank of Certain Elliptic Curves*

TONGHAI YANG
Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, U.S.A.
e-mail: thyang@math.lsa.umich.edu

(Received: 13 May 1997; accepted in final form: 24 February 1998)

Abstract. Let $D \equiv 7 \bmod 8$ be a positive squarefree integer, and let h_D be the ideal class number of $E_D = \mathbb{Q}(\sqrt{-D})$. Let $d \equiv 1 \bmod 4$ be a squarefree integer relatively prime to D. Then for any integer $k \geqslant 0$ there is a constant $M = M(k)$, independent of the pair (D, d), such that if $(-1)^k = \operatorname{sign}(d)$, $(2k+1, h_D) = 1$, and $\sqrt{D} > (12/\pi)d^2(\log|d| + M(k))$, then the central L-value $L(k+1, \chi_{D,d}^{2k+1}) > 0$. Furthermore, for $k \leqslant 1$, we can take $M(k) = 0$. Finally, if $D = p$ is a prime, and $d > 0$, then the associated elliptic curve $A(p)^d$ has Mordell–Weil rank 0 (over its definition field) when $\sqrt{D} > (12/\pi)d^2 \log d$.

Mathematics Subject Classifications (1991): 11G05, 11M20, 14H52.

Key words: central Hecke L-value, elliptic curves, eigenfunction, nonvanishing.

0. Introduction

Let $D \equiv 3 \bmod 4$ be a positive squarefree integer, and let $d \equiv 1 \bmod 4$ be a squarefree integer relatively prime to D. We consider Hecke characters χ of $E_D = \mathbb{Q}(\sqrt{-D})$ of conductor $d\sqrt{-D}\mathcal{O}$ satisfying

(1) $\chi(\bar{\mathfrak{A}}) = \overline{\chi(\mathfrak{A})}$ for every ideal of E_D relatively prime to the conductor, and
(2) $\chi(\alpha\mathcal{O}) = \pm\alpha$ for every principal ideal relatively prime to the conductor.

Here $\mathcal{O}$ is the ring of integers of E_D. There are h_D such Hecke characters for each pair (D, d), differing from each other by an ideal class character of E_D, where h_D is the ideal class number of E_D. We denote such a Hecke character by $\chi_{D,d}$. These Hecke characters were studied by Rohrlich ([Roh2-3]), who also allowed D or d to be even. In particular, he proved, that for almost all pairs (D, d) such that $D > |d|^{39+\varepsilon}$ and the root number of $\chi_{D,d}$ is one, the central L-value $L(1, \chi_{D,d}) \neq 0$. Here ε is any positive number. Rohrlich and Montgomery ([MR]) also proved a more definite result asserting that $L(1, \chi_{D,1}) \neq 0$ if and only if the root number of $\chi_{D,1}$ is one. Rodriguez Villegas further gave a nice formula in

* Partially supported by the NSF grant DMS-9304580.

https://doi.org/10.1023/A:1000934108242 Published online by Cambridge University Press

[RV1-2] for the central L-value $L(1, \chi_{D,1})$ for $D \equiv 7 \bmod 8$. From this formula the nonvanishing of the central L-value becomes obvious. A formula for the central L-value $L(k+1, \chi_{D,1}^{2k+1})$ was obtained by similar technique in [RVZ]. Using a different method developed in [Ya1], Rodriguez Villegas and the author discovered that similar formula is valid for Hecke characters $\chi_{D,d}^{2k+1}$, where every prime divisor of d splits in E_D. In fact, such a formula exists for a whole class of Hecke characters of a CM number field of any degree ([RVY]). In this paper, we use the same method to derive a formula for $L(k+1, \chi_{D,d}^{2k+1})$ without any condition on d (Theorem 2.1), which enables us to prove the following nonvanishing result in Section 3.

MAIN THEOREM. *Let* $D \equiv 7 \bmod 8$ *be a positive squarefree integer, and let* h_D *be the ideal class number of* $E_D = \mathbb{Q}(\sqrt{-D})$. *Let* $d \equiv 1 \bmod 4$ *be a squarefree integer relatively prime to* D. *Then for any integer* $k \geqslant 0$ *there is a constant* $M = M(k)$, *independent of the pair* (D, d), *such that if* $(-1)^k = \operatorname{sign}(d)$, $(2k+1, h_D) = 1$, *and* $\sqrt{D} > (12/\pi)d^2(\log|d| + M(k))$, *then the central* L*-value* $L(k+1, \chi_{D,d}^{2k+1}) > 0$. *Furthermore, for* $0 \leqslant k \leqslant 1$ *we can take* $M(k) = 0$.

Refinements and comments will also be given in Section 3. The Hecke characters considered here are arithmetic in nature; each such character has an associated CM motive (see, for example, [Sha]). In particular, when $k = 0$ and $D = p$ is a prime, the character $\chi_{p,d}$ is very closely related to the elliptic curve $A(p)^d$ over a number field F studied by Gross ([Gro]), using Shimura's theory on CM Abelian varieties ([Sh1-2]). He proved, in particular, by means of descent theory that $A(p)$ has Mordell–Weil rank 0 over F. Combining a theorem of Rubin ([Ru, Corollary 2.2]) with the main theorem, one has

COROLLARY. *Let* $p \equiv 7 \bmod 8$ *be a prime, and let* $d \equiv 1 \bmod 4$ *be a positive squarefree integer not divisible by* p *such that* $\sqrt{p} > (12/\pi)d^2 \log d$. *Then the elliptic curve* $A(p)^d$ *has Modell–Weil rank* 0 *over* $\mathbb{Q}(j)$. *Here* $j = j(1 + \sqrt{-p}/2)$ *is the* j*-invariant of* $A(p)$.

1. Eigenfunctions of Weil Representations

In this section, we will explicitly construct eigenfunctions of the local Weil representation of the unitary group of one variable in terms of a Schrödinger model. They are needed in the next section to derive an explicit formula for the central Hecke L-value $L(k+1, \chi_{D,d}^{2k+1})$ from the main formula in [Ya1]. We consider general local fields instead of just $\mathbb{Q}_p$, since it is not much harder. In the real case, the eigenfunctions are essentially classical Hermite functions as we will see in Lemma 1.1. For the p-adic case ($p \neq 2$), eigenfunctions were explicitly constructed in [Ya2] by means of a lattice model. So we only need to transfer the results to the Schrödinger model. We will state the results in this section and give the proof, which is quite technical and lengthy in the appendix.

Let F be a local field and let $E = F(\delta)$ be a quadratic extension of F. Assume that $\bar{\delta} = -\delta$ and $\Delta = \delta^2 \in F$. Let ψ be a fixed nontrivial character of F and let $\psi_E = \psi \circ \mathrm{tr}_{E/F}$. Given $\alpha \in F^*$, and a character χ of E^* such that $\chi|_{F^*} = \varepsilon$ is the quadratic character of F^* associated to E/F, there is a well-defined Weil representation $\omega_{\alpha,\chi}$ of $G = U(1) = E^1$ on the space $S(F)$ of Schwartz functions on F (also depending on δ and ψ) ([Ku], see also the appendix). By the epsilon dichotomy ([HKS, Corollary 8.5]), one has

$$S(F) = \oplus \mathbb{C}\,\phi_{\tilde{\eta}}, \tag{1.1}$$

where the sum runs over all characters η of E^1 satisfying

$$\varepsilon(\tfrac{1}{2}, \chi\tilde{\eta}, \tfrac{1}{2}\psi_E)\chi\tilde{\eta}(\delta) = \varepsilon(\alpha) \tag{1.2}$$

and ϕ_η is an eigenfunction of $(G, \omega_{\alpha,\chi})$ with eigencharacter η. Here $\tilde{\eta}(z) = \eta(z/\bar{z})$. The task is to give an explicit formula for ϕ_η. First we consider the case $F = \mathbb{R}$. Recall that every character of $\mathbb{C}^*$ is of the form $\chi_n(z) = (z/|z|)^n$, and that every character of $\mathbb{C}^1$ is of the form $\eta_l(z) = z^l$.

LEMMA 1.1 ([Ya1, Thm. 2.18]). *Let $F = \mathbb{R}$ and $\psi(x) = \mathrm{e}^{2\pi i x}$. Assume $\chi\tilde{\eta}(z) = (|z|/z)^{2m+1}$ and $\delta \in i\mathbb{R}_{>0}$. Then $\bar{\eta}$ occurs in $\omega_{\alpha,\chi}$ if and only if $k = m\ \mathrm{sign}(\alpha) - 1 - \mathrm{sign}(\alpha)/2 \geqslant 0$. When $k \geqslant 0$, $\phi^k_{|\delta|,|\alpha|}(x) = \phi^k(\sqrt{|\delta^3\alpha|}x)$ is an eigenfunction of $\omega_{\alpha,\chi}$ with eigencharacter $\bar{\eta}$. Here*

$$\phi^0 = \mathrm{e}^{-\pi x^2}, \quad \textit{and} \quad \phi^k(x) = \frac{1}{2^k}\left(x - \frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}x}\right)^k \phi^0(x).$$

Moreover

$$\left\langle \phi^k_{|\delta|,|\alpha|}, \phi^k_{|\delta|,|\alpha|} \right\rangle = \frac{1}{\sqrt{2|\delta^3\alpha|}} \frac{k!}{(4\pi)^k}.$$

Notice that there is a unique polynomial $H_k(x)$ of degree k (the kth Hermite polynomial) such that

$$\phi^k(x) = H_k(x)\phi^0(x). \tag{1.3}$$

It is easy to check that $H_0 = 1$ and $H_1 = x$. In general, H_k has the same parity as k.

