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Groups with infinite FC-center have the Schmidt property

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Abstract. We show that every countable group with infinite finite conjugacy (FC)-center has the Schmidt property, that is, admits a free, ergodic, measure-preserving action on a standard probability space such that the full group of the associated orbit equivalence relation contains a non-trivial central sequence. As a consequence, every countable, inner amenable group with property (T) has the Schmidt property.

Key words: inner amenable groups, discrete p.m.p. equivalence relations, central sequences
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1. *Introduction*

Let G be a countable group. Throughout the paper, we equip each countable group with the discrete topology unless stated otherwise. We say that G is *inner amenable* if there exists a sequence (ξ_n) of non-negative unit vectors in $\ell^1(G)$ such that, for each $g \in G$, we have $\|\xi_n^g - \xi_n\|_1 \to 0$ and $\xi_n(g) \to 0$, where the function ξ_n^g on G is defined by $\xi_n^g(h) = \xi_n(ghg^{-1})$ for $h \in G$. Inner amenability was introduced by Effros [**Ef**] as a necessary condition for the group von Neumann algebra of G to have property Gamma when G satisfies the infinite conjugacy class (ICC) condition. Inner amenability also arises in the context of p.m.p. actions of G. For brevity, by a *p.m.p.* action of G we mean a measure-preserving action of G on a standard probability space, where 'p.m.p.' stands for 'probability-measure-preserving'. Let us say that a free ergodic p.m.p. action of G is *Schmidt* if the associated orbit equivalence relation admits a non-trivial central sequence in its full group. We say that G has the *Schmidt property* if G has a free ergodic p.m.p. action which is Schmidt. While the Schmidt property of G implies inner amenability of G [**JS**, pp. 113], the converse remains an open problem which was first posed by Schmidt [**Sc**,

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Problem 4.6]. Recent advances have led to the resolution of some related long-standing problems concerning the relationship between inner amenability of groups and various kinds of central sequences [**Ki1**, **V**].

If the functions ξ_n witnessing the inner amenability of G are further required to be G-conjugation invariant, that is, they each satisfy $\xi_n^g = \xi_n$ for all $g \in G$, then an algebraic constraint is imposed on G. In fact, the existence of such a sequence (ξ_n) is equivalent to G having infinite FC-center. The *FC-center* of G is defined as the subgroup of elements $g \in G$ whose centralizer, denoted by $C_G(g)$, is of finite index in G. The FC-center of G is a normal (in fact, a characteristic) subgroup of G.

In studying the structure of inner amenable groups, the second author [**TD**] introduced the *AC-center* of G, which is defined as the subgroup of elements $g \in G$ for which the quotient group $G/\bigcap_{h\in G} hC_G(g)h^{-1}$ is amenable. The AC-center of G is also a characteristic subgroup of G and contains the FC-center of G. If G has infinite AC-center, then G is inner amenable; this follows from the fact that, for each element g in the AC-center of G, the conjugation action of G on the conjugacy class of g factors through an action of the amenable group $G/\bigcap_{h\in G} hC_G(g)h^{-1}$. If G is linear, or, more generally, fulfills a certain chain condition on its subgroups, then inner amenability of G is equivalent to G having infinite AC-center; in this case, the AC-center plays a crucial role in describing the structure of G, and this resulting structure can, in turn, be used to deduce that G has the Schmidt property [**TD**, Theorems 14 and 15]. However, there are many groups with infinite AC-center or FC-center that do not satisfy the relevant chain condition, so the results of [**TD**] do not apply to these groups. In this paper, we solve Schmidt's problem for them affirmatively.

THEOREM 1.1. *Every countable group with infinite AC-center has the Schmidt property.*

In fact, the Schmidt property for groups with infinite AC-center but finite FC-center follows from the constructions in [**TD**] (see §3.1). Thus, most of the proof of Theorem 1.1 is devoted to the case of groups with infinite FC-center.

The following corollary is an immediate consequence of Theorem 1.1 because every inner amenable group with property (T) has infinite FC-center.

COROLLARY 1.2. *Every countable, inner amenable group with property (T) has the Schmidt property.*

It is widely known that property (T) is useful for constructing interesting examples regarding the *non-existence* of non-trivial central sequences in various contexts (e.g., [**DV**, **Ki1**, **KTD**, **PV**, **V**]). By contrast, Corollary 1.2 says that there do not exist any counterexamples to Schmidt's question among groups with property (T).

As mentioned above, the proof of Theorem 1.1 is reduced to that for a countable group G with infinite FC-center. We present two constructions of a free p.m.p. Schmidt action of G. The first construction, given throughout §§2–5, stems from analyzing central sequences for translation groupoids associated with (not necessarily free) p.m.p. actions. This analysis is of independent interest and yields by-products (Theorems 1.3 and 1.5) which do not follow from the second construction. The second construction, given in §6, is by way of ultraproducts of p.m.p. actions. While the first construction splits into cases depending on the structure of G, the second construction does not split into cases and is more direct than the first.

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A summary of the first construction. Let us describe some of the ingredients and by-products of the first construction. The construction is divided into two cases, depending on whether the FC-center has finite or infinite center. Let G be a countable group with infinite FC-center R. If R has finite center C, then G admits a (not necessarily free) profinite action $G \curvearrowright (X, \mu)$ such that the quotient group R/C, which is infinite by assumption, acts freely. This action of R/C leads us to find a central sequence in the full group of the groupoid $G \ltimes (X, \mu)$, similarly to the construction of Popa and Vaes [**PV**] for residually finite groups with infinite FC-center. We need a further task to conclude that G has the Schmidt property since the action $G \curvearrowright (X, \mu)$ is not necessarily free. We will return to this point after discussing the other case.

In the other case, the FC-center of G has infinite center. The following construction is carried out after choosing some infinite abelian normal subgroup A of G contained in the FC-center of G. The group A is not necessarily the center of the FC-center of G. We set $\Gamma = G/A$ and fix a section of the quotient map from G onto Γ. The 2-cocycle $\sigma\colon \Gamma \times \Gamma \to A$ is then associated. The heart of the construction is to introduce the groupoid extension

$$1 \to \mathcal{U} \to \mathcal{G}_{\tilde{\sigma}} \to X \rtimes \Gamma \to 1,$$

which is defined as follows. For some appropriate compact abelian metrizable group L, let X be the group of homomorphisms from A into L and let μ be the normalized Haar measure on X. The conjugation $\Gamma \curvearrowright A$ induces the p.m.p. action $\Gamma \curvearrowright (X, \mu)$. We set $\mathcal{U} = X \times L$ and regard it as the bundle over X with fiber L. Let $X \rtimes \Gamma$ be the translation groupoid and let $(X \rtimes \Gamma)^{(2)}$ be the set of composable pairs of $X \rtimes \Gamma$. The 2-cocycle $\tilde{\sigma}\colon (X \rtimes \Gamma)^{(2)} \to \mathcal{U}$ is then defined by

$$\tilde{\sigma}((\tau, g), (g^{-1}\tau, h)) = (\tau, \tau(\sigma(g, h)))$$

for $\tau \in X$ and $g, h \in \Gamma$ (see [**J**, Theorem 1.1] for a related construction). This 2-cocycle $\tilde{\sigma}$ associates the groupoid $\mathcal{G}_{\tilde{\sigma}}$ that fits into the above exact sequence. Let G act on X via the quotient map from G onto Γ. We then have a natural homomorphism $\eta\colon X \rtimes G \to \mathcal{G}_{\tilde{\sigma}}$ such that $\eta(\tau, a) = (\tau, \tau(a)) \in \mathcal{U}$ for each $\tau \in X$ and $a \in A$. A crucial point is that if we prepare a free p.m.p. action $\mathcal{G}_{\tilde{\sigma}} \curvearrowright (Z, \zeta)$, then we can let $X \rtimes G$ and thus G act on (Z, ζ) via η, so that the action of A factors through the action of $\mathcal{U}$, which is easily handled since L is compact. Moreover we can describe the stabilizer of a point of Z in G in terms of $\ker \eta$, which is contained in $X \rtimes A$.

Compact groups and their p.m.p. actions are utilized in many constructions of Schmidt actions such as in [**DV**, **Ki2**, **Ki3**, **KTD**, **PV**, **TD**]. They are useful on the basis of the following simple fact. For each p.m.p. action $K \curvearrowright (X, \mu)$ of a continuous (rather than compact) group K, each sequence converging to the identity in K also converges to the identity in the automorphism group of (X, μ) in the weak topology. This weak convergence is necessary for a sequence in the full group to be central and is also sufficient if the sequence asymptotically commutes with each element of the acting group G.

Returning to the general set-up, let G be an arbitrary countable group with infinite FC-center. Independent of whether the FC-center of G has finite or infinite center, the above construction yields a p.m.p. action $G \curvearrowright (W, \omega)$ and a central sequence (T_n) in the

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full group of the translation groupoid $G \ltimes (W, \omega)$. The sequence (T_n) is non-trivial in the sense that the automorphism of W induced by T_n is nowhere the identity. We cannot yet conclude that G has the Schmidt property because the action $G \curvearrowright (W, \omega)$ is not necessarily free.

Let us now simplify the set-up as follows. Let G be a countable group with a normal subgroup M and a p.m.p. action $G \curvearrowright (X, \mu)$ such that M acts on X trivially and the quotient group G/M acts on X freely. Suppose that the groupoid $\mathcal{G} := G \ltimes (X, \mu)$ is Schmidt, that is, admits a central sequence (T_n) in its full group such that the automorphism of X induced by T_n is nowhere the identity. Under several additional assumptions, we then construct a *free* p.m.p. Schmidt action of G as follows. After replacing (T_n) by another central sequence appropriately, we obtain the product subgroupoid $M \times \mathcal{R} < \mathcal{G}$ such that $\mathcal{R}$ is the groupoid generated by all T_n and is also principal and hyperfinite. Pick a free p.m.p. action $M \curvearrowright (Y, \nu)$, let $M \times \mathcal{R}$ act on (Y, ν) via the projection from $M \times \mathcal{R}$ onto M, and co-induce the action $\mathcal{G} \curvearrowright (Z, \zeta)$ from the action $M \times \mathcal{R} \curvearrowright (Y, \nu)$. Then we have the lift of (T_n) into the translation groupoid $\mathcal{G} \ltimes (Z, \zeta)$. This lifted sequence is shown to be central in the full group by using that T_n acts on Y trivially (see Proposition 2.4 for treatment of this fact in a more general framework). Moreover, we can naturally define the p.m.p. action $G \curvearrowright (Z, \zeta)$ such that the associated groupoid $G \ltimes (Z, \zeta)$ is identified with $\mathcal{G} \ltimes (Z, \zeta)$. The action $G \curvearrowright (Z, \zeta)$ is free since the action $M \curvearrowright (Y, \nu)$ is free. Thus we obtain a free p.m.p. Schmidt action of G. This construction is flexible enough to apply to the more general set-up, and we are able to deduce the Schmidt property for all groups with infinite FC-center. It also yields the following by-products.

THEOREM 1.3. (Corollary 2.16) *Let G be a countable group and let M be a finite central subgroup of G. Let $G/M \curvearrowright (X, \mu)$ be a free ergodic p.m.p. action and let G act on (X, μ) through the quotient map from G onto G/M. Suppose that the translation groupoid $G \ltimes (X, \mu)$ is Schmidt. Then G has the Schmidt property.*

Remark 1.4. Let G be a countable group and let M be a finite central subgroup of G. It remains unsolved whether the Schmidt property of G/M implies the Schmidt property of G [**KTD**, Question 5.16]. If G/M has infinite AC-center, then G also has the same property and thus has the Schmidt property (see Proposition 3.3 (ii) and related Remark 2.18).

Theorem 1.3 might be used to answer this question affirmatively: if there exists a free ergodic p.m.p. action $G/M \curvearrowright (X, \mu)$ which is Schmidt, along with a non-trivial central sequence in the full group of $(G/M) \ltimes (X, \mu)$ which lifts to a central sequence in the full group of $G \ltimes (X, \mu)$, then we can apply Theorem 1.3 and conclude that G has the Schmidt property. While this lifting problem of central sequences is unsolved in full generality, we note that it is solved affirmatively for stability sequences in [**Ki4**].

A sequence (g_n) of elements of a countable group G is called *central* if, for each $h \in G$, g_n commutes with h for all sufficiently large n.

THEOREM 1.5. (Corollary 2.17) *If a countable group G admits a central sequence diverging to infinity, then G has the Schmidt property.*

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Remark 1.6. Let G be a countable group that admits a central sequence diverging to infinity. If G has trivial center, then the Schmidt property for G can be proved immediately as follows [**Ke2**, Proposition 9.5]. Let G act on the set $G \setminus \{e\}$ by conjugation, which induces the p.m.p. action of G on the product space $X := \prod_{G\setminus\{e\}}[0, 1]$ equipped with the product measure μ of the Lebesgue measure. Then a central sequence in G gives rise to a central sequence in the full group of $G \ltimes (X, \mu)$, and the action $G \curvearrowright (X, \mu)$ is essentially free since G has trivial center.

Let G be a countable group with infinite FC-center. Then, given a sequence (g_n) in its FC-center diverging to infinity, each centralizer $C_G(g_n)$ is of finite index in G, although the index of $C_G(g_n)$ in G possibly grows to infinity. In a sense, the g_n may become less and less central in G as n increases. In this case, the above Bernoulli-like action of G via conjugation $G \curvearrowright G \setminus \{e\}$ is not suitable for establishing the Schmidt property, and another approach must be taken.

1.1. *An organization of the paper.* In §2, we fix notation and terminology for discrete p.m.p. groupoids and describe co-induction of p.m.p. actions of discrete p.m.p. groupoids, extending the co-induction construction for actions of countable groups. As an application, we deduce the Schmidt property for a countable group G under the assumption that G admits a (not necessarily free) p.m.p. action $G \curvearrowright (X, \mu)$ such that the translation groupoid $G \ltimes (X, \mu)$ is Schmidt, together with some additional assumptions. In §3, we collect elementary properties of groups with infinite AC-center and reduce the proof of Theorem 1.1 to that for groups with infinite FC-center. Sections 4 and 5 are devoted to the first proof that groups with infinite FC-center have the Schmidt property. The proof in these two sections is divided into several cases depending on the existence and structure of an infinite abelian normal subgroup of G contained in the FC-center of G. An outline of the proof is given in §3.2. In §3.3, we exhibit examples of groups G corresponding to each of the cases considered in §§4 and 5.

In §6, for a countable group with infinite FC-center, we give the second construction of a free p.m.p. Schmidt action, by way of ultraproducts.

In Appendix A, given an arbitrary countable abelian group A, we present a countable group with property (T) whose center is isomorphic to A. Our construction relies on the construction of Cornulier [**C**] and property (T) of the group $SL_3(\mathbb{Z}[t]) \ltimes \mathbb{Z}[t]^3$, where $\mathbb{Z}[t]$ is the polynomial ring over $\mathbb{Z}$ in one indeterminate t. This result is useful in constructing interesting examples of groups with infinite FC-center, along with Examples 3.6 and 3.7, although it is not necessary for proving Theorem 1.1.

Throughout the paper, unless otherwise mentioned, all relations among Borel sets and maps are understood to hold up to null sets. Let $\mathbb{N}$ denote the set of positive integers.

2. *Central sequences in translation groupoids*

2.1. *Groupoids.* We fix notation and terminology. Let $\mathcal{G}$ be a groupoid. We denote by $\mathcal{G}^0$ the unit space of $\mathcal{G}$ and denote by $r, s\colon \mathcal{G} \to \mathcal{G}^0$ the range and source maps of $\mathcal{G}$, respectively. For $x \in \mathcal{G}^0$, we set $\mathcal{G}^x = r^{-1}(x)$ and $\mathcal{G}_x = s^{-1}(x)$. For a subset $A \subset \mathcal{G}^0$, we set $\mathcal{G}_A = r^{-1}(A) \cap s^{-1}(A)$. The set $\mathcal{G}_A$ is then a groupoid with unit space A with respect

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to the product inherited from $\mathcal{G}$. A groupoid $\mathcal{G}$ is called *Borel* if $\mathcal{G}$ is a standard Borel space, $\mathcal{G}^0$ is a Borel subset of $\mathcal{G}$ and the following maps are all Borel: the range and source maps, the multiplication map $(\gamma, \delta) \mapsto \gamma\delta$ defined for $\gamma, \delta \in \mathcal{G}$ with $s(\gamma) = r(\delta)$, and the inverse map $\gamma \mapsto \gamma^{-1}$. If the range and source maps are further countable-to-one, then $\mathcal{G}$ is called *discrete*. We mean by a *discrete p.m.p. groupoid* a pair $(\mathcal{G}, \mu)$ of a discrete Borel groupoid $\mathcal{G}$ and a Borel probability measure μ on $\mathcal{G}^0$ such that $\int_{\mathcal{G}^0} c_x^r \, d\mu(x) = \int_{\mathcal{G}^0} c_x^s \, d\mu(x)$, where c_x^r and c_x^s are the counting measures on $\mathcal{G}^x$ and $\mathcal{G}_x$, respectively. The space $\mathcal{G}$ is then equipped with this common measure $\int_{\mathcal{G}^0} c_x^r \, d\mu(x) = \int_{\mathcal{G}^0} c_x^s \, d\mu(x)$.

A discrete p.m.p. groupoid is called *principal* if the map $\gamma \mapsto (r(\gamma), s(\gamma))$ is injective. Let $\mathcal{R}$ be a p.m.p. countable Borel equivalence relation on a standard probability space (X, μ). Then the pair $(\mathcal{R}, \mu)$ is naturally a principal discrete p.m.p. groupoid with unit space $\mathcal{R}^0 = \{(x, x) \mid x \in X\}$, which is simply identified with X itself when there is no cause for confusion. The range and source maps are given by $r(x, y) = x$ and $s(x, y) = y$, respectively, and the multiplication and inverse operations are given by $(x, y)(y, z) = (x, z)$ and $(x, y)^{-1} = (y, x)$, respectively. By a *discrete p.m.p. equivalence relation* on a standard probability space (X, μ), we mean a p.m.p. countable Borel equivalence relation on (X, μ) equipped with this structure of a discrete p.m.p. groupoid.

Let $(\mathcal{G}, \mu)$ be a discrete p.m.p. groupoid. A Borel subset $A \subset \mathcal{G}^0$ is called *$\mathcal{G}$-invariant* if $r(\mathcal{G}_x) \subset A$ for μ-almost every $x \in A$. We say that $(\mathcal{G}, \mu)$ is *ergodic* if each $\mathcal{G}$-invariant Borel subset A of $\mathcal{G}^0$ is μ-null or μ-conull. A *local section* of $\mathcal{G}$ is a Borel map $\phi \colon \mathrm{dom}(\phi) \to \mathcal{G}$, where $\mathrm{dom}(\phi)$ is a Borel subset of $\mathcal{G}^0$, such that $\phi(x) \in \mathcal{G}_x$ for each $x \in \mathrm{dom}(\phi)$ and the associated map $\phi^\circ \colon \mathrm{dom}(\phi) \to \mathcal{G}^0$, given by $\phi^\circ = r \circ \phi$, is injective. Two local sections are identified if their domains and values agree up to a μ-null set. For two local sections $\phi \colon A \to \mathcal{G}$, $\psi \colon B \to \mathcal{G}$, the *composition* of them is the local section $\psi \circ \phi \colon (\phi^\circ)^{-1}(\phi^\circ(A) \cap B) \to \mathcal{G}$ defined by $(\psi \circ \phi)(x) = \psi(\phi^\circ(x))\phi(x)$. The *inverse* of a local section $\phi \colon A \to \mathcal{G}$ is the local section $\phi^{-1} \colon \phi^\circ(A) \to \mathcal{G}$ defined by $\phi^{-1}(x) = \phi((\phi^\circ)^{-1}(x))^{-1}$.

We denote by $[\mathcal{G}]$ the group of all local sections ϕ of $\mathcal{G}$ with $\mathrm{dom}(\phi) = \mathcal{G}^0$, and we call $[\mathcal{G}]$ the *full group* of $(\mathcal{G}, \mu)$. If the measure μ should be specified, then we denote it by $[(\mathcal{G}, \mu)]$. In fact, the full group is a group such that the product and inverse operations are given by the composition and inverse, respectively. For $\phi \in [\mathcal{G}]$ and a positive integer n, let ϕ^n denote the n times composition of ϕ with itself, and let ϕ^{-n} denote the inverse of ϕ^n. Let ϕ^0 denote the trivial element of $[\mathcal{G}]$, that is, the identity map on $\mathcal{G}^0$. We draw attention to distinction between the trivial element ϕ^0 of $[\mathcal{G}]$ and the associated map $\phi^\circ = r \circ \phi$.

To each action $G \curvearrowright X$ of a group G on a set X, the *translation groupoid* $\mathcal{G} = G \ltimes X$ is associated as follows. The set of groupoid elements is defined as $\mathcal{G} = G \times X$ with unit space $\{e\} \times X$, which is identified with X if there is no cause for confusion. The range and source maps $r, s \colon \mathcal{G} \to \mathcal{G}^0$ are given by $r(g, x) = gx$ and $s(g, x) = x$, respectively. The multiplication and inverse operations are given by $(g, hx)(h, x) = (gh, x)$ and $(g, x)^{-1} = (g^{-1}, gx)$, respectively. Suppose that G is a countable group and X is a standard Borel space equipped with a Borel probability measure μ. If the action $G \curvearrowright X$ is further Borel and preserves μ, then the pair $(G \ltimes X, \mu)$ is a discrete p.m.p. groupoid and is denoted by $G \ltimes (X, \mu)$. It is also denoted by $G \ltimes X$ for brevity if μ is understood from the context.

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If the action $G \curvearrowright (X, \mu)$ is essentially free, that is, the stabilizer of almost every point of X is trivial, then the groupoid $G \ltimes (X, \mu)$ is isomorphic to the associated orbit equivalence relation $\{(gx, x) \mid g \in G,\ x \in X\}$ via the map $(g, x) \mapsto (gx, x)$.

For each action $G \curvearrowright X$, we similarly define the groupoid $X \rtimes G$ such that the set of groupoid elements is $X \times G$ and the range and source of $(x, g) \in X \times G$ are x and $g^{-1}x$, respectively. Then $X \rtimes G$ is isomorphic to $G \ltimes X$ via the map $(x, g) \mapsto (g, g^{-1}x)$.

Let $p \colon G \times X \to G$ be the projection. Then each local section ϕ of the groupoid $G \ltimes X$ is completely determined by the composed map $p \circ \phi \colon \mathrm{dom}(\phi) \to G$. Thus we will abuse notation and identify ϕ with $p \circ \phi$ if there is no cause for confusion. The group G embeds into $[G \ltimes X]$ via the map $g \mapsto \phi_g$, where $\phi_g \colon X \to G$ is the constant map with value g.

2.2. *Central sequences.* Let $(\mathcal{G}, \mu)$ be a discrete p.m.p. groupoid. A sequence (A_n) of Borel subsets of the unit space $\mathcal{G}^0$ is called *asymptotically invariant* for $(\mathcal{G}, \mu)$ if

$$\mu(T^\circ A_n \triangle A_n) \to 0$$

for every $T \in [\mathcal{G}]$. A sequence (T_n) in the full group $[\mathcal{G}]$ is called *central* in $[\mathcal{G}]$ if T_n asymptotically commutes with every $S \in [\mathcal{G}]$, that is,

$$\mu(\{x \in \mathcal{G}^0 \mid (T_n \circ S)x \neq (S \circ T_n)x\}) \to 0$$

for every $S \in [\mathcal{G}]$.

Remark 2.1. Let G be a countable subgroup of $[\mathcal{G}]$ and suppose that G generates $\mathcal{G}$, that is, the minimal subgroupoid of $\mathcal{G}$ containing G in its full group is equal to $\mathcal{G}$. Then a sequence (A_n) of Borel subsets of $\mathcal{G}^0$ is asymptotically invariant for $(\mathcal{G}, \mu)$ if $\mu(gA_n \triangle A_n) \to 0$ for every $g \in G$ [**JS**, pp. 93]. Moreover, a sequence (T_n) in $[\mathcal{G}]$ is central if and only if T_n asymptotically commutes with every $g \in G$ and $\mu(T_n^\circ A \triangle A) \to 0$ for every Borel subset $A \subset X$ ([**JS**, Remark 3.3] or [**Ki4**, Lemma 2.3]). While these assertions are verified only for translation groupoids $G \ltimes (X, \mu)$ in the cited papers, the same proof is available for the above generalization.

We say that a discrete p.m.p. groupoid $(\mathcal{G}, \mu)$ is *Schmidt* if there exists a central sequence (T_n) in $[\mathcal{G}]$ such that $\mu(\{x \in X \mid T_n^\circ x \neq x\}) \to 1$. We say that a p.m.p. action $G \curvearrowright (X, \mu)$ of a countable group G is *Schmidt* if the groupoid $G \ltimes (X, \mu)$ is Schmidt. If a countable group G admits a free ergodic p.m.p. action which is Schmidt, then we say that G has the *Schmidt property*. (N.B. A countable group, being a discrete p.m.p. groupoid on a singleton, is never Schmidt.) The following lemma implies that the Schmidt property of G follows once we find a free p.m.p. Schmidt action of G which may not be ergodic. We refer to [**H**, §6] for the ergodic decomposition of discrete p.m.p. groupoids.

LEMMA 2.2. *Let $(\mathcal{G}, \mu)$ be a discrete p.m.p. groupoid with the ergodic decomposition map $\pi \colon (\mathcal{G}^0, \mu) \to (Z, \zeta)$ and the disintegration $\mu = \int_Z \mu_z \, d\zeta(z)$. Suppose that $(\mathcal{G}, \mu)$ is Schmidt and let (T_n) be a central sequence in $[(\mathcal{G}, \mu)]$ such that $\mu(\{x \in X \mid T_n^\circ x \neq x\}) \to 1$. Then there exists a subsequence (T_{n_i}) of (T_n) such that, for ζ-almost every $z \in Z$, (T_{n_i}) is a central sequence in $[(\mathcal{G}, \mu_z)]$ such that $\mu_z(\{x \in X \mid T_{n_i}^\circ x \neq x\}) \to 1$. Thus, for ζ-almost every $z \in Z$, the ergodic component $(\mathcal{G}, \mu_z)$ is Schmidt.*

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Proof. Let $\mathcal{B}$ be the sigma field of Borel subsets of $\mathcal{G}^0$. Let $\{A_k\}$ be a countable subfamily of $\mathcal{B}$ which generates $\mathcal{B}$. Then, for every $z \in Z$, the family $\{A_k\}$ generates a dense subfield in $\mathcal{G}^0$ with respect to μ_z. Since (T_n) is central in $[(\mathcal{G}, \mu)]$, we have $\int_Z \mu_z(T_n^\circ A_k \triangle A_k)\, d\zeta(z) = \mu(T_n^\circ A_k \triangle A_k) \to 0$ for each k. Thus, after passing to a subsequence of (T_n), for ζ-almost every $z \in Z$, we have $\mu_z(T_n^\circ A_k \triangle A_k) \to 0$ for each k.