For the rest of this section, we assume that F is a p-adic local field with $p \neq 2$ and that δ is a uniformizer or a unit of E depending on whether E/F is ramified or not. Let $\psi' = (\alpha\delta/4)\psi_E$, and let $n(\psi')$ be the conductor of ψ'. Let

$$L = \begin{cases} \pi_E^n \mathcal{O}_E & \text{if } n(\psi') = 2n, \\ \pi^n \delta \mathcal{O}_F \oplus \pi^{n-1}\mathcal{O}_F & \text{if } n(\psi') = 2n - 1. \end{cases} \tag{1.4}$$

Then the Weil representation of G has a lattice model realization ω on

$$S(L,\psi)=\{\phi\in S(E)\colon \phi(z+l)=\psi'(z\bar{l})f(z)\quad \text{for all } l\in L\}. \tag{1.5}$$

Decomposition of $S(L,\psi)$ is well-understood in [Ya2]. Define

$$\rho\colon S(L,\psi)\to S(F),\quad \rho(f)(x)=\int_{F/F\cap L} f(x\delta+y)\psi'(-\delta xy)\,dy. \tag{1.6}$$

Then there is a constant $c>0$ such that $\langle\rho(f),\rho(f)\rangle=c\langle f,f\rangle$, for any $f\in S(L,\psi)$. Through ρ, ω gives a Weil representation of G on $S(F)$. So there is a unique character ξ of G such that such that

$$\omega_{\alpha,\chi}(g)\circ\rho=\xi(g)\rho\circ\omega(g),\quad g\in G. \tag{1.7}$$

PROPOSITION 1.2. *Write $g=x+y\delta\in G$. Let $G'=\{x+y\delta\in G : y\in\pi\mathcal{O}\}$ and $G_k=\{g\in G : g\equiv 1 \bmod \pi^k\}$, where π is a uniformizer of F.*

(1) *If E/F is ramified. Then*

$$\xi(g)=\begin{cases}\chi(\delta(g-1))(\Delta,-y)_F & \text{if } g\in G_1,\\ \chi(\delta\alpha)\varepsilon(\frac12,\varepsilon_{E/F},\psi) & \text{if } g\in G-G_1.\end{cases} \tag{1.8}$$

In particular, when the conductor $n(\chi)$ of χ is equal to 1, one has

$$\xi=\begin{cases}\textit{trivial} & \text{if } \varepsilon(\frac12,\chi,\frac12\psi_E)\chi(\delta)=\varepsilon(\alpha),\\ \text{sign} & \textit{otherwise},\end{cases} \tag{1.9}$$

where sign *is the nontrivial character of $G/G_1=\{\pm1\}$.*

(2) *If E/F is unramified and $n(\psi')=n(\psi)-\mathrm{ord}_F(\alpha)=2n$ is even. Then*

$$\xi(g)=\begin{cases}\chi(\delta(g-1))(\Delta,-y)_F & \text{if } g\in G_1,\\ \chi(\delta(g-1)) & \textit{otherwise}.\end{cases} \tag{1.10}$$

In particular, if χ is unramified, then ξ is trivial.

(3) *If E/F is unramified and $n(\psi')=n(\psi)-\mathrm{ord}_F(\alpha)=2n-1$ is odd. Then*

$$\xi(g)=\begin{cases}\chi(\delta(g-1))(\Delta,-y)_F & \text{if } g\in G_1,\\ \chi(\delta(g-1))\left(\dfrac{-\Delta}{\bar F}\right) & \text{if } g\in G'-G_1,\\ \chi(\delta(g-1))\left(\dfrac{2\Delta(x-1)}{\bar F}\right) & \text{if } g\in G-G'.\end{cases} \tag{1.11}$$

In particular, if χ is unramified, then $\xi=\eta_0$, where $\eta_0(z/\bar z)=\tilde\eta_0(z)=(\pi,z\bar z)_F$.

(4) *If* $n(\chi) \leqslant 1$, *then* ξ *is trivial on* G_1.

COROLLARY 1.3. *Let* $\phi_{\eta'} \in S(L, \psi)$ *be an eigenfunction of* ω *with eigencharacter* η'. *Then* $\rho(\phi_{\eta'})$ *is an eigenfunction of* $\omega_{\alpha,\chi}$ *with eigencharacter* $\xi\eta'$, *where* ξ *is given in Proposition* 1.2.

Applying this to [Ya2, Cor. 2.5], one gets

COROLLARY 1.4. (1) *Assume* E/F *is ramified and let* $n(\psi') = 2n$. *Then* $\mathrm{char}(\pi^{[n/2]}\mathcal{O}_F)$ *is an eigenfunction of* $(G, \omega_{\alpha,\chi})$ *with eigencharacter* ξ, *where* $[x]$ *denotes the largest integer less than or equal to* x.

(2) *Assume that* E/F *is unramified and that* $n(\psi') = 2n$ *is even. Then* $\mathrm{char}(\pi^n\mathcal{O}_F)$ *is an eigenfunction of* $(G, \omega_{\alpha,\chi})$ *with eigencharacter* ξ.

Given a character η of $G = E^1$ satisfying (1.2), we denote $\eta' = \eta\xi^{-1}$.

PROPOSITION 1.5. *Assume that* E/F *is unramified and that* $n(\psi') = 2n - 1$ *is odd. Assume further that* $n(\eta') \leqslant 1$. *Then* η *occurs in* $\omega_{\alpha,\chi}$ *if and only if* $\eta' \neq \eta_0$. *Write* $\psi'' = (\Delta\alpha/2)\pi^{2n-2}\psi$ *and view it as a character of the residue field* $\bar{F} = \mathcal{O}_F/\pi$.

(1) *If* $\eta'(-1) = (-1/\bar{F})$, *let*

$$\phi'_\eta(u) = \mathrm{char}(\pi\mathcal{O}_F)(u) + \frac{1}{2G(\psi'')} \times$$

$$\times \sum_{A^2 - B^2 \equiv \Delta \bmod \pi} \eta'\left(\frac{A+\delta}{B}\right)\left(\frac{B}{\bar{F}}\right)\psi''\left(\frac{\Delta\alpha}{2}\mathrm{Au}^2\right)\mathrm{char}(\mathcal{O}_F)(u).$$

Then $\phi_\eta(u) = \phi'_\eta(\pi^{1-n}u)$ *is an eigenfunction of* $(G, \omega_{\alpha,\chi})$ *with eigencharacter* η. *Here* $G(\psi'')$ *is the Gauss sum of the character* ψ'' *of* $\bar{F}$.

(2) *If* $\eta'(-1) = -(-1/\bar{F})$ *and* $\eta' \neq \eta_0$, *let* $a \in \mathcal{O}_F^*$, *and*

$$\phi'_{\eta,a}(u) = \mathrm{char}(a + \pi\mathcal{O}_F)(u) - \mathrm{char}(-a + \pi\mathcal{O}_F)(u) +$$

$$+ \frac{1}{G(\psi'')}\sum \eta'\left(\frac{A+\delta}{B}\right)\left(\frac{B}{\bar{F}}\right) \times$$

$$\times \psi''(\mathrm{Au}^2 - 2Bau + Aa^2)\,\mathrm{char}(\mathcal{O}_F)(u).$$

Here the sum runs over $(A, B) \in \bar{F}^2$ *with* $A^2 - B^2 \equiv \Delta \bmod \pi$. *Then* $\phi_{\eta,a}(u) = \phi'_{\eta,a}(\pi^{1-n}u) \neq 0$ *is an eigenfunction of* $(G, \omega_{\alpha,\chi})$ *with eigencharacter* η.

The following two propositions will not be needed in this paper. However, we include them here without proof for completeness and for their own rights. For

$z \in E$, we write $z = R(z) + I(z)\delta$ with $R(z)$ and $I(z) \in F$.

PROPOSITION 1.6. *Assume that $n(\psi') = 2n$ is even. Then η occurs in $\omega_{\alpha,\chi}$ if and only if there is $w \in \pi_E^{-k+n}\mathcal{O}_E^*$ such that $\eta'(g) = \psi'(-w\bar{w}g)$ for every $g \in G_k$. In such a case*

$$\phi_\eta(u) = \sum_{g \in G/G_k} \eta'(g)\psi\left(\frac{\Delta\alpha}{2}R(wg)I(wg)\right)\psi(-\Delta\alpha R(wg)u) \times$$
$$\times \begin{cases} \mathrm{char}(I(wg) + \pi^n\mathcal{O}_F)(u) & \text{if } E/F \text{ is unramified,} \\ \mathrm{char}(I(wg) + \Delta^{[n/2]}\mathcal{O}_F)(u) & \text{if } E/F \text{ is ramified.} \end{cases}$$

is an eigenfunction of $(G, \omega_{\alpha,\chi})$ with eigencharacter η. In particular, $\mathrm{Supp}\,\phi_\eta \subset \pi^{-k+n}\mathcal{O}_F$ *if E/F is unramified, and* $\mathrm{Supp}\,\phi_\eta \subset \pi^{[-k+n/2]}\mathcal{O}_F$ *if E/F is ramified. Here* $\mathrm{Supp}\,\phi$ *denotes the support of the function ϕ.*

PROPOSITION 1.7. *Assume that E/F is unramified, $n(\psi') = 2n - 1$ is odd, and that $n(\eta') > 1$. Then η occurs in $\omega_{\alpha,\chi}$ if and only if $n(\eta') = 2k - 1$ is odd. In such a case, there is $w \in \pi^{-k+n}\mathcal{O}_E^* - \pi^{-k+n}(\delta\mathcal{O}_F + \pi\mathcal{O}_F)$ such that $\eta'(g) = \psi'(-w\bar{w}g)$ for $g \in G_k$. Moreover*

$$\begin{aligned}\phi_{\eta,w}(u) = &\sum_{g \in G'/G_k} (\eta'(g)\lambda(g))^{-1}\psi\left(\frac{\Delta\alpha}{2}I(wg^{-1})R(wg^{-1})\right) \times \\ &\times\psi(-\Delta\alpha R(wg^{-1})u)\,\mathrm{char}(I(wg^{-1}) + \pi^n\mathcal{O}_F)(u) + \\ &+\frac{1}{\sqrt{q}}\sum_{g \in (G-G')/G_k} (\eta'(g)\lambda(g))^{-1}\psi(-\Delta\alpha R(w)I(w)) \times \\ &\times\psi\left(\frac{\Delta\alpha}{2y}(xI(w)^2 - 2I(w)u + xu^2)\right) \times \\ &\times\mathrm{char}(I(wg^{-1}) + \pi^{n-1}\mathcal{O}_F)(u)\end{aligned}$$

is an eigenfunction of $(G, \omega_{\alpha,\chi})$ with eigencharacter η.

2. The Central *L*-value

Let $D \equiv 7 \bmod 8$ be a squarefree positive integer and let $d \equiv 1 \bmod 4$ be a squarefree integer relatively prime to D. Then there is unique decomposition $d = d_1d_2$ such that d_i are fundamental discriminants and that every prime divisor of d_1 (d_2) is split (inert) in $E = \mathbb{Q}(\sqrt{-D})$. It is allowed d or $d_i = 1$. We view E as a subfield of $\mathbb{C}$, and fix $\delta = \sqrt{-D} = i\sqrt{D} \in i\mathbb{R}_{>0}$. Let $\chi_{D,d}$ be a Hecke character of E defined in the introduction. Then there is a decomposition $\chi_{D,d} = \chi_{D,1}\tilde{\eta}$ where $\chi_{D,1}$ is a canonical character of E and $\tilde{\eta} = (\frac{d}{\cdot}) \circ N_{E/\mathbb{Q}}$ ([Roh2-3]). Since $\tilde{\eta}|_{\mathbb{Q}_\mathbb{A}^*}$ is trivial, there is a character η of $E^1\backslash E_\mathbb{A}^1$ such that $\tilde{\eta}(z) = \eta(z/\bar{z})$. Let $\chi = \chi_{\mathrm{can}}|\,|_\mathbb{A}^{1/2}$, then

$\chi|_{\mathbb{Q}^*_{\mathbb{A}}} = \varepsilon = \Pi\varepsilon_l$ is the quadratic Hecke character of $\mathbb{Q}$ associated to the Dirichlet character $(\frac{-D}{\cdot})$. For every integer $k \geqslant 0$, let $\eta_k = \eta\chi^k|_{E^1_{\mathbb{A}}}$, then $\tilde{\eta}_k = \chi^{2k}\tilde{\eta}$. We assume that the global root number of $\chi^{2k+1}\tilde{\eta}$ is ONE, i.e., $(-1)^k = \mathrm{sign}(d)$. Then there is unique decomposition (up to order) $D = D_1D_2$ with $D_i > 0$ and

$$\varepsilon(\tfrac{1}{2}, (\chi\tilde{\eta}_k)_l, \tfrac{1}{2}\psi_{E_l})(\chi\tilde{\eta}_k)_l(\delta) = \varepsilon_l\left(\frac{2d_2}{D_2}\right) = \varepsilon_l\left(\frac{2d_1}{D_1}\right), \tag{2.1}$$

for every prime $l \leqslant \infty$ (see [RVY, Lem. 3.1]). Here $\psi_{E_l} = \psi_l \circ \mathrm{tr}_{E_l/\mathbb{Q}_l}$ and ψ_l is a 'canonical' additive character of $\mathbb{Q}_l$ given by

$$\psi_l(x) = \begin{cases} \mathrm{e}^{2\pi i x} & \text{if } l\infty, \\ \mathrm{e}^{-2\pi i \lambda_0(x)} & \text{if } l \nmid \infty, \end{cases}$$

where $\lambda_0\colon \mathbb{Q}_l \to \mathbb{Q}_l/\mathbb{Z}_l \hookrightarrow \mathbb{Q}/\mathbb{Z}$. For a prime $l|d_2$ with $l \equiv 1 \bmod 4$, we define $\phi_l \in S(\mathbb{Z}_l) \subset S(\mathbb{Q}_l)$ via

$$\begin{aligned}\phi_l(u) \;=\; & \mathrm{char}(l\mathbb{Z}_l)(u) + \frac{1}{2G(\psi'')} \times \\ & \times \sum_{A^2-B^2\equiv -D \bmod l} \left(\frac{B}{l}\right) \psi_l''(\mathrm{Au}^2)\,\mathrm{char}(\mathbb{Z}_l)(u).\end{aligned} \tag{2.2}$$

Here $\psi_l'' = -(2D_1/d_2D_2)\psi_l$. For a prime $l|d_2$ with $l \equiv -1 \bmod 4$, we define $\phi_l \in S(\mathbb{Z}_l) \subset S(\mathbb{Q}_l)$ via

$$\begin{aligned}\phi_l(u) \;=\; & \mathrm{char}(1 + l\mathbb{Z}_l)(u) - \mathrm{char}(-1 + l\mathbb{Z}_l)(u) + \frac{1}{G(\psi'')} \times \\ & \times \sum_{A^2-B^2\equiv -D \bmod l} \left(\frac{B}{l}\right) \psi_l''(\mathrm{Au}^2 - 2Bu + A)\,\mathrm{char}(\mathbb{Z}_l)(u).\end{aligned} \tag{2.3}$$

For an integer $a > 0$, we also define a theta function

$$\begin{aligned}\theta_{d,k,a}(z) \;=\; & (\mathrm{Im}\, z)^{-(k/2)} \sum_{(x,d_1)=1} \left(\frac{d_1}{x}\right) \times \\ & \times \prod_{l|d_2} \phi_l\left(\frac{x}{4d_1D_1a}\right) H_k(x\sqrt{\mathrm{Im}\, z})\, \mathrm{e}^{\pi i x^2 z}.\end{aligned} \tag{2.4}$$

Here H_k is kth Hermite polynomial defined by (1.3). Notice that $\theta_{d,k,a}$ is very simple and independent of a when $d_2 = 1$.

THEOREM 2.1. *Let* $\mathrm{CL}(E)$ *be the ideal class group of* E*, and let* $s = s(d)$ *be the*

https://doi.org/10.1023/A:1000934108242 Published online by Cambridge University Press

number of prime factors of d. For every ideal class $C \in \mathrm{CL}(E)$, choose a primitive ideal $\mathfrak{A} \in C^{-1}$ relatively prime to $2d$, and write

$$\mathfrak{A}^2 = \left[a^2, \frac{-b+\sqrt{-D}}{2}\right], \quad a > 0,$$

with $b \equiv r \bmod 8d_1^2$, $b \equiv 0 \bmod D_1 d_2$, where r is a fixed square root of $-D \bmod 16d_1^2$. Then

$$L(k+1, \chi_{D,d}^{2k+1}) = \kappa \left| \sum_{C \in \mathrm{CL}(E)} \frac{(\frac{d_1}{a})}{(\chi_{D,d})^{2k+1}(\bar{\mathfrak{A}})} \theta_{d,k,a}(\tau_{\mathfrak{A},D_1}) \right|^2. \tag{2.5}$$

Here

$$\kappa = \frac{2^{(1/2)-s}\pi^{k+1}\prod_{l|d_2}(1+l^{-1})}{k!\sqrt{D}\prod_{l|d_2}\langle\phi_l,\phi_l\rangle}\left(\frac{\sqrt{D_2}}{d_1^2 d_2\sqrt{D_1}}\right)^{2k+1/2}$$

and

$$\tau_{\mathfrak{A},D_1} = \frac{b+\sqrt{-D}}{4d_1^2 d_2 D_1 a^2} \in E.$$

Proof (sketch). The proof is similar to that of [RVY, Thm. 3.2] and is based on the main formula in [Ya1]. Applying [Ya1, Thm. 2.15] to the datum $(\chi, \eta_k, \delta, \psi, \alpha = 4/d_2 D_2)$, one has

$$\frac{L(k+1, \chi_{D,d}^{2k+1})}{L(1, (\frac{-D}{}))} = \frac{L(\frac{1}{2}, \chi\tilde{\eta}_k)}{L(1, (\frac{-D}{}))} = c|\theta_\phi(\eta_k)(1)|^2.$$

Here c is an explicit constant, $\phi = \Pi\phi_l \in S(\mathbb{Q}_\mathbb{A})$ is a Schwartz function on $\mathbb{Q}_\mathbb{A}$ given below, and $\theta_\phi(\eta_k)(1)$ is an integral over $E^1\backslash E^1_\mathbb{A}$ given by theta lifting from unitary group of one variable to itself. When l is split, ϕ_l is given by [Ya1, (2.29)–(2.30)]. When $l = \infty$, $\phi_l = \phi^k$ is given by Lemma 1.1. When l is finite and nonsplit (so $l \neq 2$), ϕ_l is given by Corollary 1.4 and Proposition 1.5. More precisely, when $l|D$, $\phi_l = (1/\sqrt{l})\,\mathrm{char}\, l^{-1}\mathbb{Z}_l$, and when $l|d_2$, ϕ_l is given by (2.2) or (2.3). Finally, if $l \nmid d_2\infty$ is inert in E, $\phi_l = \mathrm{char}(\mathbb{Z}_l)$. In [RVY, Sect. 1], we gave a method to compute $\theta_\phi(\eta_k)(1)$ in terms of ϕ ([RVY, Cor. 1.4 and Prop. 1.7]). Applying the method to this situation, we obtain the desired formula (after some computation). The case $d_2 = 1$ was computed in [RVY].