Applying the Lusin–Novikov uniformization theorem [**Ke1**, Theorem 18.10], we obtain a countable collection $\{\phi_l\}$ of local sections of $\mathcal{G}$ such that $\bigcup_l \phi_l(\mathrm{dom}(\phi_l)) = \mathcal{G}$. Similarly to the above, after passing to a subsequence of (T_n), for ζ-almost every $z \in Z$, we have $\mu_z(\{x \in X \mid (\phi_l \circ T_n)x = (T_n \circ \phi_l)x\}) \to 1$ for each l and $\mu_z(\{x \in X \mid T_n^\circ x \neq x\}) \to 1$. The first convergence together with the convergence obtained in the last paragraph implies that (T_n) is a central sequence in $[(\mathcal{G}, \mu_z)]$ for ζ-almost every $z \in Z$. □

2.3. *Co-induced actions.* Co-induction is a canonical method to obtain a p.m.p. action of a countable group from a p.m.p. action of its subgroup. We generalize this for p.m.p. actions of discrete p.m.p. groupoids.

Remark 2.3. Formally, by an action of a groupoid $\mathcal{G}$ we mean an action of $\mathcal{G}$ on a space Z fibered over $\mathcal{G}^0$ such that each $g \in \mathcal{G}$ gives rise to an isomorphism from the fiber at the source of g onto the fiber at the range of g. Then we say that $\mathcal{G}$ acts on the fibered space Z. We often obtain such an action of $\mathcal{G}$ from a groupoid homomorphism $\alpha\colon \mathcal{G} \to \mathrm{Aut}(Y)$ for some space Y, as follows. Let $Z = \mathcal{G}^0 \times Y$ and regard it as being fibered over $\mathcal{G}^0$ via the projection. Then $\mathcal{G}$ acts on Z by $g(s(g), y) = (r(g), \alpha(g)y)$. For simplicity, we will often abuse terminology of actions, and call this action on the fibered space Z an action of $\mathcal{G}$ on the space Y (which is not fibered over $\mathcal{G}^0$, however) unless there is cause for confusion.

Let $(\mathcal{G}, \mu)$ be a discrete p.m.p. groupoid and set $X = \mathcal{G}^0$. Let $\mathcal{S}$ be a Borel subgroupoid of $\mathcal{G}$ and suppose that $\mathcal{S}$ admits the measure-preserving action on a standard probability space (Y, ν) arising from a Borel homomorphism $\alpha\colon \mathcal{S} \to \mathrm{Aut}(Y, \nu)$. From this action of $\mathcal{S}$, we co-induce a p.m.p. action $\mathcal{G} \curvearrowright (Z, \zeta)$ as follows. For each $x \in X$, we set

$$Z_x = \{f : \mathcal{G}^x \to Y \mid f(gh^{-1}) = \alpha(h) f(g) \text{ for each } g \in \mathcal{G}^x \text{ and each } h \in \mathcal{S}_{s(g)}\}$$

and define Z as the disjoint union $Z = \bigsqcup_{x \in X} Z_x$. The set Z is fibered with respect to the projection $p\colon Z \to X$ sending each element of Z_x to x. The groupoid $\mathcal{G}$ acts on Z by

$$(gf)(g') = f(g^{-1}g')$$

for $g \in \mathcal{G}_x$, $g' \in \mathcal{G}^{r(g)}$ and $f \in Z_x$ with $x \in X$.

A measure-space structure on Z is defined as follows. We have the decomposition of the unit space, $X = \bigsqcup_{m \in \mathbb{N} \cup \{\infty\}} X_m$, into the $\mathcal{G}$-invariant Borel subsets X_m such that the index of $\mathcal{S}_{X_m}$ in $\mathcal{G}_{X_m}$ is the constant m. First, suppose that $X = X_m$ for some $m \in \mathbb{N} \cup \{\infty\}$. Let $\{\psi_i\}_{i=1}^m$ be a family of choice functions for the inclusion $\mathcal{S} < \mathcal{G}$, that is, a family of Borel maps $\psi_i\colon X \to \mathcal{G}$ such that, for each $x \in X$, we have $\psi_i(x) \in \mathcal{G}^x$ and the family $\{\psi_i(x)\}_{i=1}^m$ is a complete set of representatives of all the equivalence classes in $\mathcal{G}^x$, where the equivalence relation on $\mathcal{G}^x$ is associated to the inclusion $\mathcal{S} < \mathcal{G}$ as follows: two elements $g, h \in \mathcal{G}^x$ are equivalent if and only if $g^{-1}h \in \mathcal{S}$. Then Z is identified

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with the product space $X \times \prod_{i=1}^{m} Y$ under the map sending each $f \in Z_x$ with $x \in X$ to $(x, (f(\psi_i(x)))_i)$. The measure-space structure on Z is induced by this identification, where the space $X \times \prod_{i=1}^{m} Y$ is equipped with the product measure $\mu \times \prod_{i=1}^{m} \nu$. The action of $\mathcal{G}$ on Z is Borel and preserves the probability measure on Z.

If X is not necessarily equal to X_m for some $m \in \mathbb{N}$, then, as already stated, we have the decomposition $X = \bigsqcup_{m \in \mathbb{N} \cup \{\infty\}} X_m$ into $\mathcal{G}$-invariant Borel subsets. The set Z is decomposed into the $\mathcal{G}$-invariant subsets $p^{-1}(X_m)$, on which the measure-space structure is given in the way described in the previous paragraph. Then the measure-space structure is also induced on Z, so that each $p^{-1}(X_m)$ is Borel and the projection $p\colon Z \to X$ is measure-preserving.

Let ζ be the induced probability measure on Z. We define a discrete p.m.p. groupoid $(\mathcal{G}, \mu) \ltimes (Z, \zeta) = (\tilde{\mathcal{G}}, \tilde{\mu})$ as follows. The set of groupoid elements is the fibered product $\tilde{\mathcal{G}} := \mathcal{G} \times_X Z$ with respect to the source map $s\colon \mathcal{G} \to X$ and the projection $p\colon Z \to X$. The unit space is $\tilde{\mathcal{G}}^0 := Z$ with measure $\tilde{\mu} := \zeta$. The range and source maps are given by $\tilde{r}(g, z) = gz$ and $\tilde{s}(g, z) = z$, respectively, with groupoid operations given by $(gh, z) = (g, hz)(h, z)$ and $(g, z)^{-1} = (g^{-1}, gz)$. Each element $T \in [\mathcal{G}]$ lifts to the element $\tilde{T} \in [\tilde{\mathcal{G}}]$ defined by $\tilde{T}z = (Tx, z)$ for $z \in Z_x$ with $x \in X$.

Let us recall the following fact from the proof of [**TD**, Theorem 15] or [**KTD**, Example 8.8]. Let G be a countable group, let C be a central subgroup of G and let $C \curvearrowright (Y, \nu)$ be a p.m.p. action. We define $G \curvearrowright (Z, \zeta)$ as the action co-induced from the action $C \curvearrowright (Y, \nu)$. Then each sequence of elements of C that converges to the identity in $\mathrm{Aut}(Y, \nu)$ is central in the full group of the groupoid $G \ltimes (Z, \zeta)$. We generalize this fact to give the following proposition.

PROPOSITION 2.4. *Let $(\mathcal{G}, \mu)$ be a discrete p.m.p. groupoid and set $X = \mathcal{G}^0$. Let $\mathcal{S}$ be a Borel subgroupoid of $\mathcal{G}$, let (Y, ν) be a standard probability space and let $\alpha\colon \mathcal{S} \to \mathrm{Aut}(Y, \nu)$ be a Borel homomorphism. Let $\mathcal{G} \curvearrowright (Z, \zeta)$ denote the action co-induced from the action $\mathcal{S} \curvearrowright (X \times Y, \mu \times \nu)$ via α. Let (T_n) be a central sequence in $[\mathcal{G}]$ such that each T_n belongs to $[\mathcal{S}]$ and, for each Borel subset $B \subset Y$,*

$$\int_X \nu(\alpha(T_n x)B \,\triangle\, B)\, d\mu(x) \to 0$$

as $n \to \infty$. Then the sequence $(\tilde{T}_n)$ of the lifts of T_n is central in the full group of the groupoid $(\mathcal{G}, \mu) \ltimes (Z, \zeta)$ defined above.

Proof. Since (T_n) is central in $[\mathcal{G}]$, by the definition of lifts, $\tilde{T}_n$ asymptotically commutes with the lift of each $S \in [\mathcal{G}]$, that is, $\zeta(\{z \in Z \mid (\tilde{S} \circ \tilde{T}_n)z \neq (\tilde{T}_n \circ \tilde{S})z\}) \to 0$ for each $S \in [\mathcal{G}]$. Hence it suffices to show that, for each Borel subset $C \subset Z$, we have $\zeta(\tilde{T}_n^{\circ} C \,\triangle\, C) \to 0$ (Remark 2.1). We may suppose that the index of $\mathcal{S}$ in $\mathcal{G}$ is the constant $m \in \mathbb{N} \cup \{\infty\}$. Let $\{\psi_i\}_{i=1}^{m}$ be a family of choice functions for the inclusion $\mathcal{S} < \mathcal{G}$ and identify Z with the product space $X \times \prod_{i=1}^{m} Y$ as before the proposition. Then it suffices to show that $\zeta(\tilde{T}_n^{\circ} C \,\triangle\, C) \to 0$ for each cylindrical subset

$$C = \{(x, (y_i)_{i=1}^{m}) \in X \times \textstyle\prod_{i=1}^{m} Y \mid x \in A \text{ and } y_i \in B_i \text{ for each } i \in \{1, \ldots, l\}\},$$

where $A \subset X$ and $B_1, \ldots, B_l \subset Y$ are Borel subsets and l is a positive integer with $l \leq m$.

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Let $\varepsilon > 0$. We set $\bar{\psi}_i = s \circ \psi_i$ and set $\phi_i(x) = \psi_i(x)^{-1}$ for $x \in X$. Since ϕ_i is the union of local sections of $\mathcal{G}$, the assumption on the central sequence (T_n) implies that there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then:

(1) $\mu(T_n^\circ A \triangle A) < \varepsilon$;

(2) $\int_X \nu(\alpha(T_n(\bar{\psi}_i(x)))B_i \triangle B_i)\, d\mu(x) < \varepsilon/l$ for each $i \in \{1, \ldots, l\}$; and

(3) $\mu(A_1) > \mu(A) - \varepsilon$,

where A_1 is defined as the set of all elements $x \in A$ such that $(\phi_i \circ T_n)x = (T_n \circ \phi_i)x$ for each $i \in \{1, \ldots, l\}$. Fix $n \in \mathbb{N}$ with $n \geq N$. We show that $\tilde{T}_n^\circ f \in C$ if f belongs to the set C_1, which is slightly smaller than C, of all elements $(x, (y_i)_{i=1}^m) \in X \times \prod_{i=1}^m Y$ such that:

- $x \in A_1 \cap (T_n^\circ)^{-1} A$; and
- $y_i \in \alpha(T_n(\bar{\psi}_i(x)))^{-1} B_i \cap B_i$ for each $i \in \{1, \ldots, l\}$.

We pick $f = (x, (y_i)_{i=1}^m) \in C_1$ and set $y = T_n^\circ x$. For each $i \in \{1, \ldots, l\}$, regarding f as a map from $\mathcal{G}^x$ to Y belonging to the set Z_x,

$$\begin{aligned}(\tilde{T}_n^\circ f)(\psi_i(y)) &= f((T_n x)^{-1}\psi_i(y)) = f(\psi_i(x) T_n(\bar{\psi}_i(x))^{-1}) \\ &= \alpha(T_n(\bar{\psi}_i(x))) f(\psi_i(x)),\end{aligned}$$

where the second equation follows from $x \in A_1$ and $\phi_i^\circ(x) = \bar{\psi}_i(x)$. The right-hand side belongs to B_i because $f(\psi_i(x)) = y_i \in \alpha(T_n(\bar{\psi}_i(x)))^{-1} B_i$. Moreover, $\tilde{T}_n^\circ f \in Z_y$ and $y \in A$ because $x \in (T_n^\circ)^{-1} A$. Therefore $\tilde{T}_n^\circ f \in C$. As a result, we obtain the inequality

$$\zeta(C \cap (\tilde{T}_n^\circ)^{-1} C) \geq \zeta(C_1) = \int_{A_1 \cap (T_n^\circ)^{-1} A} \prod_{i=1}^{l} \nu(\alpha(T_n(\bar{\psi}_i(x)))^{-1} B_i \cap B_i)\, d\mu(x).$$

The left-hand side of this inequality is equal to $\zeta(C) - \zeta(\tilde{T}_n^\circ C \triangle C)/2$, and the right-hand side is equal to

$$\begin{aligned}&\int_{A_1 \cap (T_n^\circ)^{-1} A} \prod_{i=1}^{l} \bigg(\nu(B_i) - \frac{1}{2}\nu(\alpha(T_n(\bar{\psi}_i(x)))^{-1} B_i \triangle B_i)\bigg)\, d\mu(x) \\ &\quad > \zeta(C) - \mu(A \setminus (A_1 \cap (T_n^\circ)^{-1} A)) - \varepsilon/2 \\ &\quad \geq \zeta(C) - (\mu(A \setminus A_1) + \mu(A \setminus (T_n^\circ)^{-1} A)) - \varepsilon/2 > \zeta(C) - 2\varepsilon\end{aligned}$$

by (1)–(3), where, to deduce the first inequality, we use the inequality $|\prod_{i=1}^l a_i - \prod_{i=1}^l b_i| \leq \sum_{i=1}^l |a_i - b_i|$ for $a_i, b_i \in [0, 1]$. Therefore $\zeta(\tilde{T}_n^\circ C \triangle C) < 4\varepsilon$. □

2.4. *Construction of a free action.* Under the assumption that a countable group G admits a p.m.p. Schmidt action, in Theorem 2.5, we present a sufficient condition for G to admit a free p.m.p. Schmidt action. Another sufficient condition will be given in Theorem 2.14 in §2.6. We remark that the analogous problem for stability in place of the Schmidt property is solved in [**Ki3**, Theorem 1.4] with a much simpler method.

For $p \in \mathbb{N}$ and a Borel automorphism T of a standard Borel space X, we call a point $x \in X$ a *p-periodic point* of T if $T^p x = x$ and $T^i x \neq x$ for all $i \in \mathbb{N}$ less than p. If a point $x \in X$ is a p-periodic point of T for some $p \in \mathbb{N}$, then x is called a *periodic point* of T

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and the number p is called the *period* of x. For possible constraints on periods of T_n° for a central sequence (T_n) in the full group, we refer to [**KTD**, Proposition 8.7].

THEOREM 2.5. *Let G be a countable group, let $G \curvearrowright (X, \mu)$ be a p.m.p. action and let $\pi\colon (X, \mu) \to (\Omega, \eta)$ be a G-equivariant measure-preserving map into a standard probability space (Ω, η). Suppose that, for μ-almost every $x \in X$, the stabilizer of x in G depends only on $\pi(x)$ and we thus have a subgroup M_ω of G indexed by η-almost every $\omega \in \Omega$ such that, for μ-almost every $x \in X$, the stabilizer of x in G is equal to $M_{\pi(x)}$. We set $(\mathcal{G}, \mu) = G \ltimes (X, \mu)$.*

Suppose that there exists a central sequence (S_n) in $[\mathcal{G}]$ such that:

- *for all n, S_n° preserves each fiber of π, that is, we have $\pi(S_n^\circ x) = \pi(x)$ for μ-almost every $x \in X$; and*
- *$\mu(\{x \in X \mid S_n^\circ x \neq x,\ S_n x \in C_G(M_{\pi(x)})\}) \to 1$ as $n \to \infty$,*

where, for a subgroup $M < G$, we denote by $C_G(M)$ the centralizer of M in G. For $p \in \mathbb{N}$, let $A_n^p \subset X$ be the set of p-periodic points of S_n°. Suppose further that, for each $p \in \mathbb{N}$, we have $\mu(A_n^p) \to 0$ as $n \to \infty$. Then G has the Schmidt property.

The proof of this theorem will be given after proving Lemmas 2.6 and 2.7 below. For a discrete p.m.p. groupoid $(\mathcal{G}, \mu)$ and an element $T \in [\mathcal{G}]$, we say that T is *periodic* if, for μ-almost every $x \in \mathcal{G}^0$, there exists a $p \in \mathbb{N}$ such that x is a p-periodic point of T° and $T^p x = e$. We should emphasize that T is not necessarily periodic even if every point of X is a periodic point of the induced automorphism T°.

LEMMA 2.6. *Let G be a countable group, let $G \curvearrowright (X, \mu)$ be a p.m.p. action and let $\pi\colon (X, \mu) \to (\Omega, \eta)$ be a G-equivariant measure-preserving map satisfying the assumption in the first paragraph in Theorem 2.5. We set $(\mathcal{G}, \mu) = G \ltimes (X, \mu)$.*

Pick $\varepsilon > 0$ and $S \in [\mathcal{G}]$ such that S° preserves each fiber of π. Let D and E be Borel subsets of X with $D \subset E$ and suppose that the following three conditions hold.

(1) *If $x \in D$, then $S^\circ x \neq x$ and $Sx \in C_G(M_{\pi(x)})$, and if $x \in D$ is, further, a p-periodic point of S° for some $p \in \mathbb{N}$, then either $p > 1/\varepsilon$ or $S^p x = e$.*
(2) *The inequality $\mu(E \setminus D) < \varepsilon\mu(E)$ holds.*
(3) *The inclusion $S^\circ D \subset E$ holds.*

Then there exists an element $T \in [\mathcal{G}_E]$ such that:

(4) *T is periodic;*
(5) *T° preserves each fiber of π and $Tx \in C_G(M_{\pi(x)})$ for each $x \in E$; and*
(6) *$\mu(\{x \in E \mid Tx \neq Sx\}) < 5\varepsilon\mu(E)$.*

Proof. For a positive integer k, we set

$$Z_k = \{x \in D \mid S^\circ x,\ (S^\circ)^2 x, \ldots, (S^\circ)^{k-1} x \in D,\ (S^\circ)^k x \notin D\}.$$

The sets Z_k are mutually disjoint and satisfy $S^\circ Z_{k+1} \subset Z_k$ and $Z_1 = D \setminus (S^\circ)^{-1} D$. Thus

$$\mu(Z_1) = \mu(D \setminus (S^\circ)^{-1} D) = \mu(S^\circ D \setminus D) \le \mu(E \setminus D) < \varepsilon\mu(E)$$

by conditions (2) and (3).

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We define a local section T of $\mathcal{G}$ on Z_k for $k \geq 2$, on $S^\circ Z_2$ and on $Z_1 \setminus S^\circ Z_2$, respectively, as follows; it is defined so that T is periodic and equal to S on a subset as large as possible. If $x \in Z_k$ and $k \geq 2$, then we set $Tx = Sx$. For almost every $x \in S^\circ Z_2$, there is a maximal integer $k \geq 2$ such that $x \in (S^\circ)^{k-1} Z_k$, and we let $y \in Z_k$ be the point with $x = (S^\circ)^{k-1} y$ and set $Tx = (S^{k-1} y)^{-1}$. On $Z_1 \setminus S^\circ Z_2$, we set $Tx = e$ for each point x of that set. We defined the local section T on the union $Z \coloneqq \bigcup_{k=1}^{\infty} Z_k$ and have the inequality

$$\mu(\{x \in Z \mid Tx \neq Sx\}) \leq \mu(Z_1) < \varepsilon \mu(E). \tag{2.1}$$

We set $D_1 = D \setminus Z$, which is S°-invariant. Let B be the set of points of D_1 that are p-periodic points of S° for some $p \in \mathbb{N}$. Let C be the complement of B in D_1, that is, the set of aperiodic points of S° in D_1. For an integer $p \geq 2$, let B_p denote the set of p-periodic points of S° in B. Then each B_p is S°-invariant, and B is the disjoint union of the sets B_p with $p \geq 2$ since $S^\circ x \neq x$ for each $x \in D$ by condition (1).

We extend the domain of T to the set B as follows. If $p \leq 1/\varepsilon$, then, for each $x \in B_p$, we have $S^p x = e$ by condition (1) and we thus set $T = S$ on B_p, so that T is periodic on it. Otherwise, that is, if $p > 1/\varepsilon$, pick a Borel fundamental domain $B'_p \subset B_p$ of the periodic automorphism $S^\circ|_{B_p}$. We set $Tx = Sx$ for $x \in B_p \setminus (S^\circ)^{-1} B'_p$ and set $Tx = (S^{p-1}(S^\circ x))^{-1}$ for $x \in (S^\circ)^{-1} B'_p$. Then $T^p x = e$ for each $x \in B_p$, and

$$\mu(\{x \in B \mid Tx \neq Sx\}) < \varepsilon \mu(E) \tag{2.2}$$

because

$$\mu(\{x \in B \mid Tx \neq Sx\}) \leq \sum_{p > 1/\varepsilon} \mu((S^\circ)^{-1} B'_p) = \sum_{p > 1/\varepsilon} p^{-1} \mu(B_p) \leq \varepsilon \mu(B) \leq \varepsilon \mu(E).$$

We next define T on C, the set of aperiodic points of S° in D_1. Let N be a positive integer with $1/N < \varepsilon \mu(E)$. By the Rokhlin lemma, we can find a Borel subset $C_0 \subset C$ such that $C_0, S^\circ C_0, \ldots, (S^\circ)^{N-1} C_0$ are mutually disjoint and $\mu(C \setminus \bigcup_{n=0}^{N-1} (S^\circ)^n C_0) < \varepsilon \mu(E)$. We define T on C as follows. For $x \in C_0$ and $n \in \{0, 1, \ldots, N-2\}$, we set $T((S^\circ)^n x) = S((S^\circ)^n x)$ and $T((S^\circ)^{N-1} x) = (S^{N-1} x)^{-1}$. If $x \in C \setminus \bigcup_{n=0}^{N-1} (S^\circ)^n C_0$, then we set $Tx = e$. Then T is periodic on C in the sense that each $x \in C$ is a p-periodic point of T° for some $p \in \mathbb{N}$, and we then have $T^p x = e$. We also have

$$\mu(\{x \in C \mid Tx \neq Sx\}) \leq \mu((S^\circ)^{N-1} C_0) + \mu(C \setminus \textstyle\bigcup_{n=0}^{N-1} (S^\circ)^n C_0) < 2\varepsilon \mu(E). \tag{2.3}$$

Finally, we define T on $E \setminus D$ by $Tx = e$ for each $x \in E \setminus D$. By construction, T° is an automorphism of each of Z, B, C and $E \setminus D$ and hence of E. Thus we defined $T \in [\mathcal{G}_E]$, which is periodic. This is desired. Indeed, for each $x \in E$, the element Tx is either e or the product of some values of S, which belong to $C_G(M_{\pi(x)})$ by condition (1). Therefore T fulfills condition (5). By inequalities (2.1)–(2.3) and condition (2),

$$\mu(\{x \in E \mid Tx \neq Sx\}) < 4\varepsilon \mu(E) + \mu(E \setminus D) < 5\varepsilon \mu(E). \qquad \square$$

In order to state the next lemma, we prepare the following terminology. Let $(\mathcal{G}, \mu)$ be a discrete p.m.p. groupoid. For $T, S \in [\mathcal{G}]$, we say that T and S *commute* if $T \circ S = S \circ T$. Let $T = (T_1, \ldots, T_n)$ be a finite sequence of elements of $[\mathcal{G}]$ such that T_i and T_j commute

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for all i and j. For $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$, we set

$$T^k = (T_n)^{k_n} \circ \cdots \circ (T_1)^{k_1}.$$

For $l = (l_1, \ldots, l_n) \in \mathbb{N}^n$, we say that a point $x \in \mathcal{G}^0$ is (l, T)-*periodic* if the following two conditions hold.

- For every $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$, we have $(T^k)^\circ x = x$ if and only if $k_i \equiv 0$ modulo l_i for all $i \in \{1, \ldots, n\}$.
- If this equivalent condition holds, then we further have $T^k x = e$.

For a discrete p.m.p. equivalence relation $\mathcal{Q}$ on a standard probability space (X, μ), by a *Borel transversal* of $\mathcal{Q}$ we mean a Borel subset of X that meets each equivalence class of $\mathcal{Q}$ at exactly one point.

LEMMA 2.7. *With the notation and the assumption in Theorem 2.5, let $\mathcal{R}$ be the orbit equivalence relation associated with the action $G \curvearrowright (X, \mu)$. Then there exists a central sequence $(T_n)_{n\in\mathbb{N}}$ in $[\mathcal{G}]$ satisfying the following four conditions.*

(i) *We have $\mu(\{x \in X \mid T_n^\circ x \neq x\}) \to 1$.*

(ii) *For each n, T_n° preserves each fiber of π and $T_n x \in C_G(M_{\pi(x)})$ for all $x \in X$.*

(iii) *For each m and n, T_m and T_n commute.*

(iv) *Let $\mathcal{Q}_n$ be the subrelation of $\mathcal{R}$ generated by $T_1^\circ, \ldots, T_n^\circ$. Then there exists a Borel transversal $E_{n+1} \subset X$ of $\mathcal{Q}_n$ and its Borel partition $E_{n+1} = \bigsqcup_{l\in\mathbb{N}^n} E_{n+1}^l$ such that, for each $l = (l_1, \ldots, l_n) \in \mathbb{N}^n$:*
   - *every point of E_{n+1}^l is (l, T)-periodic, where $T = (T_1, \ldots, T_n)$;*
   - *$T_{n+1}^\circ E_{n+1}^l = E_{n+1}^l$; and*
   - *if $n \geq 2$, then $E_{n+1}^l \subset E_n^{(l_1,\ldots,l_{n-1})}$.*

   In particular, for each n, if $\mathcal{E}_n$ denotes the subgroupoid of $\mathcal{G}$ generated by $T_1, \ldots, T_n$ (that is, the minimal subgroupoid of $\mathcal{G}$ containing $T_1, \ldots, T_n$ in its full group), then $\mathcal{E}_n$ and $\mathcal{Q}_n$ are isomorphic under the quotient map from $\mathcal{G}$ onto $\mathcal{R}$.