Combining this with a theorem of Shimura and a trick of Rohrlich (see [Roh2] for detail), one has

THEOREM 2.2 (Notation as in Theorem 2.1). *Assume that $(2k+1, h_D) = 1$. Then the following are equivalent.*

(1) *The central L-value $L(k+1, (\chi_{D,d})^{2k+1}) = 0$.*

https://doi.org/10.1023/A:1000934108242 Published online by Cambridge University Press

(2) *For every ideal class $C \in \mathrm{CL}(E)$, and a (and any) primitive ideal $\mathfrak{A} \in C^{-1}$, relatively prime to $2d$, $\tau_{\mathfrak{A},1}$ is a root of the theta function $\theta_{d,k,a}$, $a = N\mathfrak{A}$.*

(3) *The global theta lifting $\theta_\alpha(\eta_k)$ (with respect to $(\alpha = 2/d_2D_2, \chi, \psi, \delta)$) vanishes.*

We remark that $\tau_{\mathfrak{A},1}$ does not depend on the decomposition $D = D_1D_2$ associated to formula (2.1). This is because if the central L-value for one choice of the canonical Hecke character vanishes, it will vanish for any choice of the canonical Hecke character by a theorem of Rohrlich ([Roh2]).

3. The Proof of the Main Theorem

First we notice that the functions ϕ_l defined via (2.2) and (2.3) can be viewed as functions on $\mathbb{F}_l$. Indeed, one has

$$\begin{aligned}\phi_l(u) &= \delta_{0,u} + \frac{1}{2G(\psi'')} \sum_{A^2-B^2=-D} \left(\frac{B}{l}\right) \psi''(\mathrm{Au}^2)\\ &= \delta_{0,u} + \frac{1}{2G(\frac{1}{2}\psi'')} \sum_{x\in\mathbb{F}_l^*} \left(\frac{x+\frac{D}{x}}{l}\right) \psi''\left(\frac{1}{2}\left(x-\frac{D}{x}\right)u^2\right), \qquad (3.1)\end{aligned}$$

for $l \equiv 1 \bmod 4$, and

$$\begin{aligned}\phi_l(u) &= \delta_{1,u} - \delta_{-1,u} + \frac{1}{G(\psi'')} \sum_{A^2-B^2=-D} \left(\frac{B}{l}\right) \psi''(\mathrm{Au}^2 - 2Bu + A)\\ &= \delta_{1,u} - \delta_{-1,u} + \frac{1}{G(\frac{1}{2}\psi'')} \sum_{x\in\mathbb{F}_l^*} \left(\frac{x+\frac{D}{x}}{l}\right) \times\\ &\quad \times \psi''\left(\frac{1}{2}\left(x(u-1)^2 - \frac{D(u+1)^2}{x}\right)\right), \qquad (3.2)\end{aligned}$$

for $l \equiv -1 \bmod 4$. Here $u \in \mathbb{F}_l$ and $\delta_{a,u}$ is the Kronecker symbol. Also the equality $A^2 - B^2 = -D$ is in $\mathbb{F}_l$. Recall $\psi'' = -(2/d_2D)\psi_l$ (we take $D_1 = 1$ by the remark in the end of Section 2).

LEMMA 3.1. *Assume $l \equiv 1 \bmod 4$ and write $l = a^2 + b^2$ with b being a positive even integer. Then $\phi_l(0) = 1 \pm b/\sqrt{l}$. For $u \in \mathbb{F}_l^*$, one has $\phi_l(u) \neq 0$.*

Proof. By (3.1), one has

$$\phi_l(0) = 1 \pm \frac{1}{2\sqrt{l}} \sum_{x\in\mathbb{F}_l^*} \left(\frac{x^3+Dx}{l}\right).$$

Let A_D be the elliptic curve defined by $y^2 = x^3 + Dx$. Then

$$\#A_D(\mathbb{F}_l) = l + 1 + \sum_{x \in \mathbb{F}_l^*} \left(\frac{x^3 + Dx}{l} \right).$$

On the other hand, it is well known ([Sil, page 185]) that

$$\#A_D(\mathbb{F}_l) = \begin{cases} l + 1 \pm 2a & \text{if } (-D/l) = 1, \\ l + 1 \pm 2b & \text{if } (-D/l) = -1. \end{cases}$$

Therefore $\phi_l(0) = 1 \pm b/\sqrt{l}$ in our case. For $u \in \mathbb{F}_l^*$, $\psi''((x - D/x)u^2) \equiv 1 \bmod (1 - \zeta_l)$. So

$$\pm 2G(\psi'')\phi_l(u) \equiv \sum_{x \in \mathbb{F}_l^*} \left(\frac{x + \frac{D}{x}}{l} \right) \equiv \pm 2b \not\equiv 0 \bmod (1 - \zeta_l).$$

In particular, $\phi_l(0) \neq 0$.

LEMMA 3.2. *Assume* $l \equiv -1 \bmod 4$. *Then* $\phi_l(0) = 0$ *and* $\phi_l(u) \neq 0$ *for every* $u \in \mathbb{F}_l^*$. *Moreover, there is a map* $j\colon \mathbb{F}_l^* \to \mathbb{C}^*$ *such that*

$$j(ab) = j(a)j(b)^{\sigma_{a^2}}, \tag{3.3}$$

and

$$\phi_l(u) = j(a)\phi_l\left(\frac{u}{a}\right)^{\sigma_{a^2}}, \tag{3.4}$$

for every $a, u \in \mathbb{F}_l^*$. *Here* $\sigma_a \in \mathrm{Gal}(\mathbb{Q}(\zeta_l)/\mathbb{Q})$ *is given via* $\zeta_l^{\sigma_a} = \zeta_l^a$.

Proof. Obviously

$$\phi_l(0) = \frac{1}{G(\psi'')} \sum_{A^2 - B^2 = -D} \left(\frac{B}{l}\right) \psi''(A) = 0,$$

since $(-1/l) = -1$. For every $a \in \mathbb{F}_l^*$, define

$$\phi_{l,a}(u) = \delta_{a,u} - \delta_{-a,u} + \frac{1}{G(\psi'')} \sum_{A^2 - B^2 = -D} \left(\frac{B}{l}\right) \psi''(\mathrm{Au}^2 - 2\,Bau + Aa^2), \tag{3.5}$$

for $u \in \mathbb{F}_l$. One has $\phi_{l,1} = \phi_l$. One can view $\phi_{l,a} \in S(\mathbb{Z}_l) \subset S(\mathbb{Q}_l)$ via $\phi_{l,a}(u) = \phi_{l,a}(u \bmod l)$ for $u \in \mathbb{Z}_l$. By Proposition 1.7, $\phi_{l,a} \neq 0$ is also an eigenfunction of $\omega_{\alpha,\chi,l}$ with eigencharacter $(\eta_k)_l$, where $\alpha = 4/d_2D_2$ is as in the proof of Theorem 2.1. By the multiplicity one theorem, there is a unique nonzero complex

https://doi.org/10.1023/A:1000934108242 Published online by Cambridge University Press

number $j(a)$ such that $\phi_l(u) = j(a)\phi_{l,a}(u)$, for every $u \in \mathbb{F}_l$. On the other hand, one has $\phi_{l,a}(u) = \phi_l(u/a)^{\sigma_{a^2}}$. This proves (3.4). It follows easily from (3.4) that $\phi_l(u) \neq 0$ for every $u \in \mathbb{F}_l^*$. Applying (3.4) twice, one gets (3.3) (since $\phi_l(u) \neq 0$).

The following lemma can be checked by standard method in exponential sums (see [Li, Chap. 6] for example) and is left to the reader.

LEMMA 3.3. *Let $a_i \in \mathbb{F}_p$, $i = 1, 2, 3$, and $a_1 \in \mathbb{F}_p^*$. Let ψ be a nontrivial additive character of $\mathbb{F}_p$. Then*

$$\left| \sum_{x \in \mathbb{F}_p^*} \left(\frac{x + a_1/x}{l} \right) \psi(a_2 x + a_3/x) \right| < b\sqrt{p}. \tag{3.6}$$

Here

$$b = \begin{cases} 4 & \text{if } a_2 a_3 \neq 0, \\ 2 & \text{if } a_2 = a_3 = 0, \\ 3 & \text{otherwise.} \end{cases}$$

Now we proceed to prove the main theorem in the introduction. We divide it into two theorems.

THEOREM 3.4. *Assume that $D \equiv 7 \bmod 8$ and $d \equiv 1 \bmod 4$ are two positive squarefree integers such that every positive factor of d is inert in E_D and is congruent to 1 modulo 4. Let $k \geqslant 0$ be an even integer, and let h_D be the ideal class number of $E_D = \mathbb{Q}(\sqrt{-D})$. Then there exists a constant $M = M(k)$, independent of the pair (D, d), such that if $(h_D, 2k+1) = 1$, and $\sqrt{D} \geqslant (12/\pi) d(\log d + M(k))$, then the central L-value $L(k+1, \chi_{D,d}^{2k+1}) > 0$. One can take $M(0) = 0$.*

Proof. By Theorem 2.2 and the assumption, it suffices to show that $\tau = b + \sqrt{-D}/4d$ is not a root of the theta function

$$\theta_{d,k,1}(z) = (\operatorname{Im} z)^{-(k/2)} \sum_{x \in \mathbb{Z}} \phi(x) H_k(x\sqrt{\operatorname{Im} z})\, e^{\pi i x^2 z}.$$

Here we denote $\phi(x) = \prod_{l|d} \phi_l(x/4)$. By Lemma 3.3, one has

$$|\phi(x)| \leqslant \prod_{l|d} 2 \leqslant \sqrt{d}. \tag{3.7}$$

By Lemma 3.1, one has

$$|\phi(0)| \geqslant \prod_{l|d} \frac{1}{2l} \geqslant d^{-(3/2)}. \tag{3.8}$$

Set $c = \mathrm{e}^{-(\pi\sqrt{D}/4d)} d^3$ and assume $c < 1$. Since H_k is a polynomial of x of degree k, there is a constant $C_1 = C_1(k) > 0$ such that $|H_k(x)| \geqslant C_1|x|^k$ for $|x| \geqslant \pi^{-1}$. So

$$|H_k(n\sqrt{\operatorname{Im}\tau})| \geqslant C_1 n^k \pi^{-(k/2)}(-\log c + 3\log d)^{k/2}, \tag{3.9}$$

when $c \leqslant 1/e$ and n is a positive integer. Set $C_2 = |H_k(0)| > 0$ (k is even). Combining (3.7)–(3.9), one has

$$\begin{aligned}
&(\operatorname{Im}\tau)^{(k/2)}|\theta_{d,k,1}(\tau)| \\
&\quad \geqslant C_2 d^{-(3/2)} - 2C_1\pi^{-(k/2)} d^{1/2}(-\log c + 3\log d)^{k/2} \sum_{n=1}^{\infty} n^k d^{-3n^2} c^{n^2} \\
&\quad \geqslant d^{-(3/2)} f(c),
\end{aligned}$$

where

$$f(x) = C_2 - 2C_1\pi^{-(k/2)} x (C_3 - \log x)^{k/2} \sum_{n=1}^{\infty} n^k x^{n^2-1},$$

and $C_3 \geqslant 1$ is chosen such that $3\log x < x^{2/k}$ for $x > C_3$. Here we have used the inequality

$$\frac{-\log c + 3\log d}{d^{2/k}} < C_3 - \log c.$$

Notice that $f(x)$ is independent of D or d. Since $f(0) = C_2 > 0$, there is a constant $0 < C_4 < 1/e$ such that $f(x) > 0$ for $0 < x < C_4$. Therefore, when $c < C_4$, i.e., $\sqrt{D} > (12/\pi)d(\log d - \frac{1}{3}\log C_4)$, one has $\theta_{d,k,1}(\tau) \neq 0$, and so $L(k+1, \chi_{D,d}^{2k+1}) > 0$. Taking $M(k) = -\frac{1}{3}\log C_4$, we have proved the general statement of the theorem. When $k = 0$, $H_k = 1$, similar but simpler argument gives

$$\begin{aligned}
|\theta_{d,0,1}(\tau)| \;&\geqslant\; d^{-(3/2)} - 2d^{1/2}\sum_{n=1}^{\infty} c^{n^2} \\
&\geqslant\; d^{-(3/2)}\left(1 - 2d^2\frac{c}{1-c}\right),
\end{aligned}$$

for $c = \mathrm{e}^{-(\pi\sqrt{D}/4d)} < 1$ (different from the c used above). So for $c < 1/d^3 < 1/2d^2 + 1$ (we may assume $d > 1$, the case $d = 1$ is trivial) or, equivalently, $\sqrt{D} \geqslant (12/\pi)d\log d$, one has

$$\theta_{d,0,1}(\tau) \neq 0.$$

So we can take $M(0) = 0$. This proves the theorem

THEOREM 3.5. *Let* $D \equiv 7 \bmod 8$ *be a positive squarefree integer, and let* $d \equiv 1 \bmod 4$ *be a squarefree integer not satisfying the special condition in Theorem* 3.4. *Let* $k \geqslant 0$ *be an integer. Then there is a constant* $M(k)$*, independent of the pair* (D, d)*, such that if* $\operatorname{sign}(d) = (-1)^k$, $(2k+1, h_D) = 1$*, and* $\sqrt{D} \geqslant (4/\pi) d^2(\log|d| + M(k))$*, then the central L-value* $L(k+1, \chi_{D,d}^{2k+1}) > 0$*. One can take* $M(k) = 0$ *for* $k \leqslant 1$.

Proof. When d does not satisfy the special condition in Theorem 3.4, the theta function $\theta_{d,k,1}$ does not have a constant term. As in Section 2, we write $d = d_1 d_2$ such that every prime factor of d_1 is split in E_D and every prime factor of d_2 is inert in E_D. As before, it is sufficient to prove that $\tau = b + \sqrt{-D}/4d_1^2 d_2$ is not a root of the theta function $\theta_{d,k,1}$ given by (2.4) with $D_1 = 1$. Set $\phi(x) = \Pi_{l|d} \phi_l(x/4d_1)$. By Lemma 3.3, one has $|\phi_l(x)| \leqslant 4$ and so

$$|\phi(x)| \leqslant 4|d_2| \tag{3.10}$$

($4 < l$ except for $l = 3$). Notice that $\sqrt{l}\phi_l(x)$ is an algebraic integer in the lth cyclotomic field $\mathbb{Q}(\zeta_l)$. It is not difficult to see from this fact and Lemmas 3.2 and 3.3 that

$$|\sqrt{l}\phi_l(x)| \geqslant \prod_{1 \neq \sigma \in \operatorname{Gal}(\mathbb{Q}(\zeta_l)/\mathbb{Q})} |(\sqrt{l}\phi_l(x))^\sigma|^{-1} \geqslant (4\sqrt{l})^{1-l},$$

for $x \in \mathbb{Z}_p^*$. So

$$|\phi(x)| \geqslant 4|d_2|^{-(3/2)|d_2|}, \tag{3.11}$$

for $x \in \mathbb{Z}_p^*$. Set $c = \mathrm{e}^{-(\pi\sqrt{D}/4d^2)}|d|$. As in the proof of Theorem 3.4, there is a constant $C_1 = C_1(k)$, independent of (D, d), such that

$$|H_k(n\sqrt{\operatorname{Im}\tau})| \leqslant C_1 n^k \pi^{-(k/2)} |d_2|^{k/2}(-\log c + \log|d|)^{k/2}, \tag{3.12}$$

when n is a positive integer and $0 < c < 1$. On the other hand, since H_k has only finite many roots, there are positive constants C_2 and $C_3 < 1$ such that

$$|H_k(x)| \geqslant C_2 \quad \text{for } |x| > \sqrt{-\frac{1}{\pi}\log C_3}.$$

So

$$|H_k(\sqrt{\operatorname{Im}\tau})| \geqslant C_2 \quad \text{for } c \leqslant C_3. \tag{3.13}$$

https://doi.org/10.1023/A:1000934108242 Published online by Cambridge University Press

Combining (3.10)–(3.13), one has then

$$\begin{aligned}\frac{(\operatorname{Im}\tau)^{k/2}}{2}&|\theta_{d,k,1}(\tau)|\\ &\geqslant 4C_2|d_2|^{-(3/2)|d_2|}c - 4C_1\pi^{-(k/2)}|d_2|^{k/2+1}\times\\ &\quad\times(-\log c+\log|d|)^{k/2}\sum_{n=2}^{\infty}n^k|d|^{-n^2|d_2|}c^{|d_2|n^2}\\ &\geqslant 4|d_2|^{-(3/2)|d_2|}cf_1(c,d)),\end{aligned}$$

where

$$f_1(c,d)=C_2-\pi^{-(k/2)}C_1\frac{|d_2|^{k/2+1}}{|d|^{(5/2)|d_2|-1}}\left(\frac{-\log c+\log|d|}{|d|^{2/k}}\right)^{k/2}\sum_{n=2}^{\infty}n^kc^{n^2-1}.$$

Choose $C_4\geqslant 1$ so that $\log x<x^{2/k}$ for $x>C_4$. Then

$$\frac{-\log c+\log|d|}{|d|^{2/k}}\geqslant C_4-\log c.$$

Notice that $|d_2|^{k/2+1}/|d|^{(5/2)|d_2|-1}$ is bounded above as a function of d. So there is a constant $C_5=C_5(k)$, independent of the pair (D,d), such that $f_1(c,d)\geqslant f(c)$ where

$$f(x)=C_2-C_5x^3(C_4-\log x)^{k/2}\sum_{n=2}^{\infty}n^kx^{n^2-4}.$$

Now the same argument as in the proof of Theorem 3.4 gives the general statement of this theorem. Similar argument to the last part in the proof of Theorem 3.4 (together with slightly better lower bound for $|\phi_l(x)|$) shows that one can take $M(k)=0$ for $k\leqslant 1$. We leave the detail to the reader.