Proof. Fix a decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive numbers converging to zero. We inductively construct a sequence $(T_n, E_{n+1})_{n\in\mathbb{N}}$ of pairs satisfying conditions (ii)–(iv) and the inequality $\mu(\{x \in X \mid T_n x \neq S_n x\}) < 7\varepsilon_n$ for all n. This inequality implies condition (i) and also implies that the sequence $(T_n)_{n\in\mathbb{N}}$ is central in $[\mathcal{G}]$.

In Theorem 2.5, we assume that, for each $p \in \mathbb{N}$, we have $\mu(A_n^p) \to 0$ as $n \to \infty$, where A_n^p is the set of p-periodic points of S_n°. After replacing S_1 with S_n for a large n, we may assume that $\mu(X \setminus D_1) < \varepsilon_1$, where D_1 is defined as the set of points $x \in X$ such that $S_1^\circ x \neq x$, $S_1 x \in C_G(M_{\pi(x)})$, and if x is a p-periodic point of S_1° for some $p \in \mathbb{N}$, then $p > 1/\varepsilon_1$. Letting $D = D_1$ and $E = X$, we apply Lemma 2.6. We then obtain a periodic $T_1 \in [\mathcal{G}]$ such that T_1° preserves each fiber of π, we have $T_1 x \in C_G(M_{\pi(x)})$ for almost every $x \in X$, and $\mu(\{x \in X \mid T_1 x \neq S_1 x\}) < 5\varepsilon_1 < 7\varepsilon_1$. Since T_1 is periodic, we can find a Borel fundamental domain $E_2 \subset X$ for the automorphism T_1° of X and its Borel partition $E_2 = \bigsqcup_{l\in\mathbb{N}} E_2^l$ such that $\mathcal{Q}_1 E_2^l$ is equal to the set of l-periodic points of T_1°, where $\mathcal{Q}_1$ is the subrelation of $\mathcal{R}$ generated by T_1°. This completes the first step of the induction.

Assuming that we have constructed $T_1, \ldots, T_{n-1}$ and $E_2, \ldots, E_n$, we construct T_n and E_{n+1}. By the induction hypothesis, the equivalence relation $\mathcal{Q}_{n-1}$ generated

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by $T_1^\circ, \ldots, T_{n-1}^\circ$ admits a Borel transversal $E_n \subset X$ and its Borel partition $E_n = \bigsqcup_{l \in \mathbb{N}^{n-1}} E_n^l$ such that, for each $l \in (l_1, \ldots, l_{n-1}) \in \mathbb{N}^{n-1}$, every point of E_n^l is (l, T)-periodic, where we set $T = (T_1, \ldots, T_{n-1})$. We choose a finite subset $L_n \subset \mathbb{N}^{n-1}$ such that $\mu(E_n^l) > 0$ for all $l \in L_n$ and

$$\mu(X \setminus \mathcal{Q}_{n-1} F_n) < \varepsilon_n, \tag{2.4}$$

where we set $F_n = \bigsqcup_{l \in L_n} E_n^l$. After replacing S_n with S_m for a large m, we may assume that

$$\mu(E_n^l \setminus D_n^l) < \varepsilon_n \mu(E_n^l) \tag{2.5}$$

for each $l \in L_n$ if D_n^l is defined as the set of points $x \in E_n^l$ such that:

- $x \in E_n^l \cap ((S_n^\circ)^{-1} E_n^l)$, $S_n^\circ x \neq x$ and $S_n x \in C_G(M_{\pi(x)})$;
- if x is a p-periodic point of S_n° for some $p \in \mathbb{N}$, then $p > 1/\varepsilon_n$; and
- $(S_n \circ T^k)x = (T^k \circ S_n)x$ for each $k = (k_1, \ldots, k_{n-1}) \in \Phi_l$,

where we set

$$\Phi_l = \{0, 1, \ldots, l_1 - 1\} \times \{0, 1, \ldots, l_2 - 1\} \times \cdots \times \{0, 1, \ldots, l_{n-1} - 1\}.$$

Letting $D = D_n^l$ and $E = E_n^l$, we apply Lemma 2.6 for each $l \in L_n$. Then there exists a periodic $T_n \in [\mathcal{G}_{F_n}]$ such that T_n° preserves each E_n^l with $l \in L_n$, we have $T_n x \in C_G(M_{\pi(x)})$ for almost every $x \in F_n$ and, for each $l \in L_n$,

$$\mu(\{x \in E_n^l \mid T_n x \neq S_n x\}) < 5\varepsilon_n \mu(E_n^l). \tag{2.6}$$

We extend the local section T_n to the set $\mathcal{Q}_{n-1} F_n$ so that it commutes with $T_1, \ldots, T_{n-1}$. That is, if $l \in (l_1, \ldots, l_{n-1}) \in L_n$ and $x \in E_n^l$, then we set

$$T_n((T^k)^\circ x) = ((T^k \circ T_n)x)(T^k x)^{-1}$$

for $k = (k_1, \ldots, k_{n-1}) \in \Phi_l$. We note that by condition (iv) for $T_1, \ldots, T_{n-1}$, which is an induction hypothesis, each point of $\mathcal{Q}_{n-1} F_n$ is uniquely written as $(T^k)^\circ x$ for some $k \in \Phi_l$ and $x \in E_n^l$ with $l \in L_n$. Finally, we define T_n on $X \setminus \mathcal{Q}_{n-1} F_n$ by $T_n x = e$ for each point x in that set. Then the element $T_n \in [\mathcal{G}]$ satisfies conditions (ii) and (iii). By construction, T_n° preserves each E_n^l with $l \in L_n$ and also preserves the other E_n^l with $l \in \mathbb{N}^{n-1} \setminus L_n$ since T_n° is the identity on it.

Let $\mathcal{Q}_n$ be the subrelation of $\mathcal{R}$ generated by $T_1^\circ, \ldots, T_n^\circ$. We find a Borel transversal $E_{n+1} \subset X$ of $\mathcal{Q}_n$ satisfying condition (iv). Since T_n° preserves each E_n^l with $l \in \mathbb{N}^{n-1}$ and is periodic, we can choose a Borel fundamental domain B_n^l for the automorphism T_n° of E_n^l and its Borel partition $B_n^l = \bigsqcup_{m \in \mathbb{N}} E_n^{l,m}$ such that $E_n^{l,m}$ consists of m-periodic points of T_n°. Pick $l = (l_1, \ldots, l_{n-1}) \in \mathbb{N}^{n-1}$ and $m \in \mathbb{N}$ and put $k = (l_1, \ldots, l_{n-1}, m) \in \mathbb{N}^n$. If $l \in L_n$, we set $E_{n+1}^k = E_n^{l,m}$. Otherwise, we have $B_n^l = E_n^{l,1}$. We then set $E_{n+1}^k = E_n^l$ or $E_{n+1}^k = \emptyset$, depending on $m = 1$ or $m \neq 1$, respectively, and set $E_{n+1} = \bigsqcup_{k \in \mathbb{N}^n} E_{n+1}^k$. This partition fulfills condition (iv) except for the equation involving T_{n+1}, which is still not defined.

Finally, we estimate the measure $\mu(\{x \in X \mid T_n x \neq S_n x\})$. If $x \in D_n^l$ with $l \in L_n$ and $T_n x = S_n x$, then, for each $k = (k_1, \ldots, k_{n-1}) \in \Phi_l$,

$$S_n((T^k)^\circ x) = ((T^k \circ S_n)x)(T^k x)^{-1} = ((T^k \circ T_n)x)(T^k x)^{-1} = T_n((T^k)^\circ x),$$

where the first equation follows from $x \in D_n^l$, the second one follows from $T_n x = S_n x$ and the third one holds by the definition of T_n. Hence we have $T_n = S_n$ on the equivalence class of x in $\mathcal{Q}_{n-1}$. The set $\{x \in X \mid T_n x \neq S_n x\}$ is thus contained in the union

$$(X \setminus \mathcal{Q}_{n-1} F_n) \cup \bigcup_{l \in L_n} \mathcal{Q}_{n-1}\{x \in E_n^l \mid x \notin D_n^l \text{ or } T_n x \neq S_n x\}.$$

By inequalities (2.4), (2.5) and (2.6), the measure of this union is less than

$$\begin{aligned} &\varepsilon_n + \sum_{l=(l_1,\ldots,l_{n-1}) \in L_n} (l_1 + \cdots + l_{n-1})(\mu(E_n^l \setminus D_n^l) + \mu(\{x \in E_n^l \mid T_n x \neq S_n x\})) \\ &\quad < \varepsilon_n + \sum_{l=(l_1,\ldots,l_{n-1}) \in L_n} (l_1 + \cdots + l_{n-1})(\varepsilon_n + 5\varepsilon_n)\mu(E_n^l) \leq 7\varepsilon_n, \end{aligned}$$

where the sum $\sum_l (l_1 + \cdots + l_{n-1})\mu(E_n^l)$ over $l = (l_1, \ldots, l_{n-1}) \in L_n$ is equal to $\mu(\mathcal{Q}_{n-1} F_n)$ by condition (iv) and hence is at most 1. We thus have $\mu(\{x \in X \mid T_n x \neq S_n x\}) < 7\varepsilon_n$. This completes the induction. □

Proof of Theorem 2.5. By Lemma 2.7, we obtain a central sequence (T_n) in $[\mathcal{G}]$ satisfying conditions (i)–(iv) in the lemma. Let $\mathcal{E}$ and $\mathcal{Q}$ be the unions $\bigcup_n \mathcal{E}_n$ and $\bigcup_n \mathcal{Q}_n$, respectively, where we use the symbols $\mathcal{E}_n$, $\mathcal{Q}_n$ in the lemma. Then $\mathcal{Q}$ is a subrelation of $\mathcal{R}$, and by condition (iv), $\mathcal{E}$ is a subgroupoid of $\mathcal{G}$ isomorphic to $\mathcal{Q}$ via the quotient map from $\mathcal{G}$ onto $\mathcal{R}$. Let $\mathcal{M}$ be the isotropy subgroupoid of $\mathcal{G}$, which is the bundle $\bigsqcup_{x \in X} M_{\pi(x)}$ over X. Let $\mathcal{M} \times_X \mathcal{E}$ be the fibered product with respect to the range map of $\mathcal{E}$. Then $(\mathcal{M} \times_X \mathcal{E}, \mu)$ is a discrete p.m.p. groupoid with unit space X. Indeed the range and source of $(m, (g, x)) \in \mathcal{M} \times_X \mathcal{E}$ are defined to be gx and x, respectively. The product operation in $\mathcal{M} \times_X \mathcal{E}$ is defined by $(m, (g, hx))(l, (h, x)) = (ml, (gh, x))$ for $(g, hx), (h, x) \in \mathcal{E}$ and $m, l \in M_{\pi(x)}$, where we note that $\pi(ghx) = \pi(hx) = \pi(x)$ since all T_n° preserve each fiber of π. Let $\mathcal{M} \vee \mathcal{E}$ be the subgroupoid of $\mathcal{G}$ generated by $\mathcal{M}$ and $\mathcal{E}$. By condition (ii), if $(g, x) \in \mathcal{E}$, then g commutes with each element of $M_{\pi(x)}$. Therefore the map from $\mathcal{M} \times_X \mathcal{E}$ to $\mathcal{M} \vee \mathcal{E}$ sending $(m, (g, x))$ to (mg, x) is a homomorphism and thus an isomorphism.

Let $\bar{\mathcal{M}}$ be the subgroupoid of $G \ltimes (\Omega, \eta)$ that is the bundle $\bigsqcup_{\omega \in \Omega} M_\omega$. We obtain the homomorphism from $\mathcal{M} \vee \mathcal{E}$ onto $\bar{\mathcal{M}}$ as the composition of the isomorphism from $\mathcal{M} \vee \mathcal{E}$ onto $\mathcal{M} \times_X \mathcal{E}$, with the projection from $\mathcal{M} \times_X \mathcal{E}$ onto $\bar{\mathcal{M}}$. Pick a Borel homomorphism $\alpha_0 \colon \bar{\mathcal{M}} \to \mathrm{Aut}(Y, \nu)$ with some standard probability space (Y, ν) such that the associated action of $\bar{\mathcal{M}}$ on (Y, ν) is essentially free, that is, we have $\alpha_0(m) y \neq y$ for almost every $y \in Y$ and almost every $m \in \bar{\mathcal{M}} \setminus \bar{\mathcal{M}}^0$, where $\bar{\mathcal{M}}$ is equipped with the measure $\int_\Omega c_\omega \, d\eta(\omega)$ with c_ω the counting measure on M_ω. Such α_0 is obtained as follows. Pick a free p.m.p. action $G \curvearrowright (Y, \nu)$. Via the projection from $G \ltimes (\Omega, \eta)$ onto G, we obtain the homomorphism from $G \ltimes (\Omega, \eta)$ into $\mathrm{Aut}(Y, \nu)$. Let α_0 be its restriction to $\bar{\mathcal{M}}$. Then the action α_0 is essentially free. Let $\mathcal{M} \vee \mathcal{E}$ act on (Y, ν) via the homomorphism from $\mathcal{M} \vee \mathcal{E}$ onto $\bar{\mathcal{M}}$, and denote this action by $\alpha \colon \mathcal{M} \vee \mathcal{E} \to \mathrm{Aut}(Y, \nu)$.

We now apply Proposition 2.4 by letting $\mathcal{S} = \mathcal{M} \vee \mathcal{E}$. Note that the central sequence (T_n) satisfies the assumption in the proposition, that is, for each Borel subset $B \subset Y$, we have $\int_X \nu(\alpha(T_n x) B \,\triangle\, B) \, d\mu(x) \to 0$ as $n \to \infty$ because $\mathcal{E}$ acts on Y trivially and thus

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$\alpha(T_n x)$ is the identity for every $x \in X$. By the proposition, the sequence $(\tilde{T}_n)$ of the lift of T_n is central in the full group of the groupoid $(\tilde{\mathcal{G}}, \tilde{\mu})$, where we let $\mathcal{G} \curvearrowright (Z, \zeta)$ be the action co-induced from the action $\alpha\colon \mathcal{M} \vee \mathcal{E} \to \mathrm{Aut}(Y, \nu)$ and let $(\tilde{\mathcal{G}}, \tilde{\mu}) = (\mathcal{G}, \mu) \ltimes (Z, \zeta)$ be the groupoid associated with this co-induced action, which was introduced just before the proposition. Recall that $\tilde{\mathcal{G}}$ is the fibered product $\mathcal{G} \times_X Z$ with respect to the source map $s\colon \mathcal{G} \to X$ and is a groupoid with unit space Z.

If we define an action of G on Z by $gz = (g, x)z$ for $g \in G$ and $z \in Z_x$ with $x \in X$, then this action preserves the measure ζ and $(\tilde{\mathcal{G}}, \tilde{\mu})$ is identified with the translation groupoid $G \ltimes (Z, \zeta)$ via the map $((g, x), z) \mapsto (g, z)$ for $g \in G$ and $z \in Z_x$ with $x \in X$. The action $G \curvearrowright (Z, \zeta)$ is free because the action of $\bar{\mathcal{M}}$ on (Y, ν) is free. Therefore we obtained the free p.m.p. action $G \curvearrowright (Z, \zeta)$ such that the groupoid $G \ltimes (Z, \zeta)$ is Schmidt. By Lemma 2.2, G admits a free ergodic p.m.p. action which is Schmidt. $\square$

2.5. *Central sequences and periodic points.* In Theorem 2.5, we assumed that the central sequence (S_n) satisfies the property that, for each $p \in \mathbb{N}$, the set of p-periodic points of the automorphism S_n° has measure approaching zero. On the other hand, in Theorem 2.14 in the next subsection, we focus on a central sequence (S_n) without this property. This subsection deals with such a central sequence.

In the rest of this subsection, we fix the following notation. Let G be a countable group and let M be a normal subgroup of G. Let $G/M \curvearrowright (X, \mu)$ be a free ergodic p.m.p. action and let G act on (X, μ) through the quotient map from G onto G/M. We set $(\mathcal{G}, \mu) = G \ltimes (X, \mu)$.

LEMMA 2.8. *Let $(S_n)_{n \in \mathbb{N}}$ be a central sequence in $[\mathcal{G}]$. For $n, p \in \mathbb{N}$ and $h \in M$, we set*

$$A_n^p = \{x \in X \mid x \text{ is a } p\text{-periodic point of } S_n^\circ\} \ \text{ and } \ A_n^{p,h} = \{x \in A_n^p \mid (S_n)^p x = h\}.$$

Then:

(i) *the sequence $(A_n^p)_n$ is asymptotically invariant for $\mathcal{G}$; and*

(ii) *if h is central in G, then the sequence $(A_n^{p,h})_n$ is asymptotically invariant for $\mathcal{G}$.*

Proof. Pick $\phi \in [\mathcal{G}]$. If n is large, then the set

$$\{x \in X \mid (\phi \circ (S_n)^i)x = ((S_n)^i \circ \phi)x \text{ for each } i \in \{1, \ldots, p\}\}$$

has measure close to 1. If $x \in A_n^p$ belongs to this set, then $(S_n^\circ)^i(\phi^\circ x) = \phi^\circ((S_n^\circ)^i x)$ for each $i \in \{1, \ldots, p\}$. The right-hand side of this equation is not equal to $\phi^\circ x$ if $i < p$ and is equal to $\phi^\circ x$ if $i = p$. Hence $\phi^\circ x$ is a p-periodic point of S_n° and belongs to A_n^p. We thus have $\mu(\phi^\circ A_n^p \triangle A_n^p) \to 0$ as $n \to \infty$. Assertion (i) follows.

To prove assertion (ii), we pick $g \in G$. If n is large, then the set

$$\{x \in X \mid (\phi_g \circ (S_n)^p)x = ((S_n)^p \circ \phi_g)x\}$$

has measure close to 1. If a point $x \in A_n^{p,h}$ belongs to this set, then

$$((S_n)^p(gx))g = ((S_n)^p \circ \phi_g)x = (\phi_g \circ (S_n)^p)x = gh$$

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and thus $(S_n)^p(gx) = ghg^{-1} = h$ if h is central in G. Combining this with assertion (i), we have $\mu(gA_n^{p,h} \triangle A_n^{p,h}) \to 0$ as $n \to \infty$. Assertion (ii) follows. $\square$

LEMMA 2.9. *Let $(S_n)_{n\in\mathbb{N}}$ be a central sequence in $[\mathcal{G}]$ and let N be a normal subgroup of G. Then the sequence (A_n) defined by $A_n = \{x \in X \mid S_n x \in N\}$ is asymptotically invariant for $\mathcal{G}$.*

Proof. Pick $g \in G$. If n is large, then, for every point $x \in X$ outside a set of small measure, we have $(\phi_g \circ S_n)x = (S_n \circ \phi_g)x$, that is, $g(S_n x) = (S_n(gx))g$. Therefore if, further, $x \in A_n$, then $S_n(gx)$ belongs to $gNg^{-1} = N$ and thus $gx \in A_n$. $\square$

Remark 2.10. Lemma 2.9 will be used in the proof of Lemma 2.11 by letting N be the centralizer $C_G(M)$ of M in G.

Let $(S_n)_{n\in\mathbb{N}}$ be a central sequence in $[\mathcal{G}]$ and set $A_n = \{x \in X \mid S_n x \in C_G(M)\}$. While (A_n) is asymptotically invariant for $\mathcal{G}$ by Lemma 2.9, we further have $\mu(A_n) \to 1$ if M is finitely generated. Indeed, if F is a finite generating set of M and n is large enough, then, for all $x \in X$ outside a set of small measure, we have $(\phi_g \circ S_n)x = (S_n \circ \phi_g)x$ for all $g \in F$ and hence $g(S_n x) = (S_n x)g$ since M acts on X trivially. Thus $S_n x$ commutes with every element of M.

LEMMA 2.11. *Let $(S_n)_{n\in\mathbb{N}}$ be a central sequence in $[\mathcal{G}]$ and let $p \geq 2$ be an integer. Let $h \in M$ and suppose that h is central in G. We define $A_n \subset X$ as the set of p-periodic points x of S_n° such that $(S_n)^i x \in C_G(M)$ for all $i \in \{1, \ldots, p-1\}$ and $(S_n)^p x = h$. Suppose that $\mu(A_n)$ is uniformly positive.*

Then there exists a central sequence (R_n) in $[\mathcal{G}]$ such that if we define $B_n \subset X$ as the set of p-periodic points x of R_n° such that $(R_n)^i x \in C_G(M)$ for all $i \in \{1, \ldots, p-1\}$ and $(R_n)^p x = h$, then $\mu(B_n) \to 1$.

Proof. We follow the proof of [**KTD**, Lemma 5.3], patching the restrictions $S_n|_{A_n}$ together to obtain a desired $R \in [\mathcal{G}]$ after passing to an appropriate subsequence of (S_n).

Note that the equation $S_n^\circ A_n = A_n$ holds. Indeed, let $x \in A_n$ and put $y = S_n^\circ x$. Then y is a p-periodic point of S_n°. The condition that $(S_n)^i x \in C_G(M)$ for all $i \in \{1, \ldots, p-1\}$ and $(S_n)^p x = h \in C_G(M)$ implies that the value of S_n at each point of the orbit of x under iterations of S_n° belongs to $C_G(M)$. Thus $(S_n)^i y \in C_G(M)$ for all $i \in \{1, \ldots, p-1\}$. We also have $((S_n)^p y)(S_n x) = (S_n)^{p+1} x = (S_n x)h = h(S_n x)$ and thus $(S_n)^p y = h$. Therefore $y \in A_n$ and $S_n^\circ A_n \subset A_n$. The converse inclusion follows from this because S_n° is measure-preserving or we have $(S_n^\circ)^{-1} = (S_n^\circ)^{p-1}$ on A_n.

Since A_n is asymptotically invariant for $\mathcal{G}$ by Lemmas 2.8 and 2.9, the sequence (S_n') in $[\mathcal{G}]$, defined by $S_n' = S_n$ on A_n and $S_n' x = e$ for all $x \in X \setminus A_n$, is central in $[\mathcal{G}]$. After replacing S_n with S_n', we may assume that $S_n x = e$ for all $x \in X \setminus A_n$. Then $(S_n^\circ)^p$ is the identity on X. It suffices to show that, for every $\varepsilon > 0$ and every finite subset $F \subset [\mathcal{G}]$, there exists an $R \in [\mathcal{G}]$ such that $\mu(\{g \circ R \neq R \circ g\}) < \varepsilon$ and $\mu(B) > 1 - \varepsilon$, where, for $u, v \in [\mathcal{G}]$, we let $\{u \circ v \neq v \circ u\}$ be the set of points of X on which $u \circ v$ and $v \circ u$ are not equal, and we define $B \subset X$ as the set of p-periodic points of R° such that $R^i x \in C_G(M)$ for all $i \in \{1, \ldots, p-1\}$ and $R^p x = h$.

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Passing to a subsequence of (S_n), we may assume that the following conditions hold.

(1) $\sum_n \mu(g^\circ A_n \triangle A_n) < \varepsilon$ for all $g \in F$.

(2) $\sum_n \mu(\{g \circ S_n \neq S_n \circ g\}) < \varepsilon$ for all $g \in F$.

(3) $\sum_n \sum_{k<n} \sum_{i=1}^{p-1} \mu((S_n^\circ)^i A_k \triangle A_k) < \varepsilon$.

Inequality (1) holds since the sequence (A_n) is asymptotically invariant for $\mathcal{G}$. The other two inequalities hold since the sequence (S_n) is central in $[\mathcal{G}]$. We set $C_n = \bigcup_{k<n} A_k$ and also set

$$Y_1 = A_1, \ Y_n = A_n \setminus \bigcup_{i=0}^{p-1} (S_n^\circ)^i C_n \quad \text{for } n \geq 2, \quad \text{and} \quad Y = \bigcup_{n=1}^{\infty} Y_n.$$

Note that the last union is disjoint. For each n, we have $S_n^\circ Y_n = Y_n$ because $(S_n^\circ)^p$ is the identity on X and $S_n^\circ A_n = A_n$. Then $Y_n \subset A_n \setminus C_n$ and $\sum_n \sum_{i=1}^{p-1} \mu((S_n^\circ)^i C_n \triangle C_n) < \varepsilon$ by inequality (3). Thus $\sum_n \mu((A_n \setminus C_n) \setminus Y_n) < \varepsilon$ and $\mu(\bigcup_n (A_n \setminus C_n) \setminus Y) < \varepsilon$. By the definition of C_n, we have $\bigcup_n (A_n \setminus C_n) = \bigcup_n A_n$, and this is equal to X by [**KTD**, Lemma 5.1], where we use the assumption that $\mu(A_n)$ is uniformly positive. Thus

(4) $\mu(X \setminus Y) < \varepsilon$.

We pick $g \in F$ and estimate $\sum_n \mu(g^\circ Y_n \triangle Y_n)$. Pick $y \in Y_n \setminus g^\circ Y_n$. Since $(g^\circ)^{-1} y \notin Y_n$, either $(g^\circ)^{-1} y \notin A_n$ or $(g^\circ)^{-1} y \in D_n$, where we set $D_n = \bigcup_{i=0}^{p-1} (S_n^\circ)^i C_n$. In the former case, $y \in A_n \setminus g^\circ A_n$. In the latter case,

$$y \in (g^\circ D_n \setminus D_n) \cap Y_n \subset \bigcup_{i=0}^{p-1} \bigcup_{k<n} (g^\circ (S_n^\circ)^i A_k \setminus (S_n^\circ)^i A_k) \cap Y_n.$$

Let N be a positive integer. We have

$$\sum_{n=1}^{N} \mu(Y_n \setminus g^\circ Y_n) \leq \sum_{n=1}^{N} \mu(A_n \setminus g^\circ A_n) + \sum_{i=0}^{p-1} \sum_{n=1}^{N} \sum_{k=1}^{n-1} \mu((g^\circ (S_n^\circ)^i A_k \setminus (S_n^\circ)^i A_k) \cap Y_n).$$

By inequality (1), on the right-hand side, the first term is less than ε. In general, for all Borel subsets $A, A', B, B' \subset X$,

$$\mu(A \setminus B) \leq 2\mu(A \triangle A') + \mu(B \triangle B') + \mu(A' \setminus B')$$

[**KTD**, Lemma 5.2]. This implies that the second term is less than or equal to

$$\begin{aligned} &\sum_{i=0}^{p-1} \sum_{n=1}^{N} \sum_{k=1}^{n-1} (\mu((g^\circ A_k \setminus A_k) \cap Y_n) + 3\mu((S_n^\circ)^i A_k \triangle A_k)) \\ &\quad < p \sum_{n=1}^{N} \sum_{k=1}^{n-1} \mu((g^\circ A_k \setminus A_k) \cap Y_n) + 3\varepsilon < (p+3)\varepsilon, \end{aligned}$$

where the first inequality follows from inequality (3) and the last inequality follows from inequality (1). Then $\sum_{n=1}^{N} \mu(Y_n \setminus g^\circ Y_n) < (p+4)\varepsilon$ and therefore

(5) $\sum_n \mu(Y_n \setminus g^\circ Y_n) < (p+4)\varepsilon$ for all $g \in F$.

We define a map $R\colon X \to G$, patching the restrictions $S_n|_{Y_n}$ together as follows. For each n, we set $R = S_n$ on Y_n and set $Rx = e$ if $x \in X \setminus Y$. Since S_n° preserves Y_n, the map

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R° is an automorphism of X and hence R is an element of $[\mathcal{G}]$. Let $B \subset X$ be the set of p-periodic points of R° such that $R^i x \in C_G(M)$ for all $i \in \{1, \ldots, p-1\}$ and $R^p x = h$. Since S_n° preserves Y_n again and Y_n is a subset of A_n, each point of Y_n belongs to B and therefore $Y = B$ and $\mu(B) > 1 - \varepsilon$ by inequality (4).

We pick $g \in F$ to estimate $\mu(\{g \circ R \neq R \circ g\})$. We have the following three inclusions.

$$\{g \circ R \neq R \circ g\} \subset \bigcup_n (\{g \circ R \neq R \circ g\} \cap Y_n) \cup (X \setminus Y),$$

$$\{g \circ R \neq R \circ g\} \cap Y_n \subset (\{g \circ R \neq R \circ g\} \cap (Y_n \cap (g^\circ)^{-1} Y_n)) \cup (Y_n \setminus (g^\circ)^{-1} Y_n), \text{ and}$$

$$\{g \circ R \neq R \circ g\} \cap (Y_n \cap (g^\circ)^{-1} Y_n) \subset \{g \circ S_n \neq S_n \circ g\}.$$

It follows from inequalities (2), (5) and (4) that

$$\begin{aligned} \mu(\{g \circ R \neq R \circ g\}) &\leq \sum_n (\mu(\{g \circ S_n \neq S_n \circ g\}) + \mu(Y_n \setminus (g^\circ)^{-1} Y_n)) + \mu(X \setminus Y) \\ &< \varepsilon + (p+4)\varepsilon + \varepsilon = (p+6)\varepsilon. \end{aligned}$$

The desired estimate is obtained after scaling ε. □

The following lemma is similar in appearance to the last lemma. The difference between them is the assumption on $\mu(A_n)$ and the second condition in the definition of the set B_n. The following lemma deduces a stronger conclusion from the conclusion of the last lemma.

LEMMA 2.12. *Let $(S_n)_{n\in\mathbb{N}}$ be a central sequence in $[\mathcal{G}]$ and let $p \geq 2$ be an integer. Let $h \in M$ and suppose that h is central in G. We define $A_n \subset X$ as the set of p-periodic points x of S_n° such that $(S_n)^i x \in C_G(M)$ for all $i \in \{1, \ldots, p-1\}$ and $(S_n)^p x = h$. Suppose that $\mu(A_n) \to 1$.*

Then there exists a central sequence (R_n) in $[\mathcal{G}]$ such that if we define $B_n \subset X$ as the set of p-periodic points x of R_n° such that $(R_n)^i x \in C_G(M)$ for all $i \in \{1, \ldots, p-1\}$ and $(R_n)^p x = e$, then $\mu(B_n) \to 1$.

Proof. We show that, for all large $n \in \mathbb{N}$, if we choose a sufficiently large integer $m > n$ and set $R_n = (S_m)^{-1} \circ S_n$, then the obtained sequence (R_n) works. Let $\varepsilon > 0$ and fix a large $n \in \mathbb{N}$ such that $\mu(A_n) > 1 - \varepsilon$. If m is large enough, then $\mu(A_m) > 1 - \varepsilon$ and $\mu(C) > 1 - \varepsilon$, where C is the set of points $x \in X$ such that:

- $(S_n \circ (S_m)^{-1})x = ((S_m)^{-1} \circ S_n)x$; and
- $((S_m)^{-i} \circ (S_n)^i)x = ((S_m)^{-1} \circ S_n)^i x$ for all $i \in \{1, \ldots, p\}$.

By [**KTD**, Lemma 5.6], for all $i \in \{1, \ldots, p-1\}$,

$$\mu(\{x \in X \mid (S_m^\circ)^i x = (S_n^\circ)^i x \neq x\}) \to 0$$

as $m \to \infty$. Therefore, for all $i \in \{1, \ldots, p-1\}$, since $(S_n^\circ)^i x \neq x$ for all $x \in A_n$, after replacing m with a larger integer, we may assume that there exists a Borel subset $A_n' \subset A_n$ such that $\mu(A_n \setminus A_n') < \varepsilon$ and $(S_m^\circ)^i x \neq (S_n^\circ)^i x$ for all $x \in A_n'$. We set

$$D = C \cap A_n' \cap \bigcap_{i=0}^{p-1} (S_n^\circ)^{-i} A_m.$$

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Then $\mu(D) > 1 - (3 + p)\varepsilon$. We set $R = (S_m)^{-1} \circ S_n$ and define $B \subset X$ as the set of p-periodic points of R° such that $R^i x \in C_G(M)$ for all $i \in \{1, \ldots, p-1\}$ and $R^p x = e$. We claim that $D \subset B$. This completes the proof of the lemma. Pick $x \in D$. We first show that x is a p-periodic point of R° and $R^p x = e$. For each $i \in \{1, \ldots, p-1\}$, it follows from $x \in A'_n$ that $(S_m^\circ)^i x \neq (S_n^\circ)^i x$, and it follows from $x \in C$ that

$$((S_m)^{-i} \circ (S_n)^i)^\circ x = (((S_m)^{-1} \circ S_n)^i)^\circ x = (R^i)^\circ x = (R^\circ)^i x.$$

Hence $(R^\circ)^i x \neq x$. We also have

$$R^p x = ((S_m)^{-1} \circ S_n)^p x = ((S_m)^{-p} \circ (S_n)^p) x = ((S_m)^{-p} x) h = e,$$

where the second equation follows from $x \in C$, the third equation follows from $x \in A_n$, and the last equation follows from $x \in A_m = (S_m^\circ)^p A_m$. Finally, for each $i \in \{1, \ldots, p-1\}$,

$$R^i x = ((S_m)^{-1} \circ S_n)^i x = ((S_m)^{-i} \circ (S_n)^i) x = (S_m)^{-i}((S_n^\circ)^i x)((S_n)^i x),$$

which belongs to $C_G(M)$ because $x \in A_n \cap (S_n^\circ)^{-i} A_m$ and the set A_m is preserved by S_m°, as shown in the second paragraph of the proof of Lemma 2.11. □

Combining Lemmas 2.11 and 2.12, we obtain the following corollary, which also reminds us of the notation fixed at the beginning of this subsection.

COROLLARY 2.13. *Let G be a countable group and let M be a normal subgroup of G. Let $G/M \curvearrowright (X, \mu)$ be a free ergodic p.m.p. action and let G act on (X, μ) through the quotient map from G onto G/M. We set $(\mathcal{G}, \mu) = G \ltimes (X, \mu)$. Let (S_n) be a central sequence in $[\mathcal{G}]$ and let $p \geq 2$ be an integer. Let $h \in M$ and suppose that h is central in G. We define $A_n \subset X$ as the set of p-periodic points x of S_n° such that $(S_n)^i x \in C_G(M)$ for all $i \in \{1, \ldots, p-1\}$ and $(S_n)^p x = h$. Suppose that $\mu(A_n)$ is uniformly positive.*

Then there exists a central sequence (R_n) in $[\mathcal{G}]$ such that if we define $B_n \subset X$ as the set of p-periodic points x of R_n° such that $(R_n)^i x \in C_G(M)$ for all $i \in \{1, \ldots, p-1\}$ and $(R_n)^p x = e$, then $\mu(B_n) \to 1$.

2.6. *A variant construction.* Continuing from §2.4, we present another sufficient condition for a countable group G to admit a free p.m.p. Schmidt action, under the assumption that G admits a p.m.p. Schmidt action. In the following theorem, we assume that the given p.m.p. action $G \curvearrowright (X, \mu)$ is ergodic, as opposed to Theorem 2.5. This is because the proof uses certain asymptotically invariant sequences of subsets, which are better controlled if the action is ergodic.

THEOREM 2.14. *Let G be a countable group and let M be a normal subgroup of G. Let $G/M \curvearrowright (X, \mu)$ be a free ergodic p.m.p. action and let G act on (X, μ) through the quotient map from G onto G/M. We set $(\mathcal{G}, \mu) = G \ltimes (X, \mu)$.*

Let (S_n) be a central sequence in $[\mathcal{G}]$, let $p \geq 2$ be an integer and let $L < M$ be a finite subgroup which is central in G. We define $A_n \subset X$ as the set of p-periodic points of S_n° such that $(S_n)^i x \in C_G(M)$ for all $i \in \{1, \ldots, p-1\}$ and $(S_n)^p x \in L$. Suppose that $\mu(A_n)$ is uniformly positive. Then G has the Schmidt property.

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The scheme of the proof of this theorem is the same as that for Theorem 2.5. Lemma 2.6 will be used in the following lemma, which is analogous to Lemma 2.7.

LEMMA 2.15. *With the notation and the assumption in Theorem 2.14, let $\mathcal{R}$ be the orbit equivalence relation associated with the action $G/M \curvearrowright (X, \mu)$. Then there exist a central sequence $(T_n)_{n\in\mathbb{N}}$ in $[\mathcal{G}]$ and a sequence $(E_{n+1})_{n\in\mathbb{N}}$ of Borel subsets of X satisfying conditions (i), (iii) and (iv) of Lemma 2.7 together with the following condition.*
(ii)$'$ *For each n and each $x \in X$, we have $T_n x \in C_G(M)$.*

Proof. The desired sequence $(T_n, E_{n+1})_{n\in\mathbb{N}}$ is constructed by induction, similarly to the proof of Lemma 2.7. Fix a decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive numbers converging to zero. We inductively construct a sequence $(T_n, E_{n+1})_{n\in\mathbb{N}}$ satisfying conditions (ii)$'$, (iii) and (iv) and satisfying the inequality $\mu(\{x \in X \mid T_n x \neq S_n x\}) < 7\varepsilon_n$ for all n. Let p be the integer in Theorem 2.14. Since L is finite, by Corollary 2.13, we may assume, without loss of generality, that $\mu(B_n) \to 1$, where we define $B_n \subset X$ as the set of p-periodic points x of S_n° such that $(S_n)^i x \in C_G(M)$ for all $i \in \{1, \ldots, p-1\}$ and $(S_n)^p x = e$.

To construct T_1, we set $D_1 = B_1$. After replacing S_1 with S_n for a large n, we may assume that $\mu(X \setminus D_1) < \varepsilon_1$. We apply Lemma 2.6 by letting $D = D_1$ and $E = X$ and letting Ω be a singleton. Then we obtain a periodic $T_1 \in [\mathcal{G}]$ such that $T_1 x \in C_G(M)$ for almost every $x \in X$ and $\mu(\{x \in X \mid T_1 x \neq S_1 x\}) < 5\varepsilon_1 < 7\varepsilon_1$. Since T_1 is periodic, we can find a Borel fundamental domain $E_2 \subset X$ for the automorphism T_1° of X and its Borel partition $E_2 = \bigsqcup_{l\in\mathbb{N}} E_2^l$ such that $\mathcal{Q}_1 E_2^l$ is equal to the set of l-periodic points of T_1°, where $\mathcal{Q}_1$ is the subrelation of $\mathcal{R}$ generated by T_1°. This completes the first step of the induction.

Assuming that we have constructed $T_1, \ldots, T_{n-1}$ and $E_2, \ldots, E_n$, we construct T_n and E_{n+1}. Let $\mathcal{Q}_{n-1}$ be the subrelation of $\mathcal{R}$ generated by $T_1^\circ, \ldots, T_{n-1}^\circ$. By the induction hypothesis, we have a Borel transversal $E_n \subset X$ of $\mathcal{Q}_{n-1}$ and its Borel partition $E_n = \bigsqcup_{l\in\mathbb{N}^{n-1}} E_n^l$. We choose a finite subset $L_n \subset \mathbb{N}^{n-1}$ and set $F_n = \bigsqcup_{l\in L_n} E_n^l$ as in the proof of Lemma 2.7. After replacing S_n with S_m for a sufficiently large m, for each $l \in L_n$, we define D_n^l as the set of points $x \in E_n^l \cap ((S_n^\circ)^{-1} E_n^l) \cap B_n$ such that $(S_n \circ T^k)x = (T^k \circ S_n)x$ for each $k = (k_1, \ldots, k_{n-1}) \in \Phi_l$, where we set $T^k = (T_{n-1})^{k_{n-1}} \circ \cdots \circ (T_2)^{k_2} \circ (T_1)^{k_1}$ and define Φ_l as before. Letting $D = D_n^l$ and $E = E_n^l$ and letting Ω be a singleton, we apply Lemma 2.6 for each $l \in L_n$ and obtain a periodic $T_n \in [\mathcal{G}_{F_n}]$. The rest of the construction of $T_n \in [\mathcal{G}]$, whose domain is extended to X, and a Borel transversal E_{n+1} of $\mathcal{Q}_n$ is a verbatim translation of that in the proof of Lemma 2.7. □

Proof of Theorem 2.14. The proof is a verbatim translation of that of Theorem 2.5, where we apply Lemma 2.15 in place of Lemma 2.7 and let Ω be a singleton. We note that the groupoid $\mathcal{M} \times_X \mathcal{E}$ in that proof then reduces to the direct product $M \times \mathcal{E}$. □

We now prove Theorems 1.3 and 1.5 stated in §1.

COROLLARY 2.16. *Let G be a countable group and let M be a finite central subgroup of G. Let $G/M \curvearrowright (X, \mu)$ be a free ergodic p.m.p. action and let G act on (X, μ) through the quotient map from G onto G/M. If the action $G \curvearrowright (X, \mu)$ is Schmidt, then G has the Schmidt property.*

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Proof. By assumption, we have a central sequence (S_n) in $[G \ltimes (X, \mu)]$ such that $\mu(\{x \in X \mid S_n^\circ x \neq x\}) \to 1$. We will apply Theorem 2.5 or 2.14. The most remarkable difference between the assumptions in those two theorems is the condition on the set A_n^p of p-periodic points of S_n° and its measure. Passing to a subsequence of (S_n), we may assume that either $\mu(A_n^p) \to 0$ for every integer $p \geq 2$ or there is some integer $p \geq 2$ for which the values $\mu(A_n^p)$ are uniformly positive. If the former holds, then we apply Theorem 2.5 by letting Ω be a singleton. We note that $C_G(M) = G$ since M is central in G. If the latter holds, then we apply Theorem 2.14 by letting $L = M$. Thus the corollary follows from the theorems. $\square$

Recall that a sequence (g_n) in a countable group G is called *central* if, for each $h \in G$, g_n commutes with h for all sufficiently large n. The following is an immediate application of Corollary 2.16.

COROLLARY 2.17. *If a countable group G admits a central sequence diverging to infinity, then G has the Schmidt property.*

Proof. Let G act on the set $G \setminus \{e\}$ by conjugation, which induces the p.m.p. action of G on the product space $X := \prod_{G\setminus\{e\}}[0, 1]$ equipped with the product measure μ of the Lebesgue measure. We may assume that G has finite center because otherwise the Schmidt property of G is shown in [**KTD**, Example 8.8]. Let C be the center of G. Then C acts on X trivially and the induced action $G/C \curvearrowright (X, \mu)$ is essentially free. By assumption, we have a central sequence (g_n) in G diverging to infinity, and we may assume that none of g_n belong to C. Then by Remark 2.1, (g_n) is a central sequence in the full group $[G \ltimes (X, \mu)]$ such that $\mu(\{x \in X \mid g_n x \neq x\}) = 1$ for all n. Thus Corollary 2.16 is applied to G and its finite center C. $\square$

Remark 2.18. Let G be a countable group. If M is a finite central subgroup of G and the quotient group G/M admits a central sequence diverging to infinity, then G also admits such a sequence and thus has the Schmidt property by Corollary 2.17.

To show this, choose a section $\mathsf{s}\colon G/M \to G$ of the quotient map. Let (g_n) be a central sequence in G/M diverging to infinity. For each $h \in G$, the commutator $[\mathsf{s}(g_n), h]$ belongs to M if n is large enough. Since M is finite, after passing to a subsequence, we may assume that, for each $h \in G$, the element $[\mathsf{s}(g_n), h]$ is independent of n. Then the sequence $(\mathsf{s}(g_n)\mathsf{s}(g_1)^{-1})$ is central in G and diverges to infinity.

3. *Groups with infinite AC-center*

3.1. *Reduction to the proof for groups with infinite FC-center.*

We collect basic properties of groups with infinite AC-center. For a subset S of a group G, we denote by $C_G(S)$ the centralizer of S in G and denote by $\langle S\rangle_G$ the normal closure of S in G, that is, the minimal normal subgroup of G containing S. If S consists of elements $g_1, \ldots, g_n$, then $C_G(S)$ and $\langle S\rangle_G$ are also denoted by $C_G(g_1, \ldots, g_n)$ and $\langle g_1, \ldots, g_n\rangle_G$, respectively.

LEMMA 3.1. *Let G be a countable group and denote by R the AC-center of G, that is, the set of elements $g \in G$ such that the quotient group $G/C_G(\langle g\rangle_G)$ is amenable. Then:*

(i) *the set R is a normal subgroup of G;*

(ii) *for each finite subset $S \subset R$, the quotient group $G/C_G(\langle S\rangle_G)$ is amenable;*

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(iii) *the group R is amenable; and*

(iv) *the group R is generated by all normal subgroups M of G such that $G/C_G(M)$ is amenable. Therefore R is equal to the AC-center introduced in [TD, 0.G].*

Proof. Although some assertions in the lemma are proved in [TD, Theorem 13], we give a proof here for the reader's convenience. To make the symbols easier, in this proof, let us write $\bar{C}(g)$ and $\bar{C}(S)$ for $C_G(\langle g\rangle_G)$ and $C_G(\langle S\rangle_G)$, respectively, given $g \in G$ and $S \subset G$. By its definition, the set R contains the trivial element and is closed under inverse. If $r, s \in R$, then $\bar{C}(r) \cap \bar{C}(s) < \bar{C}(rs)$. Thus $G/(\bar{C}(r) \cap \bar{C}(s))$ surjects onto $G/\bar{C}(rs)$ and injects into $G/\bar{C}(r) \times G/\bar{C}(s)$ diagonally. The last group is amenable and thus $rs \in R$. Hence R is a subgroup of G, and, by its definition R is normal in G. Assertion (i) follows.

If S consists of finitely many elements $r_1, \ldots, r_n \in R$, then $G/\bar{C}(S)$ diagonally injects into the direct product $G/\bar{C}(r_1) \times \cdots \times G/\bar{C}(r_n)$, which is amenable. Thus $G/\bar{C}(S)$ is amenable, and assertion (ii) follows. Moreover, the group $\langle S\rangle$ generated by S admits the homomorphism into $G/\bar{C}(S)$ induced by the inclusion into G, whose kernel is $\langle S\rangle \cap \bar{C}(S)$ and is thus abelian. Hence $\langle S\rangle$ is amenable, and assertion (iii) follows.

Let $\mathcal{M}$ be the set of normal subgroups M of G such that $G/C_G(M)$ is amenable, and let R_1 be the group generated by all members of $\mathcal{M}$. If $r \in R$, then $\langle r\rangle_G \in \mathcal{M}$ and thus $r \in R_1$. To show the converse, we note that if $M_1, M_2 \in \mathcal{M}$, then the group generated by M_1 and M_2 belongs to $\mathcal{M}$ since its centralizer in G is equal to $C_G(M_1) \cap C_G(M_2)$, and the group $G/(C_G(M_1) \cap C_G(M_2))$ diagonally injects into $G/C_G(M_1) \times G/C_G(M_2)$, which is amenable. Therefore R_1 is the union of members of $\mathcal{M}$. If $r \in R_1$, then r is contained in some $M \in \mathcal{M}$, and since $C_G(M) < \bar{C}(r)$, we have $r \in R$. Assertion (iv) follows. □

Let G be a countable group. Suppose that the AC-center of G, denoted by R, is infinite. We first assume that there exists a finite subset $S \subset R$ such that the normal closure $M := \langle S\rangle_G$ is infinite. Setting $L := C_G(M)$, we then have two commuting, normal subgroups L, M of G such that M is amenable and the quotient group $G/(LM)$ is amenable. If $L \cap M$ is finite, then the infinite group $M/(L \cap M)$ injects into the group $(LM)/L$ and hence the index of L in LM is infinite. By [TD, Theorem 18 (H1)], we conclude that G is stable and thus has the Schmidt property. If $L \cap M$ is infinite, then LM has the infinite central subgroup $L \cap M$. Since $G/(LM)$ is amenable, the construction in the proof of [TD, Theorem 15] yields an ergodic free p.m.p. action of G which is Schmidt.

We next assume that, for each finite subset $S \subset R$, the normal closure $\langle S\rangle_G$ is finite. For each $r \in R$, the normal closure $\langle r\rangle_G$ is then finite. The group G acts on $\langle r\rangle_G$ by conjugation, and some finite index subgroup of G acts on it trivially. Hence the centralizer $C_G(r)$ is of finite index in G, that is, r belongs to the FC-center of G. The AC-center R is thus contained in the FC-center of G, and they coincide after all. Let us record the following structural alternative obtained at this point.

PROPOSITION 3.2. *Let G be a countable group with infinite AC-center. Then either:*

(1) *there exist two commuting, normal subgroups L, M of G such that one of them is infinite and amenable and the quotient group $G/(LM)$ is amenable; or*

(2) *the AC-center and the FC-center of G coincide, and for each finite subset of the FC-center of G, its normal closure in G is finite.*

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As shown above, if there exists a finite subset $S \subset R$ such that the normal closure $\langle S\rangle_G$ is infinite, then case (1) occurs, and if there exists no such S, then case (2) occurs. In case (1), it has already been shown that G has the Schmidt property. Therefore, for the proof of Theorem 1.1, it remains to show that G has the Schmidt property if G has infinite FC-center and every finite subset of the FC-center has finite normal closure in G.

Finally, we point out the following permanence properties, which are concerned with the question in Remark 1.4, but are not necessary for the proof of Theorem 1.1.

PROPOSITION 3.3. *Let G be a countable group with a finite central subgroup Z. Then:*

(i) *the group G has infinite FC-center if and only if G/Z has infinite FC-center; and*

(ii) *the group G has infinite AC-center if and only if G/Z has infinite AC-center.*

Proof. For each $g \in G$, let $A_G(g)$ denote the conjugacy class of g in G. We note that an element $g \in G$ belongs to the FC-center of G if and only if the set $A_G(g)$ is finite. We set $\Gamma = G/Z$ with $\pi : G \to \Gamma$ the quotient map. Let R^0 be the FC-center of G and let R_1^0 be the FC-center of Γ. For each $g \in G$, the map π is a surjection from $A_G(g)$ onto $A_\Gamma(\pi(g))$, and is finite-to-one since Z is finite. This implies that $\pi(R^0) = R_1^0$, and assertion (i) follows.

We now prove assertion (ii). Let R be the AC-center of G and let R_1 be the AC-center of Γ. It suffices to show that $\pi(R) = R_1$. For each $g \in G$, we have $\pi(C_G(\langle g\rangle_G)) < C_\Gamma(\langle \pi(g)\rangle_\Gamma)$. We thus have the surjection from $G/C_G(\langle g\rangle_G)$ onto $\Gamma/C_\Gamma(\langle \pi(g)\rangle_\Gamma)$. Hence $\pi(R) < R_1$.

We fix $\gamma \in \Gamma$ and set $M = \langle \gamma\rangle_\Gamma$ and $L = C_\Gamma(M)$. We choose a section $\mathsf{s}\colon \Gamma \to G$ of π. Let $\mathrm{Hom}(M, Z)$ be the group of homomorphisms from M into Z such that the product of two elements $\tau_1, \tau_2 \in \mathrm{Hom}(M, Z)$ is given by the homomorphism $m \mapsto \tau_1(m)\tau_2(m)$. Since L and M commute, we obtain the homomorphism $\tau\colon L \to \mathrm{Hom}(M, Z)$ defined by $\tau_l(m) = [\mathsf{s}(l), \mathsf{s}(m)]$ for $l \in L$ and $m \in M$. We set $L_1 = \ker \tau$. Then L/L_1 is abelian and hence amenable. If $g \in G$ with $\pi(g) = \gamma$, then $L_1 < \pi(C_G(\langle g\rangle_G))$ because, for each $l \in L_1$, we have $\mathsf{s}(l) \in C_G(\mathsf{s}(M)) = C_G(\langle g\rangle_G)$ and $l = \pi(\mathsf{s}(l)) \in \pi(C_G(\langle g\rangle_G))$.