Remark 3.6. When $k=0$, the main theorem claims that for all the pairs $(\sqrt{D},d)$ in the region above $\sqrt{D}>(12/\pi)d^2\log d$ with $D\equiv 7 \bmod 8$ and $d\equiv 1 \bmod 4$ squarefree, the central L-value $L(1,\chi_{D,d})>0$. This is strong considering the general belief that whether an L-function vanishes at its center is tricky and hard to tell.

When every prime factor of d is split in E_D one can drop $\log d$ from the condition ([RVY, Thm.]), and when every factor of D is congruent to 1 modulo 4 and is inert in E_D one can replace d^2 by d (Theorem 3.4). A natural question is what, if any, is the ultimate inequality to guarantee the nonvanishing of $L(1,\chi_{D,d})$. Can that be $\sqrt{D}>M\log d$ for some constant M?

Remark 3.7. Recall that $\chi_{D,d} = \chi_{D,1}\tilde{\eta}_d$ where $\tilde{\eta}_d = (\frac{d}{\cdot}) \circ N_{E_D/\mathbb{Q}}$, and $\chi_{D,1}$ is a canonical Hecke character of E_D. Although $\chi_{D,d}$ can be viewed as a quadratic twist of $\chi_{D,1}$, it might be better to view it as a 'quadratic' twist of $(\frac{d}{\cdot})$ by the imaginary quadratic field E_D (see [Lie] and [RVY] for similar ideas). The result of Montgomery and Rohrlich ([MR]) mentioned in the introduction is then that the central L-value of a 'quadratic' twist $\chi_{D,1}$ of the trivial character by E_D does not vanish unless it is forced to by its functional equation. Is the same true for other 'small' d? The following proposition gives a partial affirmative answer to the question for $d = 5$ (we don't consider the case when D is even).

PROPOSITION 3.8. *Let* $D \equiv 7 \bmod 8$ *be a squarefree positive integer relative prime to* 5. *Then the central L-value* $L(1, \chi_{D,5}) \neq 0$.

Proof. First we assume that $(D/5) = 1$, i.e., $D \equiv \pm 1 \bmod 5$. By Theorem 2.2 it is sufficient to prove that $\tau = b + \sqrt{-D}/200$ is not a root of

$$\theta_d(z) = \sum_{(x,5)=1} \left(\frac{5}{x}\right) \mathrm{e}^{2\pi x^2 z}.$$

Here b is some integer. Set $c = \mathrm{e}^{-\pi\sqrt{D}/100}$. Since $D \equiv 7 \bmod 8$ and $D \equiv \pm 1 \bmod 5$, one has $D \geqslant 31$. This implies $c < 0.84$, which is enough to guarantee

$$\tfrac{1}{2}|\theta(\tau)| > c - \sum_{n>1} c^{n^2} > c - c^4 - c^9 \sum_{n=0}^{\infty} c^{7n} > 0.$$

So $L(1, \chi_{D,5}) \neq 0$ in this case. Now we turn to the case $(D/5) = -1$, i.e., $D \equiv a \bmod 5$ with $a = \pm 2$. one can show by (3.1) that

$$\phi_5(u) = \begin{cases} 1 + \dfrac{a}{\sqrt{5}} & \text{if } u \equiv 0 \bmod 5, \\ -\dfrac{a}{\sqrt{5}} \cos \dfrac{\pi}{\sqrt{5}} & \text{if } u \not\equiv 0 \bmod 5. \end{cases} \tag{3.14}$$

By Theorem 2.2, it is sufficient to prove that $\tau = b + \sqrt{-D}/20$ is not a root of the theta function $\theta_{5,0,1}(z) = \sum_{x\in\mathbb{Z}} \phi_5(x/4)\, \mathrm{e}^{\pi i x^2 z}$. Here b is any integer satisfying $b \equiv 0 \bmod 5$ and $b^2 \equiv -D \bmod 16$.

Set $c = \mathrm{e}^{-(\pi\sqrt{D}/20)} \leqslant \mathrm{e}^{-(\pi\sqrt{7}/20)} < 0.66$, and $b' = b/5 \in \mathbb{Z}$. Write $\operatorname{Im} z$ for the imaginary part of the complex number z. By (3.14), $\phi_5(x)$ depends only on whether $x \equiv 0 \bmod 5$, and $|\phi_5(0)| < 3|\phi_5(1)|$. So

$$\begin{aligned} &\operatorname{Im}(\theta_{5,0,1}(\tau)) \\ &\quad = 2\phi_5(1) \sum_{(n,5)=1,n>0} \operatorname{Im}(\mathrm{e}^{\pi i n^2 b'/4}) c^{n^2} + 2\phi_5(0) \sum_{5|n,n>0} \operatorname{Im}(\mathrm{e}^{\pi i n^2 b'/4}) c^{n^2}. \end{aligned}$$

https://doi.org/10.1023/A:1000934108242 Published online by Cambridge University Press

Notice that $\mathrm{Im}(e^{\pi i n^2 b'/4}) = 0$ or $\pm\sqrt{2}/2$ depending on whether 2 divides n or not. Therefore $(c > 0.66)$

$$\frac{1}{\sqrt{2}|\phi_5(1)|}|\,\mathrm{Im}(\theta_{5,0,1}(\tau))| > c - 3\sum_{n=3}^{\infty} c^{n^2} > c - 3c^9 \sum_{n=0}^{\infty} c^{7n} > 0.$$

In particular, $\theta_{5,0,1}(\tau) \neq 0$, and thus $L(1, \chi_{D,5}) \neq 0$.

Appendix. Proof of Propositions 1.2 and 1.5

Let the notation be as in Section 1. We first recall some basic facts on the Weil representation $\omega_{\alpha,\chi}$ of $G = U(1)$ on $S(F)$. First, there is an embedding

$$\iota_\alpha\colon G \to \mathrm{Sp}(1) = \mathrm{SL}_2(F), \qquad g = x + y\delta \mapsto \begin{pmatrix} x & \Delta^2\alpha y \\ \dfrac{y}{\Delta\alpha} & x \end{pmatrix}. \tag{A1}$$

Let r_S be Rao's standard section of Sp(1) on $S(F)$ and let c be the corresponding standard 2-cocycle ([Rao]). For

$$g_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \mathrm{Sp}(1),$$

with $g_1 g_2 = g_3$, one has ([Rao, Cor. 4.3])

$$c(g_1, g_2) = \begin{cases} 1 & \text{if } c_1c_2c_3 = 0, \\ \gamma_F(\frac{1}{2}c_1c_2c_3\psi) & \text{otherwise}. \end{cases} \tag{A2}$$

Here γ_F is the local Weil index ([Wei], [Rao, App.]). For $g = x + y\delta \in G$, define

$$\mu(g) = \chi(\delta(g-1))\gamma_F(\alpha y(1-x)\psi)(\Delta, -2y(1-x))_F. \tag{A3}$$

Then $\omega_{\alpha,\chi}(g) = \mu(g) r_S(\iota_\alpha(g))$ defines a Weil representation of G ([Ku, Prop. 4.8]). Finally, when $n(\psi') = 2n - 1$ is odd, let ([Ya2, Thm. 3.5])

$$\lambda(g) = \begin{cases} \left(\dfrac{x}{\bar{F}}\right) & \text{if } g \in G', \\ \gamma_F\left(\dfrac{\Delta\alpha y}{2}\psi\right) & \text{if } g \notin G'. \end{cases} \tag{A4}$$

Then $\omega(g) = \lambda(g)^{-1} r_L(g)$ is a Weil representation of G on $S(L, \psi)$, where r_L is the action of G on $S(L, \psi)$ defined via [Ya2, (3.2)]. When $n(\psi') = 2n$ is even, the Weil representation ω of G on $S(L, \psi)$ is just the right translation.

LEMMA A1.

(a) *If E/F is ramified, then* $\xi(g) = \mu(g)c(\iota_\alpha(g), w)$.

(b) *If E/F is unramified and $n(\psi')$ is even, then*

$$\xi(g) = \begin{cases} \mu(g)c(\iota_\alpha(g), w) & \text{if } g \in G', \\ \mu(g) & \text{if } g \notin G'. \end{cases}$$

(c) *If E/F is unramified and $n(\psi')$ is odd, then*

$$\xi(g) = \begin{cases} \lambda(g)\mu(g)c(\iota_\alpha(g), w) & \text{if } g \in G', \\ \lambda(g)\mu(g) & \text{if } g \notin G'. \end{cases}$$

Proof. We only prove Claim (c). The proof of Claims (a) and (b) is similar (simpler) and is left to the reader. First note $L \cap F = \pi^{n-1}\mathcal{O}_F$. Let f_0 be the characteristic function of L. An easy calculation shows that $\rho(f_0)$ is the characteristic function of $\pi^n\mathcal{O}_F$.