Suppose that $\gamma \in R_1$ and pick $g \in G$ with $\pi(g) = \gamma$. We show that $g \in R$, which implies that the inclusion $R_1 < \pi(R)$. We set $N = C_G(\langle g\rangle_G)$. The group G/N is isomorphic to $\Gamma/\pi(N)$ via π. Since $L_1 < \pi(N)$, we have the surjection from Γ/L_1 onto $\Gamma/\pi(N)$, which surjects onto Γ/L because $\pi(N) < L$. It follows from $\gamma \in R_1$ that Γ/L is amenable. Since L/L_1 is also amenable, so are Γ/L_1, $\Gamma/\pi(N)$ and G/N, and thus $g \in R$. □

3.2. *An outline of §§4 and 5.* Let G be a countable group with infinite FC-center R. Suppose that every finite subset of R has finite normal closure in G. The proof of the Schmidt property of G will be given throughout §§4 and 5. In this subsection, we outline the proof along with a preliminary lemma on the structure of R.

In §4, we show that G has the Schmidt property under the assumption that the center of R is finite. If we set $N = \bigcap_{r\in R} C_G(r)$, then $N \cap R$ is the center of R. Since $C_G(r)$ is of finite index in G for all $r \in R$, the group G/N is residually finite and thus admits a free profinite action. Moreover, G/N has infinite FC-center because the FC-center of G/N

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contains $(RN)/N$. Following Popa–Vaes [PV, Theorem 6.4] and Deprez–Vaes [DV, §3], we construct a free profinite Schmidt action $G/N \curvearrowright (X, \mu)$ (after passing to some finite index subgroup of G). We then apply Theorems 2.5 and 2.14 to the translation groupoid $G \ltimes (X, \mu)$ and conclude that G has the Schmidt property. We remark that the proof in §4 does not use the condition that every finite subset of R has finite normal closure in G.

In §5, we assume that the center of R is infinite. We then have an infinite abelian subgroup $A < R$ normalized by G. This subgroup A will appropriately be chosen and is not necessarily the center of R. Since each finite subset of R has finite normal closure in G, there exists a strictly increasing sequence $A_1 < A_2 < \cdots$ of finite subgroups of A such that each A_n is normalized by G. Let us draw attention to the following condition.

($\star$) For every $N \in \mathbb{N}$, we have $\lim_n |F_{n,N}|/|A_n| = 1$, where $F_{n,N}$ is the set of elements of A_n whose order is more than N.

For example, if $A_n = \mathbb{Z}/2^n\mathbb{Z}$ and we embed A_n into A_{n+1} arbitrarily, then the sequence $A_1 < A_2 < \cdots$ fulfills this condition. In §5.3, we assume condition ($\star$) and show that G has the Schmidt property. In §5.4, we deal with the case where condition ($\star$) is not fulfilled. In this case, by applying Lemma 3.4 below, after replacing (A_n), we may assume, without loss of generality, that, for some prime number p, each A_n is isomorphic to the direct sum of copies of $\mathbb{Z}/p\mathbb{Z}$.

LEMMA 3.4. *Let G be a countable group and let A be an infinite abelian normal subgroup of G contained in the FC-center of G. Suppose that each finite subset of A has finite normal closure in G and let $A_1 < A_2 < \cdots$ be a strictly increasing sequence of finite subgroups of A such that each A_n is normalized by G. Suppose further that, for this sequence, condition ($\star$) does not hold. Then there exist a prime number p and a strictly increasing sequence $B_1 < B_2 < \cdots$ of finite subgroups of A such that each B_n is normalized by G and is isomorphic to the direct sum of copies of $\mathbb{Z}/p\mathbb{Z}$.*

Proof. Since condition ($\star$) does not hold, after passing to a subsequence of (A_n), we may assume that there exists $N \in \mathbb{N}$ such that the ratio $|A_n \setminus F_n|/|A_n|$ is uniformly positive, where F_n denotes the set of elements of A_n whose order is more than N. Let $\mathcal{P}$ be the set of prime numbers. Then A_n is isomorphic to the direct sum $\bigoplus_{p \in \mathcal{P}} A_n^p$, where A_n^p is the subgroup of elements of A_n whose order is a power of p. This direct sum decomposition is canonical and is thus preserved under G-conjugation. We aim to show that, for some $p \in \mathcal{P}$, the number of elements of A_n^p whose order is p diverges to infinity after passing to a subsequence of (A_n).

Let C_n^p be the set of elements of A_n^p whose order is less than or equal to N. Then C_n^p is a subgroup of A_n^p. We claim that, for some $p \in \mathcal{P}$, after passing to a subsequence of (A_n), we have $|C_n^p| \to \infty$ as $n \to \infty$. Otherwise, for each $p \in \mathcal{P}$, the sequence $(|C_n^p|)_{n \in \mathbb{N}}$ would be bounded. Therefore $|C_n^p|$ is uniformly bounded among all n and all $p \in \mathcal{P}$ with $p \leq N$. This is absurd with the condition that $|A_n \setminus F_n|/|A_n|$ is uniformly positive and $|A_n| \to \infty$ because each element of A_n whose order is less than or equal to N is a sum of elements of C_n^p with $p \leq N$.

Since C_n^p is isomorphic to a direct sum of groups $\mathbb{Z}/p^k\mathbb{Z}$ for some positive integers k with $p^k \leq N$, it follows from $|C_n^p| \to \infty$ that the number of elements of C_n^p whose order

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is p diverges to infinity. This is the claim that we aim to show. Note that elements of A of order p are preserved under G-conjugation. Note also that each finite set of elements of A of order p generates a group whose elements other than the trivial one have order p, which is isomorphic to the direct sum of finitely many copies of $\mathbb{Z}/p\mathbb{Z}$. Hence we obtain a desired sequence $B_1 < B_2 < \cdots$ of subgroups inductively as follows. Choose an element of $\bigcup_n A_n$ of order p and let B_1 be its normal closure in G. Having defined B_n, choose an element a of $\bigcup_n A_n$ of order p which does not belong to B_n and let B_{n+1} be the normal closure of $B_n \cup \{a\}$ in G. □

3.3. *Examples.* We present examples of groups with infinite FC-center such that their Schmidt property does not follow from known results in [**PV**, **KTD**] immediately. Let us recall those results.

(1) If a countable group G has infinite FC-center and is residually finite, then G has the Schmidt property ([**PV**, Theorem 6.4]; see also [**KTD**, Example 8.10]).
(2) Suppose that a countable group Γ acts on a countably infinite amenable group A by automorphisms and suppose further that each Γ-orbit in A is finite. Then the semi-direct product $\Gamma \ltimes A$ is stable [**KTD**, Example 8.11] and therefore has the Schmidt property.

Here we recall that a free ergodic p.m.p. action of a countable group is called *stable* if the associated orbit equivalence relation absorbs the ergodic p.m.p. hyperfinite equivalence relation on an atomless standard probability space, under direct product. If a countable group G admits a free ergodic p.m.p. action that is stable, then G is called *stable*.

Example 3.5. Let Γ be the group of Ershov [**Er**]. This is a countable, residually finite group with property (T) whose FC-center R is not virtually abelian. (Note that these conditions imply that $R \neq \Gamma$. Otherwise $R = \Gamma$ would be amenable by Lemma 3.1 (iii) and hence finite by property (T) of Γ, but this is absurd as R is not virtually abelian.) Let H be a countable, non-residually-finite group and define G as the amalgamated free product $G = \Gamma *_R (H \times R)$, where R is identified with the subgroup $\{e\} \times R$ of $H \times R$. Then the FC-center of G is equal to R, which is proved in the next paragraph, and G is not residually finite. Moreover, G is not stable, as shown in Corollary 3.10 below.

We prove that the FC-center of G is equal to R. Pick $r \in R$. We naturally identify H with the subgroup $H \times \{e\}$ of $H \times R$. Let $p\colon G \to \Gamma$ be the surjection onto the first factor. Then $\ker p = \langle H \rangle_G$. Since R is a normal subgroup of G, it follows from $H < C_G(R)$ that $\ker p < C_G(R) < C_G(r)$. On the other hand, since p is the identity on Γ, G is identified with the semi-direct product $\Gamma \ltimes \ker p$. Then $C_G(r)$ is identified with $C_\Gamma(r) \ltimes \ker p$, which is of finite index in $\Gamma \ltimes \ker p$. Thus r belongs to the FC-center of G. We have shown that R is contained in the FC-center of G. The converse inclusion holds because the quotient group G/R is isomorphic to the free product $(\Gamma/R) * H$ whose FC-center is trivial.

Example 3.6. We set $\Gamma = SL_m(\mathbb{Z})$ with $m \geq 2$. The group $\mathbb{Z}[1/2]/\mathbb{Z}$ is identified with the increasing union $\bigcup_n \mathbb{Z}/2^n\mathbb{Z}$, where the element $1 \in \mathbb{Z}/2^n\mathbb{Z}$ is identified with the element $1/2^n + \mathbb{Z} \in \mathbb{Z}[1/2]/\mathbb{Z}$. We set $A_n = (\mathbb{Z}/2^n\mathbb{Z})^m$ and $A = (\mathbb{Z}[1/2]/\mathbb{Z})^m = \bigcup_n A_n$. The

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group Γ acts on each A_n by automorphisms, and the increasing sequence $A_1 < A_2 < \cdots$ fulfills condition ($\star$) in §3.2.

The semi-direct product $\Gamma \ltimes A$ is not residually finite. In fact, the group $\mathbb{Z}[1/2]/\mathbb{Z}$ has no finite index subgroup other than itself, which is proved as follows. Let B be a finite index subgroup of $\mathbb{Z}[1/2]/\mathbb{Z}$ and pick $r \in \mathbb{Z}[1/2]$. Find $m \in \mathbb{N}$ with $2^m r \in \mathbb{Z}$. Since B is of finite index, there exist $k, l \in \mathbb{N}$ such that $2^{-k}r - 2^{-l}r + \mathbb{Z} \in B$ and $k - l > m$. Then the element $2^{m+l}(2^{-k}r - 2^{-l}r) + \mathbb{Z} = 2^{m+l-k}r + \mathbb{Z}$ belongs to B and so does $r + \mathbb{Z}$. Thus we have $B = \mathbb{Z}[1/2]/\mathbb{Z}$.

Let E be a countable group with property (T) containing A as a central subgroup. We define G as the amalgamated free product $G = (\Gamma \ltimes A) *_A E$. Then the FC-center of G is equal to A, and G is not stable (Corollary 3.10).

We obtain such a group E as follows, relying on the construction of Cornulier [C] (see Appendix A for the construction of analogous groups). Let H be the subgroup of $SL_5(\mathbb{Z}[1/2])$ consisting of matrices of the form

$$\begin{pmatrix} 1 & * & * \\ 0 & h & * \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.1}$$

where h runs through elements of $SL_3(\mathbb{Z}[1/2])$. Then H has property (T) [C, Proposition 2.7]. The center C of H consists of matrices such that each diagonal entry is 1 and the $(1, 5)$-entry is the only off-diagonal entry that is possibly non-zero. Let Z be the subgroup of C consisting of matrices whose $(1, 5)$-entry belongs to $\mathbb{Z}$. Then the group $E := (H/Z)^m$ is a desired one. Indeed, $(C/Z)^m$ is a central subgroup of E isomorphic to A, and E has property (T) since H has property (T).

Example 3.7. Let p be a prime number and set $A = \bigoplus_{\mathbb{N}} \mathbb{Z}/p\mathbb{Z}$. For $n \in \mathbb{N}$, we define A_n as the group of elements $(a_i)_{i\in\mathbb{N}} \in A$ such that $a_i = 0$ if $i > n$. Every non-trivial element of A has order p. Thus the increasing sequence $A_1 < A_2 < \cdots$ does not fulfill condition ($\star$) in §3.2. Let $\mathcal{N}$ be the group of matrices $(a_{ij})_{i,j\in\mathbb{N}}$ with coefficient in $\mathbb{Z}/p\mathbb{Z}$ such that $a_{ii} = 1$ for all $i \in \mathbb{N}$ and $a_{ij} = 0$ for all $i > j$. The group $\mathcal{N}$ acts on the vector space A by linear automorphisms, preserving the subspace A_n. We equip $\mathcal{N}$ with the topology of pointwise convergence as automorphisms of A. Then $\mathcal{N}$ is a compact group.

Let Γ be a countable dense subgroup of $\mathcal{N}$. In the paragraph after next, we will prove that the FC-center of the semi-direct product $\Gamma \ltimes A$ is equal to A. As in Example 3.6, let E be a countable group with property (T) containing A as a central subgroup, and define G as the amalgamated free product $G = (\Gamma \ltimes A) *_A E$. Then the FC-center of G is equal to A, and G is not stable (Corollary 3.10).

We find such a group E, relying on the construction of Cornulier [C] again. Let $\mathbb{F}_p$ be the field of order p and let $\mathbb{F}_p[t]$ be the ring of polynomials over $\mathbb{F}_p$ in one indeterminate t. We define E as the subgroup of $SL_5(\mathbb{F}_p[t])$ consisting of matrices of the form (3.1) with h running through elements of $SL_3(\mathbb{F}_p[t])$. Then E has property (T) by [C, Lemma 2.2]. The center of E is isomorphic to $\mathbb{F}_p[t]$ and to A.

Let R be the FC-center of $\Gamma \ltimes A$. We prove that R is equal to A. For each n, the group of elements of Γ acting on A_n trivially is of finite index in Γ. Thus $A_n < R$ and $A < R$. For the converse inclusion, it suffices to show that if an element $g \in \Gamma$ centralizes a finite

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index subgroup of Γ, then g is trivial. Suppose otherwise by way of a contradiction. Write $g = (g_{ij})_{i,j\in\mathbb{N}}$ as a matrix and pick positive integers $k < l$ such that $g_{kl} \neq 0$ and $g_{kj} = 0$ if $1 < j < l$. Since Γ is dense in $\mathcal{N}$ and g commutes with some finite index subgroup of Γ, there exists an open neighborhood V of the identity in $\mathcal{N}$ such that g commutes with each element of V. Then there exists an $m \in \mathbb{N}$ such that if a matrix $h = (h_{ij})_{i,j} \in \mathcal{N}$ satisfies $h_{ij} = 0$ for all $1 \leq i < j < m$, then h belongs to V. We may assume that $m > l$. Let $h \in V$ be the matrix such that the (l, m)-entry is 1 and the other off-diagonal entries are 0. Then the (k, m)-entries of gh and hg are $g_{kl} + g_{km}$ and g_{km}, respectively. We thus have $gh \neq hg$, which is a contradiction.

We present a sufficient condition for a countable group not to be stable, and we apply it to the groups in the above examples. We say that a mean on a countable group G is *diffuse* if its value on each finite subset of G is zero.

PROPOSITION 3.8. *Let G be a countable group and let A be a subgroup of G. Suppose that each diffuse, G-conjugation-invariant mean on G is supported on A and that the pair (G, A) has property (T). Then G is not stable.*

Proof. Suppose that G admits a free ergodic p.m.p. action $G \curvearrowright (X, \mu)$ which is stable. Then we have a central sequence (T_n) in the full group $[G \ltimes (X, \mu)]$ and an asymptotically invariant sequence (A_n) for $G \ltimes (X, \mu)$ such that $T_n^\circ A_n \cap A_n = \emptyset$ (and hence $\mu(A_n) = 1/2$) for all n (see Remark 3.9 below). Property (T) of the pair (G, A) implies that there exists an A-invariant Borel subset $B_n \subset X$ such that $\mu(A_n \triangle B_n) \to 0$. Since the functions on G defined by $g \mapsto \mu(\{x \in X \mid T_n x = g\})$ are asymptotically G-conjugation invariant, the assumption on G-conjugation-invariant means on G implies that there exists a Borel subset $D_n \subset X$ such that $T_n x \in A$ for all $x \in D_n$ and $\mu(D_n) \to 1$. Then

$$T_n^\circ B_n \setminus B_n \subset (T_n^\circ(D_n \cap B_n) \setminus B_n) \cup T_n^\circ(X \setminus D_n) = T_n^\circ(X \setminus D_n),$$

where the last equation holds since B_n is A-invariant and $T_n x \in A$ for all $x \in D_n$. Thus $\mu(T_n^\circ B_n \triangle B_n) \leq 2\mu(X \setminus D_n) \to 0$ and $\mu(T_n^\circ A_n \triangle A_n) \to 0$, which is a contradiction. □

Remark 3.9. Let the group $\bigoplus_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ act on the compact group $X_0 = \prod_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ by translation, equip X_0 with the Haar measure and let $\mathcal{R}_0$ denote the associated orbit equivalence relation. For each $n \in \mathbb{N}$, let $\bar{T}_n \in [\mathcal{R}_0]$ be the element of $\bigoplus_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ such that its coordinate indexed by n is 1 and the other coordinates are 0, and let $\bar{A}_n \subset X_0$ be the subset consisting of points whose coordinate indexed by n is 0. Then $(\bar{T}_n)$ is central in $[\mathcal{R}_0]$, $(\bar{A}_n)$ is asymptotically invariant for $\mathcal{R}_0$, and $\bar{T}_n \bar{A}_n \cap \bar{A}_n = \emptyset$ for all n.

If a discrete p.m.p. equivalence relation $\mathcal{R}$ is stable, then we obtain similar sequences as follows. By stability, we have a decomposition $\mathcal{R} = \mathcal{R}_0 \times \mathcal{R}_1$, where $\mathcal{R}_1$ is some discrete p.m.p. equivalence relation on a standard probability space (X_1, μ_1). Define $T_n \in [\mathcal{R}]$ by $T_n(x, y) = (\bar{T}_n(x), y)$ for $x \in X_0$ and $y \in X_1$, and set $A_n = \bar{A}_n \times X_1$. Then (T_n) is central in $[\mathcal{R}]$, (A_n) is asymptotically invariant for $\mathcal{R}$, and $T_n A_n \cap A_n = \emptyset$ for all n.

COROLLARY 3.10. *None of the groups G in Examples 3.5–3.7 are stable.*

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Proof. Let $G = \Gamma *_R (H \times R)$ be the group in Example 3.5. Then G surjects onto the free product $(\Gamma/R) * H$ with kernel R. Since each conjugation-invariant mean on $(\Gamma/R) * H$ is supported on the trivial element [**BH**, Théorème 5 (c)], each G-conjugation-invariant mean on G is supported on R. Since Γ has property (T), so does the pair (G, R). Thus Proposition 3.8 applies.

Let $G = (\Gamma \ltimes A) *_A E$ be the group in Example 3.6 or 3.7. It similarly turns out that each G-conjugation-invariant mean on G is supported on A. Since E has property (T), so does the pair (G, A). Thus Proposition 3.8 applies. □

Remark 3.11. Let Γ be a countable group acting on a countably infinite amenable group A by automorphisms. The semi-direct product $G := \Gamma \ltimes A$ then acts on A by affine transformations, that is, Γ acts on A by automorphisms and A acts on A by left multiplication. If the action of G on A admits an invariant mean, then the pair (G, A) does not have property (T). Indeed, the associated unitary representation of G on $\ell^2(A)$ weakly contains the trivial representation, but has no A-invariant unit vector.

If each Γ-orbit in A is finite, then the action of G on A admits an invariant mean (see the proof of [**TD**, Theorem 13, ii]). Therefore, for the stable group $G = \Gamma \ltimes A$ reviewed at the beginning of this subsection, the pair (G, A) does not have property (T). We refer to [**DV**, Proposition 3.1], [**Ki3**, Theorem 1.1] and [**TD**, 0.H] for other relationships between stability and relative property (T).

4. *Groups with non-commutative FC-center*

Let G be a countable group with infinite FC-center R. Suppose that the center of R is finite. In this section, we aim to prove that G has the Schmidt property.

We set $N = \bigcap_{r \in R} C_G(r)$. Then R and N commute and $N \cap R$ is exactly the center of R. We may assume, without loss of generality, that $N \cap R$ is central in G after passing to some finite index subgroup of G. Indeed, the subgroup $G_0 := \bigcap_{r \in N \cap R} C_G(r)$ is of finite index in G since $N \cap R$ is finite, and G_0 commutes with $N \cap R$. Since $N \cap R$ is central in R, we have $R < G_0$ and hence the FC-center of G_0 is equal to R. If we set $N_0 = \bigcap_{r \in R} C_{G_0}(r)$, then $N_0 = N \cap G_0$ and hence $N_0 \cap R$ is finite and central in G_0. In general, for a finite index inclusion $\Lambda < \Gamma$ of countable groups, if Λ admits a free ergodic p.m.p. action that is Schmidt, then the action of Γ induced (not co-induced) from it is also Schmidt. Therefore, after replacing G with G_0, we may assume that $N \cap R$ is central in G.

Let $G = H_0 > H_1 > H_2 > \cdots$ be a decreasing sequence of finite index subgroups of G such that $\bigcap_n H_n = N$. We can choose a sequence $(r_n)_{n \in \mathbb{N}}$ of elements of $R \setminus N$ such that:

(i) if $n \neq m$, then r_n and r_m are distinct in the quotient group $R/(N \cap R)$; and
(ii) for each $n \in \mathbb{N}$, r_n belongs to $C_G(r_1, \ldots, r_{n-1}) \cap H_{n-1}$.

Indeed, we first note that $R/(N \cap R)$ is infinite since R is infinite and $N \cap R$ is finite. Let r_1 be an arbitrary element of $R \setminus N$. If $r_1, \ldots, r_{n-1}$ are chosen, then $C_G(r_1, \ldots, r_{n-1}) \cap H_{n-1}$ is of finite index in G and hence its image in $G/(N \cap R)$ is of finite index. The intersection of that image with $R/(N \cap R)$ is of finite index in $R/(N \cap R)$ and hence infinite. If we let r_n be an element of $R \setminus N$ whose image in $R/(N \cap R)$ belongs to that intersection and is distinct from the images of $r_1, \ldots, r_{n-1}$, then conditions (i) and (ii) are

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fulfilled. For an integer $n \geq 2$, we set

$$G_n = C_G(r_1, \ldots, r_{n-1}) \cap H_{n-1} \cap C_G(r_n).$$

Let $G \curvearrowright (X, \mu)$ be the ergodic p.m.p. action obtained as the inverse limit of the system of the p.m.p. actions $G \curvearrowright G/G_n$ given by left multiplication. Then N acts on X trivially, and the induced action $G/N \curvearrowright (X, \mu)$ is free because $\bigcap_n H_n = N$.

We show that the translation groupoid $(\mathcal{G}, \mu) := G \ltimes (X, \mu)$ admits a central sequence (T_n) in its full group such that $T_n^\circ x \neq x$ and $T_n x \in R$ for all n and all $x \in X$. Let $p_n : X \to G/G_n$ be the projection obtained from the inverse limit construction. We define a map $T_n : X \to G$ by $T_n x = g r_n g^{-1}$ for $x \in p_n^{-1}(gG_n)$ and $g \in G$. This does not depend on the choice of g because r_n commutes with every element of G_n by the definition of G_n. Since r_n belongs to G_n by condition (ii), T_n° preserves the subset $p_n^{-1}(gG_n)$ for each $g \in G$. Therefore T_n belongs to $[\mathcal{G}]$ and we have $\mu(T_n^\circ A \triangle A) \to 0$ for every Borel subset $A \subset X$. For each $h \in G$, T_n commutes with the element $\phi_h \in [\mathcal{G}]$ defined as the constant map with value h. Indeed, if $x \in p_n^{-1}(gG_n)$ with $g \in G$, then $(T_n \circ \phi_h)x = T_n(hx)h = hgr_ng^{-1}$, which is equal to $(\phi_h \circ T_n)x$. Therefore (T_n) is a central sequence in $[\mathcal{G}]$, and we have $T_n^\circ x \neq x$ for every $x \in X$ because r_n does not belong to N.

We thus obtained the ergodic p.m.p. action $G \curvearrowright (X, \mu)$ such that N acts on X trivially, the induced action of G/N on X is free and there exists a central sequence (T_n) in the full group $[G \ltimes (X, \mu)]$ such that $T_n x \neq x$ and $T_n x \in R$ for all n and all $x \in X$. Recall also that R is contained in the centralizer $C_G(N)$ and that $N \cap R$ is finite and central in G. In order to apply Theorem 2.5 or 2.14, we check that at least one of the assumptions in those two theorems is fulfilled. For $p \in \mathbb{N}$, let $A_n^p \subset X$ be the set of p-periodic points of T_n°. If every $p \in \mathbb{N}$ satisfies $\mu(A_n^p) \to 0$ as $n \to \infty$, then, letting Ω be a singleton and $M_\omega = N$ in Theorem 2.5, we apply it and conclude the Schmidt property for G. Suppose otherwise, that is, suppose that, for some integer $p \geq 2$, the measure $\mu(A_n^p)$ does not converge to zero as $n \to \infty$. After passing to a subsequence, we may assume that $\mu(A_n^p)$ is uniformly positive. If $x \in A_n^p$, then $(T_n^\circ)^p x = x$ and hence $(T_n)^p x \in N$ and $(T_n)^p x \in N \cap R$. Letting $M = N$ and $L = N \cap R$ in Theorem 2.14, we apply it and conclude the Schmidt property of G.

5. *Groups with commutative FC-center*

5.1. *Groupoid extensions.* Let G be a countable group and let A be an abelian normal subgroup of G. We set $\Gamma = G/A$ and choose a section $\mathsf{s} : \Gamma \to G$ of the quotient map, with $\mathsf{s}(e) = e$. We then have the 2-cocycle $\sigma : \Gamma \times \Gamma \to A$ defined by $\sigma(g, h)\mathsf{s}(gh) = \mathsf{s}(g)\mathsf{s}(h)$ for $g, h \in \Gamma$. The map σ satisfies the 2-cocycle identity

$$\sigma(g, h)\sigma(gh, k) = {}^g\sigma(h, k)\sigma(g, hk)$$

for all $g, h, k \in \Gamma$, where we set ${}^g a = \mathsf{s}(g) a \mathsf{s}(g)^{-1}$ for $g \in \Gamma$ and $a \in A$. Note that ${}^g a$ does not depend on the choice of the section s.

Fix a compact abelian metrizable group L. We define X as the group of homomorphisms from A into L, identified with the closed subgroup of the product group $\prod_A L$. Let μ denote the normalized Haar measure on X. The group G acts on X by $(g\tau)(a) = \tau(g^{-1} a g)$ for $g \in G$, $a \in A$ and $\tau \in X$, and this gives rise to the action of Γ on X. We set $\mathcal{U} = X \times L$

and regard it as the bundle over X with respect to the projection onto the first coordinate. We also regard $\mathcal{U}$ as the groupoid with unit space X such that the range and source maps are the projection onto X, and the product is given by $(\tau, l)(\tau, m) = (\tau, lm)$ for $\tau \in X$ and $l, m \in L$. The translation groupoid $X \rtimes \Gamma$ acts on $\mathcal{U}$ by $(\tau, g)(g^{-1}\tau, l) = (\tau, l)$ for $\tau \in X$, $g \in \Gamma$ and $l \in L$.

Let $(X \rtimes \Gamma)^{(2)}$ be the set of composable pairs of the groupoid $X \rtimes \Gamma$, that is, the set of all pairs of the form $((\tau, g), (g^{-1}\tau, h))$ for some $\tau \in X$ and $g, h \in \Gamma$. The pair of that form is also denoted by (τ, g, h) for brevity. We define the 2-cocycle $\tilde{\sigma} : (X \rtimes \Gamma)^{(2)} \to \mathcal{U}$ by

$$\tilde{\sigma}(\tau, g, h) = (\tau, \langle \tau, \sigma(g, h)\rangle), \tag{5.1}$$

where $\langle \tau, a\rangle$ stands for $\tau(a)$ for $\tau \in X$ and $a \in A$. Indeed, the map $\tilde{\sigma}$ satisfies the 2-cocycle identity

$$\tilde{\sigma}(\tau, g, h)\tilde{\sigma}(\tau, gh, k) = {}^{(\tau,g)}\tilde{\sigma}(g^{-1}\tau, h, k)\tilde{\sigma}(\tau, g, hk), \tag{5.2}$$

where we set ${}^{(\tau,g)}(g^{-1}\tau, l) = (\tau, l)$ for $(\tau, g) \in X \rtimes \Gamma$ and $l \in L$, which is the result of the action of (τ, g) on $(g^{-1}\tau, l) \in \mathcal{U}$. Let us check equation (5.2). For the first coordinate in X, both sides are τ. For the second coordinate in L, the left-hand side is

$$\begin{aligned}\langle \tau, \sigma(g, h)\rangle\langle \tau, \sigma(gh, k)\rangle &= \langle \tau, \sigma(g, h)\sigma(gh, k)\rangle = \langle \tau, {}^{g}\sigma(h, k)\sigma(g, hk)\rangle \\ &= \langle \tau, {}^{g}\sigma(h, k)\rangle\langle \tau, \sigma(g, hk)\rangle = \langle g^{-1}\tau, \sigma(h, k)\rangle\langle \tau, \sigma(g, hk)\rangle,\end{aligned}$$

which is equal to the second coordinate of the right-hand side.

We now construct the groupoid extension

$$1 \to \mathcal{U} \to \mathcal{G}_{\tilde{\sigma}} \to X \rtimes \Gamma \to 1 \tag{5.3}$$

associated with the 2-cocycle $\tilde{\sigma}$ (see [Se] for the extension associated with a 2-cocycle of an equivalence relation with coefficient in a bundle of abelian Polish groups). As a set, we define $\mathcal{G}_{\tilde{\sigma}}$ as the fibered product $\mathcal{U} \times_X (X \rtimes \Gamma)$ with respect to the range map of $X \rtimes \Gamma$. The range and source of $(u, g) \in \mathcal{G}_{\tilde{\sigma}}$ with $u \in \mathcal{U}$ and $g \in X \rtimes \Gamma$ are defined as the range and source of g, respectively. The product of $\mathcal{G}_{\tilde{\sigma}}$ is given by

$$(u, g)(v, h) = (u\,{}^{g}v\tilde{\sigma}(g, h), gh) \tag{5.4}$$

for $(u, g), (v, h) \in \mathcal{G}_{\tilde{\sigma}}$ with (g, h) composable. This product is associative. Indeed, for three elements $(u, g), (v, h), (w, k) \in \mathcal{G}_{\tilde{\sigma}}$ with (g, h) and (h, k) composable,

$$\begin{aligned}(u\,{}^{g}v\tilde{\sigma}(g, h), gh)(w, k) &= (u\,{}^{g}v\tilde{\sigma}(g, h)\,{}^{gh}w\tilde{\sigma}(gh, k), ghk) \\ &= (u\,{}^{g}v\,{}^{gh}w\,{}^{g}\tilde{\sigma}(h, k)\tilde{\sigma}(g, hk), ghk) = (u, g)(v\,{}^{h}w\tilde{\sigma}(h, k), hk).\end{aligned}$$

The inverse of an element $(u, g) \in \mathcal{G}_{\tilde{\sigma}}$ is given by

$$(({}^{g^{-1}}u)^{-1}\tilde{\sigma}(g^{-1}, g)^{-1}, g^{-1}) = (({}^{g^{-1}}(u^{-1}))({}^{g^{-1}}(\tilde{\sigma}(g, g^{-1})^{-1})), g^{-1}), \tag{5.5}$$

where the left-hand side is a left inverse of (u, g), the right-hand side is a right inverse of (u, g), and these two coincide because it follows from $\mathrm{s}(e) = e$ that $\sigma(g, e) = e = \sigma(e, g)$ for every $g \in \Gamma$, and $\sigma(g, g^{-1}) = {}^{g}\sigma(g^{-1}, g)$ by the 2-cocycle identity. All these groupoid operations are Borel, and we thus obtain a Borel groupoid $\mathcal{G}_{\tilde{\sigma}}$. We have the projection

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from $\mathcal{G}_{\tilde{\sigma}} = \mathcal{U} \times_X (X \rtimes \Gamma)$ onto $X \rtimes \Gamma$, whose kernel is identified with $\mathcal{U}$ via the inclusion of $\mathcal{U}$ into $\mathcal{G}_{\tilde{\sigma}}$, $(\tau, l) \mapsto ((\tau, l), (\tau, e))$ for $\tau \in X$ and $l \in L$. Consequently, the groupoid extension (5.3) is obtained.

An element $((\tau, l), (\tau, \gamma)) \in \mathcal{G}_{\tilde{\sigma}} = \mathcal{U} \times_X (X \rtimes \Gamma)$ is also denoted by (τ, l, γ) for brevity. We define a homomorphism $\eta \colon X \rtimes G \to \mathcal{G}_{\tilde{\sigma}}$ by

$$\eta(\tau, (a, \gamma)) = (\tau, \tau(a), \gamma)$$

for $\tau \in X$, $a \in A$ and $\gamma \in \Gamma$, where G is identified with $A \times \Gamma$ via the map $(a, \gamma) \mapsto a\mathsf{s}(\gamma)$. To check that η is indeed a homomorphism, let us recall the product of two elements of $A \times \Gamma$ inherited from G, that is,

$$(a, \gamma)(b, \delta) = (a^{\gamma} b\sigma(\gamma, \delta), \gamma\delta)$$

for $a, b \in A$ and $\gamma, \delta \in \Gamma$. If we put $g = (a, \gamma)$ and $h = (b, \delta)$ and regard them as elements of G, then, for each $\tau \in X$,

$$\begin{aligned}
\eta(\tau, gh) &= (\tau, \tau(a^{\gamma} b\sigma(\gamma, \delta)), \gamma\delta) = (\tau, \tau(a)(\gamma^{-1}\tau)(b)\tau(\sigma(\gamma, \delta)), \gamma\delta) \\
&= ((\tau, \tau(a))^{(\tau,\gamma)}(\gamma^{-1}\tau, (\gamma^{-1}\tau)(b))\tilde{\sigma}(\tau, \gamma, \delta), (\tau, \gamma\delta)) \\
&= (\tau, \tau(a), \gamma)(\gamma^{-1}\tau, (\gamma^{-1}\tau)(b), \delta) = \eta(\tau, g)\eta(\gamma^{-1}\tau, h),
\end{aligned}$$

where, in the fourth term, the element of $\mathcal{G}_{\tilde{\sigma}}$ is written as a pair of an element of $\mathcal{U}$ and an element of $X \rtimes \Gamma$. Therefore η is a homomorphism. The kernel of η is given by

$$\ker \eta = \{(\tau, a) \in X \rtimes A \mid a \in \ker \tau\}.$$

The image of $X \rtimes A$ under η is given by

$$\eta(X \rtimes A) = \{(\tau, \tau(a)) \in \mathcal{U} \mid a \in A\}.$$

5.2. *A free action from co-induction.* We keep the notation in the previous subsection, where we constructed the groupoid $\mathcal{G}_{\tilde{\sigma}}$. In this subsection, we construct a free p.m.p. action of $\mathcal{G}_{\tilde{\sigma}}$, which will be obtained as the action co-induced from the shift action of $\mathcal{U}$ onto itself. This action was not treated in §2.3 since $\mathcal{G}_{\tilde{\sigma}}$ is not necessarily discrete. We do not aim to discuss co-induced actions for non-discrete Borel groupoids in full generality.

We set $\mathcal{G} = \mathcal{G}_{\tilde{\sigma}}$ and $\mathcal{Q} = X \rtimes \Gamma$ for brevity. We have the groupoid extension

$$1 \to \mathcal{U} \to \mathcal{G} \to \mathcal{Q} \to 1.$$

Recall that $\mathcal{U} = X \times L$ is the bundle of a compact abelian metrizable group L, and denote by $\mathcal{U}_x$ the fiber of $\mathcal{U}$ at x, that is, $\{x\} \times L$. Each fiber $\mathcal{U}_x$ is often identified with L naturally if there is no cause for confusion. The bundle $\mathcal{U}$ is a groupoid on X and acts on itself by left multiplication. We co-induce this action to the action of $\mathcal{G}$ in the same manner as in §2.3 as follows. For $x \in X$, we set

$$Z_x = \{f \colon \mathcal{G}^x \to L \mid f(gu^{-1}) = uf(g) \text{ for all } g \in \mathcal{G}^x \text{ and all } u \in \mathcal{U}_{s(g)}\}$$

and define Z as the disjoint union $Z = \bigsqcup_{x \in X} Z_x$. For each $f \in Z_x$, it is natural to regard the value $f(g) \in L$ at $g \in \mathcal{G}^x$ as an element of $\mathcal{U}_{s(g)}$. The set Z is fibered with respect to

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the projection $p\colon Z \to X$ sending each element of Z_x to x. Then $\mathcal{G}$ acts on Z by

$$(gf)(h) = f(g^{-1}h)$$

for $g \in \mathcal{G}_x$, $h \in \mathcal{G}^{r(g)}$ and $f \in Z_x$ with $x \in X$.

We define a measure-space structure on Z. Recall that, as a set, $\mathcal{G}$ is the fibered product $\mathcal{U} \times_X \mathcal{Q}$ with respect to the range map of $\mathcal{Q}$. For $\gamma \in \Gamma$, we define a map $\psi_\gamma\colon X \to \mathcal{G}$ by $\psi_\gamma(x) = ((x, e), (x, \gamma))$ for $x \in X$. Then, for each $x \in X$, we have $\psi_\gamma(x) \in \mathcal{G}^x$ and the family $\{\psi_\gamma(x)\}_{\gamma\in\Gamma}$ is a complete set of representatives of all the equivalence classes in $\mathcal{G}^x$, where the equivalence relation on $\mathcal{G}^x$ is defined as follows: two elements $g, h \in \mathcal{G}^x$ are equivalent if and only if $g^{-1}h \in \mathcal{U}$. Then Z is identified with the product space $X \times \prod_\Gamma L$ under the map sending each $f \in Z_x$ with $x \in X$ to $(x, (f(\psi_\gamma(x)))_\gamma)$. The measure-space structure on Z is induced by this identification, where the space $X \times \prod_\Gamma L$ is equipped with the product measure of μ and the normalized Haar measure on L. The action of $\mathcal{G}$ on Z is Borel and preserves the probability measure on Z in the following sense.

PROPOSITION 5.1. *We use the above notation.*

(i) *For all $\gamma \in \Gamma$, $x \in X$ and $l \in L$, we have*

$$\psi_\gamma(x)^{-1}(x, l)\psi_\gamma(x) = (\gamma^{-1}x, l),$$

where we identify $\mathcal{U}$ with a subset of $\mathcal{G}$ under the injection of $\mathcal{U}$ into $\mathcal{G}$.

(ii) *We define an action of the group L on Z by $lf = (x, l)f$ for $l \in L$ and $f \in Z_x$ with $x \in X$. Then this action is Borel, p.m.p. and free.*

(iii) *For each $\gamma \in \Gamma$, the action of ψ_γ on Z is Borel and p.m.p., that is, the map from Z into itself sending each $f \in Z_x$ with $x \in X$ to $\psi_\gamma(\gamma x)f \in Z_{\gamma x}$ is Borel and p.m.p.*

(iv) *Suppose that either L is infinite and $|\Gamma| \geq 3$ or L is non-trivial and Γ is infinite. Then the action of $\mathcal{G}$ on Z is essentially free, that is, for almost every $f \in Z$, letting $x \in X$ be the point with $f \in Z_x$, we have $gf \neq f$ for each $g \in \mathcal{G}_x$ except for the unit at x.*

Proof. To prove assertion (i), we pick $\gamma \in \Gamma$, $x \in X$ and $l \in L$ and set $g = (x, \gamma) \in \mathcal{Q}$. It follows from formula (5.5) that $\psi_\gamma(x)^{-1} = (\tilde{\sigma}(g^{-1}, g)^{-1}, g^{-1})$ and therefore

$$\begin{aligned}\psi_\gamma(x)^{-1}(x, l)\psi_\gamma(x) &= (\tilde{\sigma}(g^{-1}, g)^{-1}, g^{-1})((x, l), g)\\ &= (\tilde{\sigma}(g^{-1}, g)^{-1 g^{-1}}(x, l)\tilde{\sigma}(g^{-1}, g), (\gamma^{-1}x, e)) = ((\gamma^{-1}x, l), (\gamma^{-1}x, e)),\end{aligned}$$

where the first and second equations are derived from formula (5.4). Assertion (i) follows.

We prove assertion (ii). Pick $l \in L$ and $f \in Z_x$ with $x \in X$. The element f is identified with the element of $X \times \prod_\Gamma L$ given by the pair of x and the function $\gamma \mapsto f(\psi_\gamma(x))$. Let us describe the element of $X \times \prod_\Gamma L$ corresponding to lf, which is the pair of x and the function $\gamma \mapsto (lf)(\psi_\gamma(x))$. For each $\gamma \in \Gamma$,

$$\begin{aligned}(lf)(\psi_\gamma(x)) &= f((x, l)^{-1}\psi_\gamma(x)) = f(\psi_\gamma(x)\psi_\gamma(x)^{-1}(x, l^{-1})\psi_\gamma(x))\\ &= f(\psi_\gamma(x)(\gamma^{-1}x, l^{-1})) = l(f(\psi_\gamma(x))),\end{aligned}$$

where we apply assertion (i) in the third equation. Therefore the action of l on $X \times \prod_\Gamma L$ is given by $(x, (l_\gamma)_\gamma) \mapsto (x, (ll_\gamma)_\gamma)$, and the action of L on Z is Borel, p.m.p. and free.

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We prove assertion (iii). Pick $\gamma \in \Gamma$ and $f \in Z_x$ with $x \in X$. The element f is identified with the element of $X \times \prod_\Gamma L$ given by the pair of x and the function $\delta \mapsto f(\psi_\delta(x))$. The element $\psi_\gamma(\gamma x)f$ corresponds to the pair of γx and the function

$$\delta \mapsto (\psi_\gamma(\gamma x)f)(\psi_\delta(\gamma x)) = f(\psi_\gamma(\gamma x)^{-1}\psi_\delta(\gamma x)).$$

We set $g = (\gamma x, \gamma)$ and $h = (\gamma x, \delta)$. By formula (5.5), $\psi_\gamma(\gamma x)^{-1} = (\tilde{\sigma}(g^{-1}, g)^{-1}, g^{-1})$. For each $\delta \in \Gamma$, if we define $k \in L$ by

$$(x, k) = \tilde{\sigma}(g^{-1}, g)^{-1}\tilde{\sigma}(g^{-1}, h), \tag{5.6}$$

then we have

$$\begin{aligned}\psi_\gamma(\gamma x)^{-1}\psi_\delta(\gamma x) &= (\tilde{\sigma}(g^{-1}, g)^{-1}, g^{-1})((\gamma x, e), h)\\ &= (\tilde{\sigma}(g^{-1}, g)^{-1}\tilde{\sigma}(g^{-1}, h), g^{-1}h) = (x, k)\psi_{\gamma^{-1}\delta}(x)\\ &= \psi_{\gamma^{-1}\delta}(x)\psi_{\gamma^{-1}\delta}(x)^{-1}(x, k)\psi_{\gamma^{-1}\delta}(x) = \psi_{\gamma^{-1}\delta}(x)((\gamma^{-1}\delta)^{-1}x, k),\end{aligned}$$

where the second equation follows from formula (5.4) and the fifth equation follows from assertion (i). Therefore

$$f(\psi_\gamma(\gamma x)^{-1}\psi_\delta(\gamma x)) = k^{-1}(f(\psi_{\gamma^{-1}\delta}(x))),$$

and the action of ψ_γ on $X \times \prod_\Gamma L$ is given by

$$(x, (l_\delta)_\delta) \mapsto (\gamma x, (k_{\gamma,\delta,x}^{-1} l_{\gamma^{-1}\delta})_\delta),$$

where the element $k = k_{\gamma,\delta,x} \in L$ is determined by equation (5.6). By the definition of $\tilde{\sigma}$ in (5.1), the function $x \mapsto k_{\gamma,\delta,x}$ is Borel. Hence the action of ψ_γ is Borel and also p.m.p. by the above description of the action. Assertion (iii) follows.

We prove assertion (iv). Recall that each $g \in \mathcal{G}_x$ with $x \in X$ is written as $(\gamma x, l, \gamma)$ for some $\gamma \in \Gamma$ and $l \in L$. By assertion (ii), it suffices to show that, for each non-trivial $\gamma \in \Gamma$, there exists a conull subset $\bar{Z} \subset Z$ such that, for all $f \in \bar{Z}$ and all $l \in L$, letting $x \in X$ be the point with $f \in Z_x$, we have $(\gamma x, l, \gamma)f \neq f$. We fix a non-trivial $\gamma \in \Gamma$. The action of $g = (\gamma x, l, \gamma)$ on $X \times \prod_\Gamma L$ is described as

$$(x, (l_\delta)_\delta) \mapsto (\gamma x, (lk_{\gamma,\delta,x}^{-1} l_{\gamma^{-1}\delta})_\delta).$$

Thus if g fixes the point $(x, (l_\delta)_\delta)$, then $lk_{\gamma,\delta,x}^{-1} l_{\gamma^{-1}\delta} = l_\delta$ for all $\delta \in \Gamma$.

Suppose that L is infinite and $|\Gamma| \geq 3$. Pick a non-trivial element $\gamma_1 \in \Gamma$ with $\gamma_1 \neq \gamma^{-1}$. We fix $x \in X$. If a point $(l_\delta)_\delta$ is such that, for some $l \in L$, we have $lk_{\gamma,\delta,x}^{-1} l_{\gamma^{-1}\delta} = l_\delta$ for all $\delta \in \Gamma$, then $lk_{\gamma,e,x}^{-1} l_{\gamma^{-1}} = l_e$ and $lk_{\gamma,\gamma_1,x}^{-1} l_{\gamma^{-1}\gamma_1} = l_{\gamma_1}$. Deleting l, we thus obtain

$$l_{\gamma_1} = l_e l_{\gamma^{-1}}^{-1} l_{\gamma^{-1}\gamma_1} k_{\gamma,e,x} k_{\gamma,\gamma_1,x}^{-1}, \tag{5.7}$$

which says that l_{γ_1} is determined if l_e, $l_{\gamma^{-1}}$ and $l_{\gamma^{-1}\gamma_1}$ are determined. The element γ_1 is distinct from all of e, γ^{-1} and $\gamma^{-1}\gamma_1$. Hence, by Fubini's theorem, the set of points $(l_\delta)_\delta$ satisfying equation (5.7) is null, where we use the assumption that L is infinite and thus each singleton subset of L is null. Since x is an arbitrary point of X, by Fubini's theorem again, the set of points $(x, (l_\delta)_\delta) \in X \times \prod_\Gamma L$ satisfying equation (5.7) is null. Thus it suffices to let $\bar{Z}$ be the complement of that null set.

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Next suppose that L is non-trivial and Γ is infinite. Then there exists an infinite subset $S \subset \Gamma$ such that S and $\gamma^{-1}S$ are disjoint. We fix $x \in X$. Let $(l_\delta)_\delta$ be a point such that, for some $l \in L$, we have $lk_{\gamma,\delta,x}^{-1}l_{\gamma^{-1}\delta} = l_\delta$ for all $\delta \in \Gamma$. As in the previous paragraph, for all distinct $\gamma_0, \gamma_1 \in S$, we then have

$$l_{\gamma_1} = l_{\gamma_0} l_{\gamma^{-1}\gamma_0}^{-1} l_{\gamma^{-1}\gamma_1} k_{\gamma,\gamma_0,x} k_{\gamma,\gamma_1,x}^{-1}. \tag{5.8}$$

The element γ_1 is distinct from all of γ_0, $\gamma^{-1}\gamma_0$ and $\gamma^{-1}\gamma_1$. Hence, by Fubini's theorem, for all distinct $\gamma_0, \gamma_1 \in S$, the set of points $(l_\delta)_\delta$ satisfying equation (5.8) has measure less than one, where we use the assumption that L is non-trivial and thus each singleton subset of L has measure less than one. Since we have mutually disjoint, infinitely many pairs of distinct elements of S, the set of points $(l_\delta)_\delta$ satisfying equation (5.8) for all distinct $\gamma_0, \gamma_1 \in S$ is null. We thus obtain $\bar{Z}$ as well, as before, and assertion (iv) follows. □

5.3. *The case where condition* ($\star$) *holds.* Let G be a countable group and let A be an infinite abelian normal subgroup of G contained in the FC-center of G. Suppose that each finite subset of A has finite normal closure in G and let $A_1 < A_2 < \cdots$ be a strictly increasing sequence of finite subgroups of A such that each A_n is normalized by G. Suppose, further, that condition ($\star$) introduced in §3.2 holds, that is, for all $N \in \mathbb{N}$, we have $\lim_n |F_{n,N}|/|A_n| = 1$, where $F_{n,N}$ is the set of elements of A_n whose order is more than N. Under these assumptions, we aim to construct a free p.m.p. Schmidt action of G. We may assume that G/A is infinite because otherwise G is amenable. This assumption will be used in applying Proposition 5.1 (iv) later, and not used for other purposes.

We set $\Gamma = G/A$ and choose a section $\mathsf{s}\colon \Gamma \to G$ of the quotient map with $\mathsf{s}(e) = e$. We then obtain the 2-cocycle $\sigma : \Gamma \times \Gamma \to A$. We define X as the dual group $\widehat{A}$ of A, that is, the group of homomorphisms from A into the torus $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. Let μ be the normalized Haar measure on X. We recall the construction in §5.1. Define the action of G on X by $(g\tau)(a) = \tau(g^{-1}ag)$ for $g \in G$, $a \in A$ and $\tau \in X$, which induces the action of Γ on X. Let $\mathcal{U} := X \times \mathbb{T}$ be the bundle over X, which is a groupoid with unit space X. Then we obtain the 2-cocycle $\tilde{\sigma}\colon (X \rtimes \Gamma)^{(2)} \to \mathcal{U}$ by formula (5.1) and obtain the groupoid extension

$$1 \to \mathcal{U} \to \mathcal{G}_{\tilde{\sigma}} \to X \rtimes \Gamma \to 1$$

together with the homomorphism $\eta\colon X \rtimes G \to \mathcal{G}_{\tilde{\sigma}}$ such that

$$\ker \eta = \{(\tau, a) \in X \rtimes A \mid a \in \ker \tau\}$$

and $\eta(\tau, a) = (\tau, \tau(a)) \in \mathcal{U}$ for all $a \in A$ and $\tau \in X$.

Let $\mathcal{G}_{\tilde{\sigma}} \curvearrowright (Z, \zeta)$ be the free p.m.p. action constructed in §5.2, that is, the action co-induced from the shift action of $\mathcal{U}$ on itself. The space Z is fibered over X. The fiber at $\tau \in X$ is denoted by Z_τ. For $n \in \mathbb{N}$, let Γ_n be the group of elements of Γ acting on A_n trivially. Let $\Gamma \curvearrowright (Y, \nu)$ be the profinite p.m.p. action associated with the system of the p.m.p. action $\Gamma \curvearrowright \Gamma/\Gamma_n$ given by left multiplication. Through the quotient map from $\mathcal{G}_{\tilde{\sigma}}$ onto Γ factoring through $X \rtimes \Gamma$, we obtain the p.m.p. action $\mathcal{G}_{\tilde{\sigma}} \curvearrowright (Y, \nu)$. Then $\mathcal{G}_{\tilde{\sigma}}$ acts on $Y \times Z$ diagonally, where $Y \times Z$ is fibered over X with respect to the map sending each element of $Y \times Z_\tau$ to τ for each $\tau \in X$.

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Through the homomorphism $\eta\colon X \rtimes G \to \mathcal{G}_{\tilde{\sigma}}$, we obtain the p.m.p. action of $X \rtimes G$ on the product space $(W, \omega) := (Y \times Z, \nu \times \zeta)$. We then obtain the p.m.p. action $G \curvearrowright (W, \omega)$ given by $g(y, z) = (g\tau, g)(y, z)$ for $g \in G$, $y \in Y$ and $z \in Z_\tau$ with $\tau \in X$. The action of A on W is given by $a(y, z) = (y, (\tau, \tau(a))z)$ for each $a \in A$. Recall that we defined the action of $\mathbb{T}$ on Z by $tz = (\tau, t)z$ for $t \in \mathbb{T}$ and $z \in Z_\tau$ with $\tau \in X$ in Proposition 5.1 (ii). Thus, with respect to this action, the element $(y, (\tau, \tau(a))z)$ is written as $(y, \tau(a)z)$.