Assume first that $g = x + y\delta \in G'$. Then $y \in \pi\mathcal{O}_F$ and $x \in \mathcal{O}_F^*$. Let

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and write

$$\iota_\alpha(g)w = \begin{pmatrix} x^{-1} & \Delta^2 xy \\ 0 & x \end{pmatrix} w \begin{pmatrix} 1 & -\dfrac{y}{\Delta\alpha x} \\ 0 & 1 \end{pmatrix} = AwB, \tag{A5}$$

where A and B have the obvious meanings. Set

$$f_1(u) = r_S(w)\rho(f_0)(u) = \int_{\pi^n\mathcal{O}_F} \psi(-uv)\,\mathrm{d}v,$$

where $\mathrm{d}v$ is the self-dual Haar measure on F with respect to ψ. Straightforward calculation using [Rao, Thm. 3.6] gives $r_S(AwB)f_1(u) = \rho(f_0)(u)$. Therefore

$$\begin{aligned} \omega_{\alpha,\chi}(g)\rho(f_0)(u) &= \mu(g)c(\iota_\alpha(g), w)r_S(\iota_\alpha(g)w)f_1(u) \\ &= \mu(g)c(\iota_\alpha(g), w)r_S(AwB)f_1(u) \\ &= \mu(g)c(\iota_\alpha(g), w)\rho(f_0)(u) \\ &= \mu(g)c(\iota_\alpha(g), w)\lambda(g)^{-1}\rho(\omega(g)f_0)(u). \end{aligned}$$

Combining this with (1.7), one has $\xi(g) = \lambda(g)\mu(g)c(\iota_\alpha(g), w)$.

Next, assume that $g = x + y\delta \notin G'$, so $y \in \mathcal{O}_F^*$. In this case

$$\iota_\alpha(g) = \begin{pmatrix} \dfrac{\Delta\alpha}{y} & x \\ 0 & \dfrac{y}{\Delta\alpha} \end{pmatrix} w \begin{pmatrix} 1 & \dfrac{\Delta\alpha x}{y} \\ 0 & 1 \end{pmatrix}. \tag{A6}$$

Direct calculation using [Rao, Thm. 3.6] and (A6) gives

$$\omega_{\alpha,\chi}(g)\rho(f_0)(u) = \frac{\mu(g)}{\sqrt{q}}\psi\left(\frac{\Delta\alpha x}{2y}u^2\right)\mathrm{char}(\pi^{n-1}\mathcal{O}_F). \tag{A7}$$

On the other hand, By [Ya2, Lem. 3.2 and Thm. 3.5], one has

$$\omega(g)f_0(z) = \begin{cases} 0 & \text{if } z \notin L_{n-1}, \\ \dfrac{\lambda^{-1}(g)}{\sqrt{q}}\psi'(\delta uv)\psi'(\delta u^2 xy^{-1}) & \text{if } z = u\delta + v \in L_{n-1}. \end{cases}$$

So one has by (1.6)

$$\rho(\omega(g)f_0)(u) = \frac{1}{\lambda(g)\sqrt{q}}\psi\left(\frac{\Delta\alpha x}{2y}u^2\right)\mathrm{char}(\pi^{n-1}\mathcal{O}_F)(u). \tag{A8}$$

Combining (1.7) with (A7) and (A8), one has $\xi(g) = \lambda(g)\mu(g)$. Claim (c) is proved.

Proof of Proposition 1.2. First, we assume $g \in G_1$, so $x \equiv 1 \bmod \pi$, $y \equiv 0 \bmod \pi$, and $x - 1 = \Delta y^2/(x+1)$. So

$$\begin{aligned} \xi(g) &= \chi(\delta(g-1))\gamma_F(-2\alpha xy\Delta\psi)\gamma_F(2\alpha xy\Delta\psi)(\Delta, \Delta y) \\ &= \chi(\delta(g-1))(\Delta, -y). \end{aligned}$$

In particular, if $n(\chi) \leqslant 1$, then $g - 1 = \delta y(1 + \delta y/x + 1)$. So

$$\chi(\delta(g-1)) = \chi(\Delta y) = (\Delta, \Delta y) = (\Delta, -y).$$

Therefore $\xi(g) = 1$. This proves (4) and the first part of the first three claims.

Next, we assume $g \in G' - G_1$, i.e., $g \equiv -1 \bmod \pi_E$ or, equivalently, $x \equiv -1 \bmod \pi$, $y \equiv 0 \bmod \pi$. In such a case, one has

$$\mu(g)c(\iota_\alpha(g), w) = \chi(\delta(g-1))\gamma_F(2\alpha y\psi)\gamma(-2\alpha y\Delta\psi)(\Delta, -y)_F. \tag{A9}$$

When E/F is ramified, we may assume $\pi = \Delta$. One has by (A9) and Lemma A1

$$\xi(g) = \chi(\delta(g-1))\gamma_F(-\Delta\psi)\gamma_F(\psi)(\Delta, -2\alpha).$$

Given a character ψ of F of conductor n, one defines a character

$$\bar{\psi}\colon \mathcal{O}_F/\pi O_F \to \mathbb{C}^*, \qquad x \bmod \pi\mathcal{O}_F \mapsto \psi(\pi^{n-1}x).$$

Write $G(\bar{\psi})$ for the Gauss sum of $\bar{\psi}$. If n is odd, then $n(\Delta\psi)$ is even. By [Rao, A11, A2] one has

$$\gamma_F(-\Delta\psi)\gamma_F(\psi) = (\varepsilon(-1))^{1-n}\frac{G(\bar{\psi})}{|G(\bar{\psi})|} = \varepsilon(\tfrac{1}{2}, \varepsilon_{E/F}, \psi).$$

This proves (1.8). When $n(\chi) \leqslant 1$, $\chi(\delta(g-1)) = \chi(-2\delta)$, and so

$$\xi(g) = \chi(\delta\alpha)\varepsilon(\tfrac{1}{2}, \varepsilon_{E/F}, \psi).$$

Applying [Roh1, Props 3 and 8], one has (1.9). The unramified case is similar and is left to the reader.

Finally, we assume that $g \in G - G'$. So $x \pm 1 \in \mathcal{O}_F^*$ and $y \in \mathcal{O}_F^*$. Also E/F must be unramified in this case by [Ya2, Lem. 1.1].

If $n(\psi') = n(\alpha\psi)$ is even, then one has by [Rao, App.]

$$\begin{aligned}\xi(g) = \mu(g) &= \chi(\delta(g-1))\gamma_F(y(1-x)\alpha\psi)(\Delta, -2y(1-x))_F \\ &= \chi(\delta(g-1)).\end{aligned}$$

If $n(\psi') = n(\alpha\psi)$ is odd, then one has by [Rao, App.]

$$\begin{aligned}\xi(g) &= \chi(\delta(g-1))\gamma_F(y(1-x)\alpha\psi)\gamma_F(2y\Delta\alpha\psi)(\Delta, -2y(1-x))_F \\ &= \chi(\delta(g-1))\gamma_{\bar{F}}(y(1-x)\overline{\alpha\psi})\gamma_{\bar{F}}(2y\Delta\overline{\alpha\psi}) \\ &= \chi(\delta(g-1))\left(\frac{2\Delta(x-1)}{\bar{F}}\right).\end{aligned}$$

This completes the proof of Proposition 1.2.

Proof of Proposition 1.5. For $z \in E$, we write $z = R(z) + I(z)\delta$ with $R(z)$ and $I(z) \in F$. Given $w \in E$, let f_w be the unique function in $S(L, \psi)$ such that $\mathrm{Supp}(f_w) = w + L$ and $f_w(w) = 1$. Integrating (1.6) for f_w, one has

$$\begin{aligned}\rho(f_w)(u) &= \psi\left(\frac{\Delta\alpha}{2}R(w)I(w)\right)\psi(-\Delta\alpha R(w)u) \times \\ &\quad \times \begin{cases}\mathrm{char}(I(w) + \pi^n\mathcal{O}_F)(u) & \text{if } E/F \text{ is unramified,} \\ \mathrm{char}(I(w) + \Delta^{[n/2]}\mathcal{O}_F)(u) & \text{if } E/F \text{ is ramified.}\end{cases}\end{aligned} \tag{A10}$$

By [Ya2, Thm. 0.4] and Corollary 1.3

$$\begin{aligned}\rho(\phi') &= \rho(f_w) + \eta'(-1)\left(\frac{-1}{\bar{F}}\right)\rho(f_{-w}) + \\ &\quad + \sum_{g\in G/G', g\neq 1}\eta(g)^{-1}\omega_{\alpha,\chi}(g)\rho\left(f_w + \eta'(-1)\left(\frac{-1}{\bar{F}}\right)f_{-w}\right)\end{aligned}$$

https://doi.org/10.1023/A:1000934108242 Published online by Cambridge University Press

is an eigenfunction of $(G, \omega_{\alpha,\chi})$ with eigencharacter η if it is *nonzero*, where $w \in L_{n-1}$. When $g = x + y\delta \in G - G'$, one has by (A6), (A11), and [Rao, Thm. 3.6]

$$\begin{aligned}
&\omega_{\alpha,\chi}(g)\rho(f_w)(u) \\
&\quad = \frac{\mu(g)\psi(-\frac{\Delta\alpha}{2}R(w)I(w))}{\sqrt{q}}\psi\left(\frac{\Delta\alpha}{2y}(xI(w)^2 - 2I(w)u + xu^2)\right) \times \\
&\qquad \times \operatorname{char}(\pi^{n-1}\mathcal{O}_F)(u).
\end{aligned}$$

Putting things together, and applying Lemma A1, one has proved that

$$\begin{aligned}
&\phi_{\eta,w}(u) \\
&\quad = \eta'(-1)\left(\frac{-1}{\bar{F}}\right)\psi(\Delta\alpha R(w)u)\operatorname{char}(-I(w) + \pi^n\mathcal{O}_F)(u) + \\
&\qquad + \psi(-\Delta\alpha R(w)u)\operatorname{char}(I(w) + \pi^n\mathcal{O}_F)(u) + \frac{\psi(-\Delta\alpha R(w)I(w))}{\sqrt{q}} \times \\
&\qquad \times \sum_{g\in G/G', g\neq 1} (\eta'(g)\lambda(g))^{-1}\left\{\psi\left(\frac{\Delta\alpha}{2y}(xu^2 - 2I(w)u + xI(w)^2)\right) + \right. \\
&\qquad\qquad\qquad + \eta'(-1)\left(\frac{-1}{\bar{F}}\right)\psi \times \\
&\qquad\qquad\qquad \left. \times \left(\frac{\Delta\alpha}{2y}(xu^2 + 2I(w)u + xI(w)^2)\right)\right\} \times \\
&\qquad\qquad\qquad \times \operatorname{char}(\pi^{n-1}\mathcal{O}_F)(u)
\end{aligned}$$

is an eigenfunction of $(G, \omega_{\alpha,\chi})$ with eigencharacter η if it is nonzero.