We now construct a central sequence (T_N) in the full group of the translation groupoid $G \ltimes (W, \omega)$. Pick $N \in \mathbb{N}$. By condition $(\star)$, for some $n = n_N \in \mathbb{N}$, we have $|F_n|/|A_n| \geq 1 - 1/N$, where F_n is the set of elements of A_n whose order is more than N. Since the dual $\widehat{A}_n$ of A_n is isomorphic to A_n [F, Corollary 4.8], if E_n denotes the set of elements of $\widehat{A}_n$ whose order is more than N, then $|E_n|/|\widehat{A}_n| \geq 1 - 1/N$. The set E_n is further Γ-invariant. The map $p_n\colon X = \widehat{A} \to \widehat{A}_n$ induced by the inclusion of A_n into A is surjective [F, Corollary 4.42]. For each $\tau \in E_n$, since its order is more than N, there exists $a_\tau \in A_n$ such that

$$0 < |\tau(a_\tau) - 1| < |\exp(2\pi i/N) - 1|. \tag{5.9}$$

We define a map $T_N\colon W \to A$ as follows. Let Y_n denote the inverse image of the coset $e\Gamma_n$ under the projection from Y onto Γ/Γ_n. For $y \in gY_n$ with $g \in \Gamma$ and $z \in Z_\tau$ with $\tau \in X$, if τ belongs to $p_n^{-1}(E_n)$, then we set

$$T_N(y, z) = {}^g a_{g^{-1}p_n(\tau)},$$

and, otherwise, we set $T_N(y, z) = e$. This is well defined because Γ_n acts on A_n and $\widehat{A}_n$ trivially. The map from W into itself, $w \mapsto (T_N w)w$, is an automorphism of W because A acts on Y trivially and preserves each fiber Z_τ with $\tau \in X$. Thus T_N is an element of the full group $[G \ltimes (W, \omega)]$.

LEMMA 5.2. *We use the above notation.*

(i) *For each $N \in \mathbb{N}$ and $g \in G$, we have $\phi_g \circ T_N = T_N \circ \phi_g$, where $\phi_g\colon X \to G$ is the element of the full group $[G \ltimes (W, \omega)]$ given by the constant map with value g.*

(ii) *For each Borel subset $B \subset W$, we have $\omega(T_N^\circ B \bigtriangleup B) \to 0$ as $N \to \infty$.*

(iii) *We define $B_N \subset W$ as the set of periodic points of T_N° whose period is more than N. Then $\omega(B_N) \geq 1 - 1/N$ for all $N \in \mathbb{N}$.*

Proof. To prove assertion (i), we pick $N \in \mathbb{N}$ and $g \in G$. Let $n = n_N \in \mathbb{N}$ be the integer chosen before to obtain the subset $E_n \subset \widehat{A}_n$. We also pick $y \in hY_n$ with $h \in \Gamma$ and $z \in Z_\tau$ with $\tau \in X$, and set $w = (y, z)$. If $\tau \in p_n^{-1}(E_n)$, then $(\phi_g \circ T_N)w = g({}^h a_{h^{-1}p_n(\tau)})$ and

$$(T_N \circ \phi_g)w = T_N(\bar{g}y, gz)g = ({}^{\bar{g}h} a_{(\bar{g}h)^{-1}p_n(g\tau)})g = g({}^h a_{h^{-1}p_n(\tau)}),$$

where $\bar{g}$ denotes the image of g in Γ. Thus $\phi_g \circ T_N = T_N \circ \phi_g$ at w. If $\tau \notin p_n^{-1}(E_n)$, then $(\phi_g \circ T_N)w = g$, and $(T_N \circ \phi_g)w = g$ because $g\tau \notin p_n^{-1}(E_n)$. Assertion (i) follows.

We prove assertion (ii). Let the group $\mathbb{T}$ act on W by $t(y, z) = (y, tz)$ for $t \in \mathbb{T}$, $y \in Y$ and $z \in Z$. Since $\mathbb{T}$ is compact, the action $\mathbb{T} \curvearrowright W$ is isomorphic to the action $\mathbb{T} \curvearrowright D \times \mathbb{T}$ given by $t(w, s) = (w, ts)$ for $t, s \in \mathbb{T}$ and $w \in D$, where D is a Borel subset of W that is the product of Y with a Borel fundamental domain for the action $\mathbb{T} \curvearrowright Z$.

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We pick $N \in \mathbb{N}$ and let $n = n_N$. For $y \in gY_n$ with $g \in \Gamma$ and $z \in Z_\tau$ with $\tau \in X$, if τ belongs to $p_n^{-1}(E_n)$, then

$$T_N^\circ(y, z) = (y, \tau({}^g a_{g^{-1} p_n(\tau)})z) = (y, \langle g^{-1}\tau, a_{g^{-1}p_n(\tau)}\rangle z), \tag{5.10}$$

and, otherwise, $T_N^\circ(y, z) = (y, z)$. This shows that, for each $y \in Y$ and $\tau \in X$, the map T_N° preserves the set $\{y\} \times Z_\tau$, and on that set, the map T_N° is equal to the transformation given by some single element of $\mathbb{T}$. Moreover, $\{y\} \times Z_\tau$ is $\mathbb{T}$-invariant. Therefore if T_N° is regarded as a automorphism of $D \times \mathbb{T}$ under the isomorphism between W and $D \times \mathbb{T}$, then T_N° preserves each orbit $\{w\} \times \mathbb{T}$ with $w \in D$, and on that orbit, the map T_N° is equal to the transformation given by some single element of $\mathbb{T}$. By inequality (5.9), those elements of $\mathbb{T}$, that is, the values $\langle g^{-1}\tau, a_{g^{-1}p_n(\tau)}\rangle$ in equation (5.10), are uniformly close to 1 if N is so large that $\exp(2\pi i/N)$ is close to 1. Thus assertion (ii) follows.

We pick $N \in \mathbb{N}$ and let $n = n_N$. If $y \in gY_n$ with $g \in \Gamma$ and $z \in Z_\tau$ with $\tau \in p_n^{-1}(E_n)$, then the value $\langle g^{-1}\tau, a_{g^{-1}p_n(\tau)}\rangle \in \mathbb{T}$ has order more than N by inequality (5.9). Moreover, freeness of the action $\mathbb{T} \curvearrowright Z$, shown in Proposition 5.1 (ii), and equation (5.10) imply that (y, z) is a periodic point of T_N° whose period is more than N. Assertion (iii) follows from this together with the inequality $|E_n|/|\widehat{A}_n| \geq 1 - 1/N$. □

We are going to apply Theorem 2.5. Let us check that the assumption in it is fulfilled for the p.m.p. action $G \curvearrowright (W, \omega)$, the G-equivariant measure-preserving map $\pi : (W, \omega) \to (X, \mu)$ and the central sequence (T_N) in the full group $[G \ltimes (W, \omega)]$, where we define the map π by $\pi(y, z) = \tau$ for $y \in Y$ and $z \in Z_\tau$ with $\tau \in X$. We first note that (T_N) is indeed central by Lemma 5.2 (i) and (ii). The stabilizer of a point of W in G depends only on its image under π. Indeed, the action $\mathcal{G}_{\tilde{\sigma}} \curvearrowright (Z, \zeta)$ is essentially free by Proposition 5.1 (iv) and thus the stabilizer of almost every $w \in W$ is equal to the kernel of $\pi(w) \in X = \widehat{A}$. As pointed out in the proof of Lemma 5.2 (ii), T_N° preserves the set of the form $\{y\} \times Z_\tau$ with $y \in Y$ and $\tau \in X$ and thus preserves each fiber of π. For each $w \in W$, since A is abelian and the kernel of $\pi(w)$ is a subgroup of A, the element $T_N w \in A$ belongs to the centralizer of the stabilizer of w in G. The inequality $\omega(B_N) \geq 1 - 1/N$ shown in Lemma 5.2 (iii) implies that $\omega(\{w \in W \mid T_N^\circ w \neq w\}) \to 1$ as $N \to \infty$. By Lemma 5.2 (iii) again, for each $p \in \mathbb{N}$, if $B_N^p \subset W$ denotes the set of p-periodic points of T_N°, then $\omega(B_N^p) \to 0$ as $N \to \infty$. Thus the assumption in Theorem 2.5 is fulfilled, and, by the theorem, G has the Schmidt property.

5.4. *The other case.* Let G be a countable group and let A be an infinite abelian normal subgroup of G contained in the FC-center of G. Suppose that each finite subset of A has finite normal closure in G and let $A_1 < A_2 < \cdots$ be a strictly increasing sequence of finite subgroups of A such that each A_n is normalized by G. In this subsection, we suppose that condition $(\star)$ in §3.2 does not hold for this sequence and then construct a free p.m.p. Schmidt action of G. By Lemma 3.4, we may assume, without loss of generality, that there exists a prime number p such that each A_n is isomorphic to the direct sum of finitely many copies of $\mathbb{Z}/p\mathbb{Z}$. We may also assume that $A = \bigcup_n A_n$ and that G/A is infinite as in the previous subsection.

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We set $\Gamma = G/A$ and choose a section $\mathsf{s}\colon \Gamma \to G$ of the quotient map with $\mathsf{s}(e) = e$. We then obtain the 2-cocycle $\sigma\colon \Gamma \times \Gamma \to A$. We define X as the group of homomorphisms from A into the direct product $L := \prod_{\mathbb{N}} \mathbb{Z}/p\mathbb{Z}$, while X denoted the dual group $\widehat{A}$ of A in the previous subsection. Let μ be the normalized Haar measure on X. Note that if we fix an embedding of $\mathbb{Z}/p\mathbb{Z}$ into the torus $\mathbb{T}$, then the dual $\widehat{A}$ is identified with the group of homomorphisms from A into $\mathbb{Z}/p\mathbb{Z}$ since all elements of $A = \bigcup_n A_n$, except for the trivial one, have order p. Under this identification, we often identify X with the product group $\prod_{\mathbb{N}} \widehat{A}$ unless there is cause for confusion.

We recall the construction in §5.1. Define the action of G on X by $(g\tau)(a) = \tau(g^{-1}ag)$ for $g \in G$, $a \in A$ and $\tau \in X$, which induces the action of Γ on X. Let $\mathcal{U} = X \times L$ be the bundle over X, which is a groupoid with unit space X. Then we obtain the 2-cocycle $\tilde{\sigma}\colon (X \rtimes \Gamma)^{(2)} \to \mathcal{U}$ by formula (5.1) and obtain the groupoid extension

$$1 \to \mathcal{U} \to \mathcal{G}_{\tilde{\sigma}} \to X \rtimes \Gamma \to 1$$

together with the homomorphism $\eta\colon X \rtimes G \to \mathcal{G}_{\tilde{\sigma}}$ such that

$$\ker \eta = \{(\tau, a) \in X \rtimes A \mid a \in \ker \tau\}$$

and $\eta(\tau, a) = (\tau, \tau(a)) \in \mathcal{U}$ for all $a \in A$ and $\tau \in X$.

LEMMA 5.3. *We use the above notation.*

(i) *For each $N \in \mathbb{N}$, the set of points $\tau = (\tau_i)_{i\in\mathbb{N}} \in X$ such that $\bigcap_{i=1}^N \ker \tau_i = \ker \tau$ is μ-null.*

(ii) *For μ-almost every $\tau \in X$, we have $\ker \tau = \{e\}$. Therefore the groupoid $X \rtimes A$ embeds into $\mathcal{U}$ via η if it is restricted to some μ-conull subset of X.*

Proof. The set in assertion (i) is written as

$$\bigsqcup_{\tau_1,\ldots,\tau_N\in\widehat{A}} \{\tau_1\} \times \cdots \times \{\tau_N\} \times \prod_{i=N+1}^{\infty} \{\xi \in \widehat{A} \mid \textstyle\bigcap_{j=1}^N \ker \tau_j < \ker \xi\}. \tag{5.11}$$

We note that if a is a non-trivial element of A, then the subgroup $\{\xi \in \widehat{A} \mid a \in \ker \xi\}$ is of index p in $\widehat{A}$ and thus has measure $1/p$, where $\widehat{A}$ is equipped with the normalized Haar measure. Then, for each $\tau_1, \ldots, \tau_N \in \widehat{A}$, the set $\{\xi \in \widehat{A} \mid \bigcap_{i=1}^N \ker \tau_i < \ker \xi\}$ has measure at most $1/p$ because this is contained in the set $\{\xi \in \widehat{A} \mid a \in \ker \xi\}$ if a is chosen to be a non-trivial element of $\bigcap_{i=1}^N \ker \tau_i$. By Fubini's theorem, the set in (5.11) is μ-null.

For each non-trivial $a \in A$, the set $\{\tau \in X \mid a \in \ker \tau\}$ is identified with the product set $\prod_{\mathbb{N}}\{\xi \in \widehat{A} \mid a \in \ker \xi\}$ and hence is μ-null. Assertion (ii) follows. □

Let $\mathcal{G}_{\tilde{\sigma}} \curvearrowright (Z, \zeta)$ be the free p.m.p. action constructed in §5.2, that is, the action co-induced from the shift action of $\mathcal{U}$ on itself. The space Z is fibered over X. The fiber at $\tau \in X$ is denoted by Z_τ. For $n \in \mathbb{N}$, let Γ_n be the group of elements of Γ acting on A_n trivially. Let $\Gamma \curvearrowright (Y, \nu)$ be the profinite p.m.p. action associated with the system of the p.m.p. action $\Gamma \curvearrowright \Gamma/\Gamma_n$ given by left multiplication. As with the previous subsection, let $\mathcal{G}_{\tilde{\sigma}}$ act on $Y \times Z$ diagonally, where $Y \times Z$ is fibered over X with respect to the map sending each element of $Y \times Z_\tau$ to τ for each $\tau \in X$. Through the homomorphism $\eta\colon X \rtimes G \to \mathcal{G}_{\tilde{\sigma}}$, we obtain the p.m.p. action of G on the product space $(W, \omega) := (Y \times Z, \nu \times \zeta)$. We note that the action $G \curvearrowright (W, \omega)$ is essentially free because the action $\mathcal{G}_{\tilde{\sigma}} \curvearrowright (Z, \zeta)$ is essentially free by Proposition 5.1 (iv) and $\ker \eta$ is trivial in the sense of Lemma 5.3 (ii).

We now construct a central sequence (T_N) in the full group of the translation groupoid $G \ltimes (W, \omega)$. Pick $N \in \mathbb{N}$. For each $a \in A$, we set

$$X_a = \{\tau = (\tau_i)_{i\in\mathbb{N}} \in X \mid \tau_1(a) = \cdots = \tau_N(a) = 0,\ \tau(a) \neq 0\}.$$

By Lemma 5.3 (i), $X = \bigcup_{a\in A} X_a$ up to null sets. Let Y_n denote the inverse image of the coset $e\Gamma_n$ under the projection from Y onto Γ/Γ_n. Then

$$X \times Y = \bigcup_{n=1}^{\infty} \bigcup_{a\in A_n\setminus A_{n-1}} \bigcup_{g\Gamma_n\in\Gamma/\Gamma_n} X_a \times gY_n,$$

where we set $A_0 = \{e\}$. If $a \in A_n \setminus A_{n-1}$ and $g, h \in \Gamma$, then $h(X_a \times gY_n) = X_{h\cdot a} \times hgY_n$ with respect to the diagonal action $\Gamma \curvearrowright X \times Y$, where the dot stands for the action of Γ on A. Thus the saturation $\Gamma(X_a \times gY_n)$ is the disjoint union of the translates $h(X_a \times gY_n)$ with h running through representatives of elements of Γ/Γ_n. Let us call such a subset a (Γ/Γ_n)-base, that is, call a Borel subset $B \subset X \times Y$ a (Γ/Γ_n)-*base* if B is Γ_n-invariant and the saturation ΓB is the disjoint union of the translates hB with h running through representatives of elements of Γ/Γ_n.

LEMMA 5.4. *With the above notation, there exist Borel subsets of X, $B_1, B_2, \ldots$, such that $X \times Y = \bigsqcup_{m=1}^{\infty} \Gamma B_m$ and each B_m is a (Γ/Γ_n)-base contained in $X_a \times gY_n$ for some $n \in \mathbb{N}$, $a \in A_n \setminus A_{n-1}$ and $g \in \Gamma$.*

Proof. For each $n \in \mathbb{N}$, let $D(n, 1), D(n, 2), \ldots, D(n, k_n)$ be an enumeration of the (Γ/Γ_n)-bases $X_a \times gY_n$ indexed by $a \in A_n \setminus A_{n-1}$ and a representative g of an element of Γ/Γ_n, with $k_n = |A_n \setminus A_{n-1}|\,|\Gamma/\Gamma_n|$. Let $(E_m)_{m\in\mathbb{N}}$ be the enumeration of the sets $D(n, k)$ with respect to the lexicographic order of the indices (n, k).

We inductively define a Borel subset $B_m \subset X \times Y$. We set $B_1 = E_1$. Suppose that $B_1, \ldots, B_{m-1}$ are defined. We set $B_m = E_m \setminus \bigcup_{i=1}^{m-1} \Gamma B_i$. Then $E_m = D(n, k)$ for some n and k and thus B_m is a (Γ/Γ_n)-base. By construction, ΓB_m and ΓB_l are disjoint for all distinct m, l. Since the sets E_m cover $X \times Y$, the sets ΓB_m cover $X \times Y$. $\square$

We define a map $T_N\colon W \to A$ as follows. Let $q\colon W \to X \times Y$ be the projection that sends a point $(y, z) \in W$ with $z \in Z_\tau$ and $\tau \in X$ to the point (τ, y). By Lemma 5.4, the set $X \times Y$ is covered by the mutually disjoint sets ΓB_m with $m \in \mathbb{N}$. For each $m \in \mathbb{N}$, we have $n_m \in \mathbb{N}$, $a_m \in A_{n_m} \setminus A_{n_m-1}$ and $g_m \in \Gamma$ such that the set B_m is a (Γ/Γ_{n_m})-base contained in $X_{a_m} \times g_m Y_{n_m}$. For $w \in q^{-1}(hB_m)$ with $h \in \Gamma$, we set

$$T_N w = h \cdot a_m.$$

This is well defined because B_m is a (Γ/Γ_{n_m})-base and a_m is fixed by Γ_{n_m}. The map from W into itself, $w \mapsto (T_N w)w$, is an automorphism of W because A preserves each fiber of q. Thus T_N is an element of the full group $[G \ltimes (W, \omega)]$.

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LEMMA 5.5. *We use the above notation.*

(i) *For every $N \in \mathbb{N}$ and $g \in G$, we have $\phi_g \circ T_N = T_N \circ \phi_g$, where $\phi_g \colon X \to G$ is the element of the full group $[G \ltimes (W, \omega)]$ given by the constant map with value g.*

(ii) *For every Borel subset $B \subset W$, we have $\omega(T_N^\circ B \bigtriangleup B) \to 0$ as $N \to \infty$.*

(iii) *For every $N \in \mathbb{N}$ and every $w \in W$, we have $T_N^\circ w \neq w$.*

Proof. We prove assertion (i). If $w \in q^{-1}(hB_m)$ with $h \in \Gamma$, then we have $(T_N \circ \phi_g)w = T_N(gw)g = ((\bar{g}h) \cdot a_m)g$ with $\bar{g}$ being the image of g in Γ, and we also have $(\phi_g \circ T_N)w = g(h \cdot a_m)$. These two coincide.

We prove assertion (ii). The proof is similar to that of Lemma 5.2 (ii). Using the action of $\mathcal{U}$ on Z, which restricts the action of $\mathcal{G}_{\tilde{\sigma}}$, we define an action of L on Z by $lf = (\tau, l)f$ for $l \in L$ and $f \in Z_\tau$ with $\tau \in X$. This is the action defined in Proposition 5.1 (ii). Let L act on W by $l(y, z) = (y, lz)$ for $l \in L$, $y \in Y$ and $z \in Z$.

Fix $N \in \mathbb{N}$. Recall that the group A acts on W via the homomorphism $\eta \colon X \rtimes G \to \mathcal{G}_{\tilde{\sigma}}$, which satisfies $\eta(\tau, a) = (\tau, \tau(a))$ for all $\tau \in X$ and $a \in A$. Hence, if $w = (y, z) \in q^{-1}(hB_m)$ with $z \in Z_\tau$, $\tau = (\tau_i)_{i \in \mathbb{N}} \in X$ and $h \in \Gamma$, then

$$T_N^\circ w = (y, \langle \tau, T_N w\rangle z) = (y, \tau(h \cdot a_m)z).$$

Since $q(w) = (\tau, y) \in hB_m$, we have $\tau \in X_{h \cdot a_m}$ and thus $\tau_1(h \cdot a_m) = \cdots = \tau_N(h \cdot a_m) = 0$ and $\tau(h \cdot a_m) \neq 0$. This says that the element $\tau(h \cdot a_m) \in L = \prod_{\mathbb{N}} \mathbb{Z}/p\mathbb{Z}$ is non-trivial and is close to the identity if N is large. The definition of $T_N w$ depends only on $q(w)$, and the action of L on W preserves each fiber of q. Hence, on each L-orbit in W, the map T_N° is equal to the transformation given by some single element of L. Assertion (ii) then follows from the existence of a Borel fundamental domain for the action $L \curvearrowright Z$ as well as in the proof of Lemma 5.2 (ii).

Assertion (iii) follows from the condition that $\tau(h \cdot a_m) \neq 0$, shown above, and freeness of the action of L on Z, which was shown in Proposition 5.1 (ii). $\square$

Therefore the groupoid $G \ltimes (W, \omega)$ is Schmidt, and so is its almost every ergodic component by Lemma 2.2. We have already shown that the action $G \curvearrowright (W, \omega)$ is essentially free, in the paragraph after Lemma 5.3. Thus G has the Schmidt property.

6. *Another construction using ultraproducts*

Let G be a countable group with infinite FC-center. We construct a free p.m.p. Schmidt action of G by way of ultraproducts. This construction is self-contained and independent of the construction given so far.

Step 1. Setting up the sequence of actions. Let A denote the FC-center of G. Then A has an infinite abelian subgroup B, which is found as follows. First, pick a non-trivial $a_1 \in A$. If $\langle a_1\rangle$ is infinite, let $B = \langle a_1\rangle$. Otherwise, pick an element a_2 of the set $C_A(a_1) \setminus \langle a_1\rangle$, which is non-empty because $C_A(a_1)$ is of finite index in A and hence infinite. If $\langle a_1, a_2\rangle$ is infinite, let $B = \langle a_1, a_2\rangle$. Otherwise, pick an element a_3 of the set $C_A(a_1, a_2) \setminus \langle a_1, a_2\rangle$, which is non-empty by the same reason. Repeat this procedure. Then either it stops in finite steps and the group $B = \langle a_1, \ldots, a_n\rangle$ for some n is infinite and abelian, or it does not stop and the group $B = \langle a_1, a_2, \ldots\rangle$ is infinite and abelian.

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We may write B as an increasing union of finitely generated subgroups $B = \bigcup_{n\in\mathbb{N}} B_n$. Let $G_n := C_G(B_n)$, so that G_n is a finite index subgroup of G which contains B. Since B is abelian, we may find a free ergodic compact action $B \curvearrowright^{\beta} (Y, \mu_Y)$ of B, where Y is a compact abelian metrizable group and $\beta\colon B \to Y$ is an injective homomorphism with dense image, and B is acting on Y by left translation via β. Let $G_n \curvearrowright^{\beta_n} (Y, \mu_Y)^{G_n/B}$ be the p.m.p. action co-induced from the action β of B. Explicitly, this is defined as follows. We pick a section $t_n\colon G_n/B \to G_n$ of the projection map $G_n \to G_n/B$ with $t_n(eB) = e$, and we let $w_n\colon G_n \times G_n/B \to B$ be the associated cocycle for the action $G_n \curvearrowright G_n/B$ given by $w_n(g, hB) := t_n(ghB)^{-1} g t_n(hB)$ for $g, h \in G_n$. Then the action $G_n \curvearrowright^{\beta_n} Y^{G_n/B}$ is given by

$$(\beta_n(g)x)(hB) := \beta(w_n(g, g^{-1}hB))x(g^{-1}hB)$$

for $g, h \in G_n$. For each n, pick a section $s_n\colon G/G_n \to G$ of the projection map $G \to G/G_n$ with $s_n(eG_n) = e$, and let $v_n\colon G \times G/G_n \to G_n$ be the associated cocycle for the p.m.p. action $G \curvearrowright (G/G_n, \mu_{G/G_n})$ (where μ_{G/G_n} is the normalized counting measure), given by $v_n(g, hG_n) := s_n(ghG_n)^{-1} g s_n(hG_n)$ for $g, h \in G$. Then we equip $Z_n := G/G_n \times Y^{G_n/B}$ with the product measure $\eta_n := \mu_{G/G_n} \times \mu_Y^{G_n/B}$ and we let $G \curvearrowright^{\alpha_n} (Z_n, \eta_n)$ be the skew product action, which is the p.m.p. action defined by

$$\alpha_n(g)(kG_n, x) := (gkG_n, \beta_n(v_n(g, kG_n))x)$$

for $g \in G$ and $(kG_n, x) \in Z_n$.

Step 2. The ultraproduct and its quotients. Fix a non-principal ultrafilter $\mathcal{V}$ on $\mathbb{N}$ and let $G \curvearrowright^{\alpha} (Z_{\mathcal{V}}, \eta_{\mathcal{V}})$ be the ultraproduct of the sequence of actions $(G \curvearrowright^{\alpha_n} (Z_n, \eta_n))_{n\in\mathbb{N}}$ with respect to $\mathcal{V}$. Thus $Z_{\mathcal{V}} = (\prod_n Z_n)/{\sim_{\mathcal{V}}}$, where $\sim_{\mathcal{V}}$ is the equivalence relation on $\prod_n Z_n$ such that $(y_n) \sim_{\mathcal{V}} (z_n)$ if and only if $\{n \in \mathbb{N} \mid y_n = z_n\} \in \mathcal{V}$; we write $[(z_n)]_{\mathcal{V}}$ for the equivalence class of the sequence (z_n). For a sequence (D_n) of Borel sets $D_n \subset Z_n$, let $[(D_n)]_{\mathcal{V}}$ be the associated basic measurable subset of $Z_{\mathcal{V}}$, that is,

$$[(D_n)]_{\mathcal{V}} = \{[(z_n)]_{\mathcal{V}} \mid \lim_{n\to\mathcal{V}} 1_{D_n}(z_n) = 1\},$$

where 1_{D_n} is the indicator function of D_n. The assignment $[(D_n)]_{\mathcal{V}} \mapsto \lim_{n\to\mathcal{V}} \eta_n(D_n)$ defines a premeasure on the algebra of all such basic measurable sets, and hence this assignment extends uniquely to a countably additive measure $\eta_{\mathcal{V}}$ on the completion $\mathcal{B}_{\mathcal{V}}$ of the sigma algebra generated by the basic measurable sets. This is how the measure $\eta_{\mathcal{V}}$ is defined. The action α, of G on $Z_{\mathcal{V}}$, is given by $\alpha(g)[(z_n)]_{\mathcal{V}} := [(gz_n)]_{\mathcal{V}}$.

Likewise, let $G \curvearrowright (X_{\mathcal{V}}, \mu_{\mathcal{V}})$ denote the ultraproduct with respect to $\mathcal{V}$, of the sequence of actions $(G \curvearrowright (G/G_n, \mu_{G/G_n}))_{n\in\mathbb{N}}$. Then the projection map $p\colon (Z_{\mathcal{V}}, \eta_{\mathcal{V}}) \to (X_{\mathcal{V}}, \mu_{\mathcal{V}})$, $[(k_nG_n, x_n)]_{\mathcal{V}} \mapsto [(k_nG_n)]_{\mathcal{V}}$, is measure-preserving and G-equivariant.

Let $G \curvearrowright (P, \mu_P)$ denote the profinite action that is the inverse limit of the finite actions $G \curvearrowright G/G_n$. Elements of P consist of sequences (g_mG_m) with $g_mG_m \supset g_{m+1}G_{m+1}$ for all m. For each $[(k_nG_n)]_{\mathcal{V}} \in X_{\mathcal{V}}$ and each $m \in \mathbb{N}$, let $\Phi_m[(k_nG_n)]_{\mathcal{V}}$ be the unique left coset gG_m of G_m for which the set $\{n \in \mathbb{N} \mid k_nG_n \subset gG_m\}$ belongs to $\mathcal{V}$. Then each $\Phi_m\colon X_{\mathcal{V}} \to G/G_m$ is G-equivariant and measure-preserving, and $\Phi_m[(k_nG_n)]_{\mathcal{V}} \supset \Phi_{m+1}[(k_nG_n)]_{\mathcal{V}}$, so we obtain the measure-preserving G-equivariant map $\Phi\colon (X_{\mathcal{V}}, \mu_{\mathcal{V}}) \to (P, \mu_P)$ given by $\Phi[(k_nG_n)]_{\mathcal{V}} = (\Phi_m[(k_nG_n)]_{\mathcal{V}})_m$.

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For each n, let $\pi_n\colon Z_n \to Y$ be the map $\pi_n(kG_n, x) := x(eB)$ projecting to the identity-coset coordinate of $x \in Y^{G_n/B}$. Let $\pi : Z_{\mathcal{V}} \to Y$ be defined by

$$\pi[(k_n G_n, x_n)]_{\mathcal{V}} := \lim_{n\to\mathcal{V}} \pi_n(k_n G_n, x_n) = \lim_{n\to\mathcal{V}} x_n(eB)$$

(note that this limit exists since Y is compact). By [**BTD**, Proposition 8.4], this map is measurable and measure-preserving, with $\eta_{\mathcal{V}}(\pi^{-1}(E) \,\triangle\, [(\pi_n^{-1}(E))]_{\mathcal{V}}) = 0$ for every Borel subset E of Y. Let $\mathcal{Y}$ denote the subalgebra of $\mathcal{B}_{\mathcal{V}}$ consisting of all sets of the form $\pi^{-1}(E)$ with $E \subset Y$ Borel, and let $\mathcal{P}$ denote the subalgebra of $\mathcal{B}_{\mathcal{V}}$ consisting of all sets of the form $(\Phi \circ p)^{-1}(C)$ with $C \subset P$ Borel.

Step 3. The central sequence. For each $b \in B$, the conjugacy class b^G of b in G is finite, and the map $T_b\colon Z_{\mathcal{V}} \to b^G$ given by

$$T_b[(k_n G_n, x_n)]_{\mathcal{V}} := \lim_{n\to\mathcal{V}} k_n b k_n^{-1}$$

is well defined, since, if $m(b) \in \mathbb{N}$ is the least such that $G_{m(b)} < C_G(b)$, then, for all $n \geq m(b)$, the conjugate $k_n b k_n^{-1}$ depends only on the coset $k_n G_n$ of G_n. Letting $(g_m G_m)_{m\in\mathbb{N}} := \Phi[(k_n G_n)]_{\mathcal{V}}$, we have $\{n \in \mathbb{N} \mid k_n G_n \subset g_{m(b)} G_{m(b)}\} \in \mathcal{V}$ and hence $T_b[(k_n G_n, x_n)]_{\mathcal{V}} = g_{m(b)} b g_{m(b)}^{-1} = \lim_{m\to\infty} g_m b g_m^{-1}$. In particular, the map T_b is $\mathcal{P}$-measurable. We have $T_b(gz) = g T_b(z) g^{-1}$ for all $g \in G$ and $z \in Z_{\mathcal{V}}$. The map $T_b^{\circ}\colon Z_{\mathcal{V}} \to Z_{\mathcal{V}}$ given by $T_b^{\circ}(z) = \alpha(T_b(z))z$ is an automorphism of $(Z_{\mathcal{V}}, \eta_{\mathcal{V}})$ which commutes with $\alpha(g)$ for all $g \in G$. Then the map p is T_b°-invariant, and, in particular, every set in $\mathcal{P}$ is T_b°-invariant.

For each $b \in B$ and $[(k_n G_n, x_n)]_{\mathcal{V}} \in Z_{\mathcal{V}}$, since the set $\{n \in \mathbb{N} \mid T_b[(k_n G_n, x_n)]_{\mathcal{V}} = k_n b k_n^{-1}\}$ belongs to $\mathcal{V}$, the transformation T_b° is given by

$$T_b^{\circ}[(k_n G_n, x_n)]_{\mathcal{V}} = [(k_n G_n, \beta_n(v_n(k_n b k_n^{-1}, k_n G_n))x_n)]_{\mathcal{V}}.$$

For all large enough n, we have $G_n < C_G(b)$, and for such n, since $B < G_n$, we have $v_n(k_n b k_n^{-1}, k_n G_n) = v_n(k_n, eG_n) v_n(b, eG_n) v_n(k_n, eG_n)^{-1} = b$. Since this holds for all large n, we obtain

$$T_b^{\circ}[(k_n G_n, x_n)]_{\mathcal{V}} = [(k_n G_n, \beta_n(b)x_n)]_{\mathcal{V}}.$$

Also, for all n with $G_n < C_G(b)$, for each $hB \in G_n/B$ we have $b^{-1}hB = hB$ and $w_n(b, b^{-1}hB) = b$, so that $(\beta_n(b)x_n)(hB) = \beta(b)x_n(hB)$, and therefore

$$\begin{aligned}\pi(T_b^{\circ}[(k_n G_n, x_n)]_{\mathcal{V}}) &= \lim_{n\to\mathcal{V}} (\beta_n(b)x_n)(eB) = \lim_{n\to\mathcal{V}} \beta(b)x_n(eB) \\ &= \beta(b)\pi([(k_n G_n, x_n)]_{\mathcal{V}}).\end{aligned} \tag{6.1}$$

Let $(b_i)_{i\in\mathbb{N}}$ be a sequence of distinct elements in B with $\beta(b_i)$ converging weakly to the identity element of Y. Then, for each Borel subset E of Y, we have $\mu_Y(\beta(b_i)E \,\triangle\, E) \to 0$ as $i \to \infty$, so it follows from (6.1) that $\eta_{\mathcal{V}}(T_{b_i}^{\circ}(\pi^{-1}(E)) \,\triangle\, \pi^{-1}(E)) \to 0$ as $i \to \infty$.

Thus both $\mathcal{P}$ and $\mathcal{Y}$ belong to the sigma subalgebra $\mathcal{D}$ of $\mathcal{B}_{\mathcal{V}}$ consisting of all $D \in \mathcal{B}_{\mathcal{V}}$ such that $\lim_{i\to\infty} \eta_{\mathcal{V}}(T_{b_i}^{\circ} D \,\triangle\, D) = 0$. Since each T_{b_i} commutes with $\alpha(G)$, the sigma algebra $\mathcal{D}$ is $\alpha(G)$-invariant.

Step 4. Ensuring essential freeness for the action of A on the upcoming separable quotient. We pick $a \in A \setminus \{e\}$ and let $F_a \subset X_{\mathcal{V}}$ be the fixed point set of a in $X_{\mathcal{V}}$. Then we have $X_{\mathcal{V}} \setminus F_a = [(C_{a,n})_n]_{\mathcal{V}}$, where $C_{a,n} := \{kG_n \in G/G_n \mid akG_n \neq kG_n\}$. We can write the set $C_{a,n}$ as a union of three pairwise disjoint sets $C_{a,n,0}$, $C_{a,n,1}$, $C_{a,n,2}$ such that $aC_{a,n,i} \cap C_{a,n,i} = \emptyset$ (indeed, let $C_{a,n,0}$ be a maximal subset of $C_{a,n}$ such that $aC_{a,n,0} \cap C_{a,n,0} = \emptyset$ and set $C_{a,n,1} := aC_{a,n,0} \cap C_{a,n}$ and $C_{a,n,2} := C_{a,n} \setminus (C_{a,n,0} \cup C_{a,n,1})$). Each of the sets $D_{a,i} := (\Phi \circ p)^{-1}([(C_{a,n,i})_n]_{\mathcal{V}})$ is T_b°-invariant for all $b \in B$ and hence belongs to $\mathcal{D}$. For $c \in a^G$, we define $F_{a,c}$ as the set of all $[(k_nG_n)]_{\mathcal{V}} \in F_a$ for which $\lim_{n\to\mathcal{V}} s_n(k_nG_n)^{-1}as_n(k_nG_n) = c$, so that $F_{a,c}$ is a basic measurable subset of $X_{\mathcal{V}}$ corresponding to the sequence of sets $\{kG_n \in G/G_n \mid s_n(kG_n)as_n(kG_n)^{-1} = c\}$ with $n \in \mathbb{N}$. The sets $F_{a,c}$ with $c \in a^G$ partition F_a. Each of the sets $p^{-1}(F_{a,c})$ is T_b°-invariant for all $b \in B$ and hence belongs to $\mathcal{D}$.

Step 5. Defining the separable quotient of the ultraproduct. Since $\mathcal{D}$ is G-invariant and both the algebras $\mathcal{P}$ and $\mathcal{Y}$ are countably generated and G is countable, we can find a countably generated G-invariant sigma subalgebra $\mathcal{D}_0$ of $\mathcal{D}$ which contains both $\mathcal{P}$ and $\mathcal{Y}$ as well as all of the sets $D_{a,i}$ and $p^{-1}(F_{a,c})$ for $a \in A \setminus \{e\}$, $c \in a^G$ and $i \in \{0, 1, 2\}$. Then we may find a point realization $G \curvearrowright (W_0, \mu_0)$ for the action of G on the measure algebra $\mathcal{D}_0$, along with a G-equivariant measure-preserving map $\varphi\colon (Z_{\mathcal{V}}, \eta_{\mathcal{V}}) \to (W_0, \mu_0)$ which is a point realization of the measure algebra inclusion $\mathcal{D}_0 \hookrightarrow \mathcal{B}_{\mathcal{V}}$. For each $b \in B$, since the map T_b is $\mathcal{P}$-measurable and $\mathcal{P} \subset \mathcal{D}_0$, T_b descends via φ to a map $S_b\colon W_0 \to b^G$, which satisfies $S_b(gw) = gS_b(w)g^{-1}$ for all $g \in G$ and $w \in W_0$. The map $S_b^{\circ}\colon W_0 \to W_0$ given by $S_b^{\circ}(w) = S_b(w)w$ is an automorphism of (W_0, μ_0) with $\varphi \circ T_b^{\circ} = S_b^{\circ} \circ \varphi$. Since $\mathcal{Y} \subset \mathcal{D}_0$ is invariant under the group $\{T_b^{\circ} \mid b \in B\}$, the map π descends to a measure-preserving map $\pi_0\colon (W_0, \mu_0) \to (Y, \mu_Y)$ with $\pi_0(S_b^{\circ}w) = \beta(b)\pi_0(w)$ for all $b \in B$. It follows that the group $\{S_b^{\circ} \mid b \in B\}$ acts essentially freely on W_0 since $\beta(B)$ acts freely on Y.

Since $\mathcal{D}_0 \subset \mathcal{D}$, it follows that $(S_{b_i})_{i\in\mathbb{N}}$ is a central sequence in the full group of the action $G \curvearrowright (W_0, \mu_0)$ with $S_{b_i}^{\circ}w \neq w$ for almost every $w \in W_0$. However, it is not clear whether this action of G is essentially free, so we take an essentially free action $G/A \curvearrowright (W_1, \mu_1)$ and let $G \curvearrowright (W_0 \times W_1, \mu_0 \times \mu_1)$ be the product action, where G acts on W_1 via the projection onto G/A. Then each $S_b\colon W_0 \to b^G$ lifts to the map $\tilde{S}_b\colon W_0 \times W_1 \to b^G$ via the projection from $W_0 \times W_1$ onto W_0, and it satisfies $\tilde{S}_b(gw) = g\tilde{S}_b(w)g^{-1}$ for all $g \in G$ and $w \in W_0 \times W_1$. The map $\tilde{S}_b^{\circ}$ is given by $\tilde{S}_b^{\circ}(w_0, w_1) = S_b(w_0)(w_0, w_1) = (S_b^{\circ}(w_0), w_1)$ and hence an automorphism of $W_0 \times W_1$, and the group $\{\tilde{S}_b^{\circ} \mid b \in B\}$ acts essentially freely on $W_0 \times W_1$. Since A acts trivially on W_1, it follows that $(\tilde{S}_{b_i})_{i\in\mathbb{N}}$ is a central sequence in the full group of the action $G \curvearrowright (W_0 \times W_1, \mu_0 \times \mu_1)$, and it satisfies $\tilde{S}_{b_i}^{\circ}w \neq w$ for almost every $w \in W_0 \times W_1$.

Thus the proof will be complete once we show that the action $G \curvearrowright (W_0 \times W_1, \mu_0 \times \mu_1)$ is essentially free. For this, it is enough to show that the action $A \curvearrowright (W_0, \mu_0)$ is essentially free.

Step 6. Verifying that the action $A \curvearrowright (W_0, \mu_0)$ is essentially free. Fix $a \in A \setminus \{e\}$. Suppose that there is some $c \in a^G$ for which the set $F_{a,c}$ has positive measure. We first show that, for almost every $z \in p^{-1}(F_{a,c})$, $\pi(\alpha(a)z)$ and $\pi(z)$ are distinct. Since $F_{a,c}$ is a subset of F_a, if $[(k_nG_n)]_{\mathcal{V}} \in F_{a,c}$, then, for $\mathcal{V}$-almost every $n \in \mathbb{N}$, we have $v_n(a, k_nG_n) =$

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$s_n(k_nG_n)^{-1}as_n(k_nG_n) = c$ and hence $c \in G_n$. Since the sequence (G_n) is decreasing, this implies that $c \in G_n$ for all $n \in \mathbb{N}$, and hence the element $\beta(w_n(c, c^{-1}B)) \in Y$ is well defined for all n. Let y_c denote the limit along $\mathcal{V}$ of this sequence, $y_c := \lim_{n\to\mathcal{V}} \beta(w_n(c, c^{-1}B)) \in Y$. For each $z = [(k_nG_n, x_n)]_{\mathcal{V}} \in p^{-1}(F_{a,c})$, we have $\alpha(a)z = [(k_nG_n, \beta_n(c)x_n)]_{\mathcal{V}}$, and hence

$$\begin{aligned} \pi(\alpha(a)z) &= \lim_{n\to\mathcal{V}} \beta(w_n(c, c^{-1}B))x_n(c^{-1}B) = y_c \lim_{n\to\mathcal{V}} x_n(c^{-1}B) \text{ and} \\ \pi(z) &= \lim_{n\to\mathcal{V}} x_n(eB). \end{aligned} \tag{6.2}$$

To see that these are almost surely distinct, we consider the two possibilities of whether $c \in B$ or $c \notin B$. If $c \in B$, then $x_n(c^{-1}B) = x_n(eB)$ and $y_c = \lim_{n\to\mathcal{V}} \beta(w_n(c, B)) = \beta(c) \neq e$, and hence $\pi(\alpha(a)z) = \beta(c)\pi(z) \neq \pi(z)$, as was to be shown. Now suppose that $c \notin B$. By [**BTD**, Proposition 8.4], the map $\pi_c \colon (Z_{\mathcal{V}}, \eta_{\mathcal{V}}) \to (Y, \mu_Y)$ defined by $\pi_c[(k_nG_n, x_n)]_{\mathcal{V}} := y_c \lim_{n\to\mathcal{V}} x_n(c^{-1}B)$ is measurable and measure-preserving, and, for each Borel subset E of Y, we have $\eta_{\mathcal{V}}(\pi_c^{-1}(E) \triangle [(\pi_{c,n}^{-1}(E))_n]_{\mathcal{V}}) = 0$, where the map $\pi_{c,n} \colon (Z_n, \eta_n) \to (Y, \mu_Y)$ is defined by $\pi_{c,n}(kG_n, x) := y_c x(c^{-1}B)$. Since $c \notin B$, the random variables π_n, $\pi_{c,n}$ are independent for every n. Therefore the random variables π, π_c are also independent. Since μ_Y is atomless, it follows that $\pi(z) \neq \pi_c(z)$ for almost every $z \in Z_{\mathcal{V}}$. By (6.2), for almost every $z \in p^{-1}(F_{a,c})$, we thus have $\pi(\alpha(a)z) = \pi_c(z) \neq \pi(z)$, as was to be shown.

It now follows that $\pi(\alpha(a)z) \neq \pi(z)$ for almost every $z \in p^{-1}(F_a)$. Since $\pi = \pi_0 \circ \varphi$ and since each of the sets $p^{-1}(F_a)$ belongs to $\mathcal{D}_0$, it follows that $\pi_0(aw) \neq \pi_0(w)$ and hence $aw \neq w$ for almost every $w \in \varphi(p^{-1}(F_a))$. In addition, since each of the sets $D_{a,i}$ for $i \in \{0, 1, 2\}$ belongs to $\mathcal{D}_0$, it follows that $aw \neq w$ for almost every $w \in W_0 \setminus \varphi(p^{-1}(F_a))$. This shows that the action of A on W_0 is essentially free.

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A. *Appendix. A Kazhdan group with prescribed center*

Given a countable abelian group A, we construct a countable group G with property (T) such that the center of G is isomorphic to A. We rely on the construction of Cornulier [**C**] as well as Examples 3.6 and 3.7. Let $R := \mathbb{Z}[t]$ be the ring of polynomials over $\mathbb{Z}$ in one indeterminate t. In the course of the construction, we will use property (T) of the group $SL_3(R)$ (e.g., [**EJZ**, Theorem 1.1] and [**M**, Theorem 1.8]) and property (T) of the pair $(SL_3(R) \ltimes R^3, R^3)$ [**Ka**, Theorem 1.9, a)]. Note that the statements in those papers are given in terms of the group generated by elementary matrices in $SL_3(R)$, which is, in fact, equal to $SL_3(R)$ by [**Su**, Corollary 6.6].

Let H be the subgroup of $SL_5(R)$ consisting of matrices of the form

$$g = \begin{pmatrix} 1 & u & c \\ 0 & h & v \\ 0 & 0 & 1 \end{pmatrix}, \tag{A.1}$$

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where $h \in SL_3(R)$, $c \in R$, and u and v are row and column vectors of R^3, respectively. Let C be the center of H, which consists of matrices g such that $h = I$, $u = 0$ and $v = 0$. Then H/C is isomorphic to the semi-direct product $\Gamma := SL_3(R) \ltimes (R^3 \times R^3)$, where $SL_3(R)$ acts on $R^3 \times R^3$ by $h(u, v) = (uh^{-1}, hv)$ for $h \in SL_3(R)$, a row vector $u \in R^3$ and a column vector $v \in R^3$. In fact, the map sending a matrix $g \in H$ of the form (A.1) to the element $(h, (u, h^{-1}v))$ of Γ induces an isomorphism.

The group Γ has property (T). To see this, recall the following fact. If G is a countable group and N is a normal subgroup of G such that the group G/N and the pair (G, N) have property (T), then G has property (T) [**BHV**, Remark 1.7.7]. Property (T) of the group $SL_3(R)$ and the pair $(SL_3(R) \ltimes R^3, R^3)$ thus implies that $SL_3(R) \ltimes R^3$ has property (T). The group Γ is written as the semi-direct product $(SL_3(R) \ltimes R^3) \ltimes R^3$, and the above fact again implies that Γ has property (T).

Hence the group H/C has property (T). The commutator subgroup $[H, H]$ contains C, and thus the abelianization $H/[H, H]$ is finite. It follows from [**BHV**, Theorem 1.7.11] that H has property (T).

We obtained the group H with property (T) whose center C is isomorphic to R and to the direct sum $\bigoplus_{\mathbb{N}} \mathbb{Z}$. Let A be an arbitrary countable abelian group. There exists a surjection from C onto A. Let C_1 be the kernel of this surjection and set $H_1 = H/C_1$. The group H_1 has property (T) and has the central subgroup C/C_1 isomorphic to A. In fact, the center of H_1 is exactly C/C_1 because $H/C \simeq \Gamma$ has trivial center.

REFERENCES

[BH] E. Bédos and P. de la Harpe. Moyennabilité intérieure des groupes: définitions et exemples. *Enseign. Math. (2)* **32** (1986), 139–157.

[BHV] B. Bekka, P. de la Harpe and A. Valette. *Kazhdan's Property (T) (New Mathematical Monographs, 11)*. Cambridge University Press, Cambridge, 2008.

[BTD] L. Bowen and R. Tucker-Drob. The space of stable weak equivalence classes of measure-preserving actions. *J. Funct. Anal.* **274** (2018), 3170–3196.

[C] Y. de Cornulier. Finitely presentable, non-Hopfian groups with Kazhdan's property (T) and infinite outer automorphism group. *Proc. Amer. Math. Soc.* **135** (2007), 951–959.

[DV] T. Deprez and S. Vaes. Inner amenability, property Gamma, McDuff II_1 factors and stable equivalence relations. *Ergod. Th. & Dynam. Sys.* **38** (2018), 2618–2624.

[Ef] E. G. Effros. Property Γ and inner amenability. *Proc. Amer. Math. Soc.* **47** (1975), 483–486.

[Er] M. Ershov. Kazhdan groups whose FC-radical is not virtually abelian. *J. Comb. Algebra* **1** (2017), 59–62.

[EJZ] M. Ershov and A. Jaikin-Zapirain. Property (T) for noncommutative universal lattices, *Invent. Math.* **179** (2010), 303–347.

[F] G. B. Folland. *A Course in Abstract Harmonic Analysis (Textbooks in Mathematics)*, 2nd edn. CRC Press, Boca Raton, FL, 2016.

[H] P. Hahn. The regular representations of measure groupoids. *Trans. Amer. Math. Soc.* **242** (1978), 35–72.

[J] Y. Jiang. A remark on $\mathbb{T}$-valued cohomology groups of algebraic group actions. *J. Funct. Anal.* **271** (2016), 577–592.

[JS] V. F. R. Jones and K. Schmidt. Asymptotically invariant sequences and approximate finiteness. *Amer. J. Math.* **109** (1987), 91–114.

[Ka] M. Kassabov. Universal lattices and unbounded rank expanders. *Invent. Math.* **170** (2007), 297–326.

[Ke1] A. S. Kechris. *Classical Descriptive Set Theory (Graduate Texts in Mathematics, 156)*. Springer, New York, NY, 1995.

[Ke2] A. S. Kechris. *Global Aspects of Ergodic Group Actions (Mathematical Surveys and Monographs, 160)*. American Mathematical Society, Providence, RI, 2010.

[Ki1] Y. Kida. Inner amenable groups having no stable action. *Geom. Dedicata* **173** (2014), 185–192.

https://doi.org/10.1017/etds.2020.146 Published online by Cambridge University Press

[Ki2] Y. Kida. Stability in orbit equivalence for Baumslag–Solitar groups and Vaes groups. *Groups Geom. Dyn.* **9** (2015), 203–235.

[Ki3] Y. Kida. Stable actions of central extensions and relative property (T). *Israel J. Math.* **207** (2015), 925–959.

[Ki4] Y. Kida. Stable actions and central extensions, *Math. Ann.* **369** (2017), 705–722.

[KTD] Y. Kida and R. Tucker-Drob. Inner amenable groupoids and central sequences. *Forum Math. Sigma* **8** (2020), e29.

[M] M. Mimura. Superintrinsic synthesis in fixed point properties. *Preprint*, 2016, arXiv:1505.06728v2.

[PV] S. Popa and S. Vaes. On the fundamental group of II_1 factors and equivalence relations arising from group actions. *Quanta of Maths (Clay Mathematics Proceedings, 11)*. American Mathematical Society, Providence, RI, 2010, pp. 519–541.

[Sc] K. Schmidt. Some solved and unsolved problems concerning orbit equivalence of countable group actions. *Proceedings of the Conference on Ergodic Theory and Related Topics, II (Georgenthal, 1986) (Teubner-Texte zur Mathematik, 94)*. Teubner, Leipzig, 1987.

[Se] C. Series. An application of groupoid cohomology. *Pacific J. Math.* **92** (1981), 415–432.

[Su] A. A. Suslin. The structure of the special linear group over rings of polynomials. *Izv. Akad. Nauk SSSR Ser. Mat.* **41** (1977), 235–252 (in Russian); translated in *Math. USSR-Izv.* **11** (1977), 221–238.

[TD] R. D. Tucker-Drob. Invariant means and the structure of inner amenable groups. *Duke Math. J.* **169** (2020), 2571–2628.

[V] S. Vaes. An inner amenable group whose von Neumann algebra does not have property Gamma. *Acta Math.* **208** (2012), 389–394.

https://doi.org/10.1017/etds.2020.146 Published online by Cambridge University Press