When $\eta'(-1) = (-1/\bar{F})$, set $w = 0$, and applying [Ya2, Thm. 3.5] for λ, one gets

$$\begin{aligned}
\tfrac{1}{2}\phi_{\eta,0}(u) \;=\; & \operatorname{char}(\pi^n\mathcal{O}_F)(u) + \frac{1}{2G(\psi'')} \times \\
& \times \sum_{g\in G/G_1, g\neq\pm 1} \eta'(g)^{-1}\left(\frac{I(g)}{\bar{F}}\right)\psi\left(\frac{\Delta\alpha R(g)}{2I(g)}u^2\right) \times \\
& \times \operatorname{char}(\pi^{n-1}\mathcal{O}_F)(u).
\end{aligned}$$

It is not difficult to see that $a \mapsto g(a) = \delta + a/\delta - a$ gives a bijection between the projective line $P^1(\bar{F})$ and G/G_1. Set $A = R(g)/I(g)$, and $B = 1/I(g)$. Then for $g = g(a)$

$$A = \frac{1}{2}\left(a + \frac{\Delta}{a}\right) \quad \text{and} \quad B = \frac{1}{2}\left(-a + \frac{\Delta}{a}\right).$$

It is easy to check that $a \mapsto (A, B)$ is a bijection between $\bar{F}^*$ and $(A, B) \in \bar{F}^2$ with $A^2 - B^2 = \Delta$. Therefore

$$\begin{aligned}\tfrac{1}{2}\phi_{\eta,0}(u) &= \mathrm{char}(\pi^n \mathcal{O}_F)(u) + \frac{1}{2G(\psi'')} \times \\ &\quad \times \sum_{A^2-B^2 \equiv \Delta \bmod \pi} \eta'\left(\frac{A+\delta}{B}\right)\left(\frac{B}{\bar{F}}\right)\psi\left(\frac{\Delta\alpha}{2}\mathrm{Au}^2\right) \times \\ &\quad \times \mathrm{char}(\pi^{n-1}\mathcal{O}_F)(u) \\ &= \phi'_\eta(\pi^{1-n}u) = \phi_\eta(u)\end{aligned}$$

is the function sought in Proposition 1.5(1). It remains to prove that it is nonzero. But

$$\phi_\eta(0) = 1 + \frac{1}{2G(\psi'')} \sum_{A^2-B^2 \equiv \Delta \bmod \pi} \eta'\left(\frac{A+\delta}{B}\right)\left(\frac{B}{\bar{F}}\right) \neq 0,$$

since $G(\psi'') \notin \mathbb{Q}(\mathrm{e}^{2\pi i/q+1})$ and the sum is in $\mathbb{Q}(\mathrm{e}^{2\pi i/q+1})$. This proves (1).

When $\eta'(-1) = -(-1/\bar{F})$, and $\eta' \neq \eta_0$, $\phi_{\eta,w} = 0$ for every $w \in L$. So there is $w = \pi^{n-1}a \in L_{n-1} - L$ with $a \in \mathcal{O}_F^*$ such that $\phi_{\eta,w} \neq 0$. A simple manipulation shows that $\phi_{\eta,w}(u) = \phi'_{\eta,a}(\pi^{n-1}u)$ is the function sought in Proposition 1.5(2). It remains to prove that $\phi'_{\eta,a} \neq 0$ for every $a \in \mathcal{O}_F^*$. We can identify $\mathrm{Gal}(\mathbb{Q}(\zeta_q, \zeta_{q+1})/\mathbb{Q}(\zeta_{q+1}))$ with $\bar{F}^*$ via $b \mapsto \sigma_b$. Here $\zeta_q^{\sigma_b} = \zeta_q^b$ for a nth primitive root ζ_q of 1. It is easy to check that $\phi'_{\eta,a}$ can be viewed as a function on $\bar{F}$ with values in $\mathbb{Q}(\zeta_q, \zeta_{q+1})$, and that

$$\phi'_{\eta,a}(u) = \phi_{\eta,1}(u/a)^{\sigma_{a^2}}. \tag{A11}$$

So one $\phi'_{\eta,a} \neq 0$ implies every $\phi'_{\eta,a} \neq 0$. This proves Claim (2).

Proposition 1.6 follows easily from Corollary 1.3 and [Ya2, Thm. 1.1]. The proof of Proposition 1.7 is similar to that of Proposition 1.5.

Acknowledgement

The author thanks P. Deligne, K. Murty, and L. Washington for their help during the preparation of this paper. He also thanks S. Kudla and D. Rohrlich for their continuing encouragement and advice. He thanks the referee for his/her comments and suggestions, which have influenced this final version. Finally, the author thanks the Institute for Advanced Study for its hospitality and financial support.

References

[Gro] Gross, B.: *Arithmetic on Elliptic Curves with Complex Multiplication*, Lecture Notes in Math. 776, Springer-Verlag, New York, 1980.

[HKS] Harris, M., Kudla, S. and Sweet, J.: Theta dichotomy for unitary groups, *J. Amer. Math. Soc*. **9** (1996), 941–1003.

[Ku] Kudla, S.: Splitting metaplectic covers of dual reductive pairs, *Israel J. Math*. **87** (1992), 361–401.

[Li] Li, Wen-Ching W.: *Number Theory with Applications*, Ser. Univ. Math. 7, World Scientific, Singapore, 1995.

[Lie] Lieman, D.: Nonvanishing of L-series associated to cubic twists of elliptic curves, *Ann. Math*. **140** (1994), 81–108.

[MR] Montgomery, H. and Rohrlich, D.: On the L-function of canonical Hecke characters of imaginary quadratic fields II, *Duke Math. J*. **49** (1982), 937–942.

[Rao] Rao, R. R.: On some explicit formulas in the theory of Weil representations, *Pacific J. Math* **157** (1993), 335–371.

[RV1] Rodriguez Villegas, F.: On the square root of special values of certain L-series, *Invent. Math.* **106** (1991), 549–573.

[RV2] Rodriguez Villegas, F.: Square root formulas for central values of Hecke L-series II, *Duke Math. J*. **72** (1993), 431–440.

[RVZ] Rodriguez Villegas, F. and Zagier, D.: Square roots of central values of Hecke L-series, in: *Proc. 3rd Conf. Canad. Number Theory Assoc., Kingston, Ontario*, 1991.

[Roh1] Rohrlich, D.: Root numbers of Hecke L-functions of CM fields, *Amer. J. Math*. **104** (1982), 517–543.

[Roh2] Rohrlich, D.: The non-vanishing of certain Hecke L-functions at the center of the critical strip, *Duke Math. J*. **47** (1980), 223–232.

[Roh3] Rohrlich, D.: On the L-functions of canonical Hecke characters of imaginary quadratic fields, *Duke Math. J*. **47** (1980), 547–557.

[RVY] Rodriguez Villegas, F. and Yang, T. H.: Central values of Hecke L-functions of CM number fields, to appear in *Duke Math. J*. (1996).

[Ru] Rubin, K.: Elliptic curves with complex multiplication and the conjecture of Birch and Swinnerton-Dyer, *Invent. Math*. **64** (1981), 455–470.

[Sch] Schappacher, N.: *Periods of Hecke Characters*, Lecture Notes in Math. 1301, Springer-Verlag, 1988.

[Sh1] Shimura, G.: *Introduction to the Arithmetic Theory of Automorphic Functions*, Publ. Math. Soc. Japan 11, 1971.

[Sh2] Shimura, G.: On the zeta-function of an Abelian variety with complex multiplication, *Ann. Math*. **94** (1971), 504–533.

[Sil] Silverman, J.: Advanced topics in the arithmetic of elliptic curves, *Grad. Texts in Math*. 151, Springer, New York, 1994.

[Wei] Weil, A.: Sur la formule de Siegel dans la théorie des groupes classiques, *Acta. Math*. **123** (1965), 1–87.

[Ya1] Yang, Tonghai: Theta liftings and L-functions of elliptic curves, Thesis, Univ. of Maryland (part of it appeared in the *Crelle* **485** (1997), 25–53) (1995).

[Ya2] Yang, Tonghai: Eigenfunctions of Weil representation of unitary groups of one variable, to appear in *Trans. Amer. Math. Soc*.