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ERGODIC MEASURES FOR THE IRRATIONAL ROTATION ON THE CIRCLE

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Abstract

Riesz products are employed to give a construction of quasi-invariant ergodic measures under the irrational rotation of $\mathbf{T}$. By suitable choice of the parameters such measures may be required to have Fourier-Stieltjes coefficients vanishing at infinity. We show further that these are the unique quasi-invariant measures on $\mathbf{T}$ with their associated Radon-Nikodym derivative.

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1. Introduction

Fix an irrational number $\alpha \in (0,1)$ and consider the action $T_\alpha(x) = x + \alpha \pmod 1$ on the circle group $\mathbf{T}$. Lebesgue measure is well known to be the unique invariant measure for this action and it is ergodic. The question arises as to whether there exist singular quasi-invariant measures which are ergodic for the action of T_α. Such measures allow the construction of non-monomial representations of the discrete Heisenberg group (see [1]):

$$H = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbf{Z} \right\}.$$

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Measures of this kind were first constructed by Michael Keane in [5]. His proof used the continued fraction expansion of α to identify the action of T_α with the odometer action on a subset of an infinite product space. For this action it is clear that appropriate infinite product measures have the desired properties. Roughly the same end can be obtained by choosing a sequence (n_k) of positive integers such that the fractional part $\{n_k\alpha\}$ satisfies $0 < \{n_k\alpha\} < 2^{-k}$ and letting ν be the infinite convolution measure

$$\nu = \mathop{\bigstar}_{k=1}^{\infty} \frac{1}{2}(\delta(-\{n_k\alpha\}) + \delta(\{n_k\alpha\}))$$

where $\delta(x)$ denotes the unit point mass at x. The measure ν is not itself quasi-invariant but it is ergodic. This is easily seen from a result of Gavin Brown and the author [3, Proposition 1]. Replacing ν by a convex combination of its translates by all numbers of the subgroup $D_\alpha = \{j\alpha : j \in \mathbf{Z}\}$ of Π, we achieve a measure of the desired type.

A more general method for obtaining quasi-invariant ergodic measures which works for suitable actions on compact spaces X has been given by Katznelson and Weiss in [4]. Specifically, they show that a homeomorphism of a compact metric space has a quasi-invariant ergodic measure if and only if it has a recurrent point, and in these circumstances it has uncountably inequivalent quasi-invariant ergodic measures. Their technique is akin to those of [3] and [5] in that it employs a Cantor-type construction.

In some recent work of Larry Baggett, Alan Carey, Arlan Ramsay and the author on non-monomial representations of nilpotent groups [1], there arose the problem of finding measures quasi-invariant and ergodic for the action of T_α, and with Fourier-Stieltjes coefficients vanishing at infinity. Ultimately, we were able to complete the work without finding a solution to this problem. Nonetheless, it remains of some interest to construct such measures, and we do this here.

For very special irrationals it is possible to follow a method analogous to that of Keane. Let α be an irrational number between 0 and 1 satisfying a quadratic equation of the form $\alpha^2 + b\alpha + c = 0$ where $b, c \in \mathbf{Z}$ and $c \neq \pm 1$. Then α^{-1} is not a Pisot number so that by a result of Salem [7, Chapter IV, Theorem II]

$$\nu = \mathop{\bigstar}_{n=1}^{\infty} \frac{1}{2}(\delta(-\alpha^n) + \delta(\alpha^n))$$

has Fourier-Stieltjes coefficients vanishing at infinity. Moreover $\alpha^n \equiv k_n\alpha$ (mod 1)) for some $k_n \in \mathbf{Z}$, so that invoking the result of [3] again we can see that ν is ergodic under T_α.

This method fails, however, when $c = \pm 1$, since α^{-1} is then a Pisot number, and so the Fourier-Stieltjes coefficients do not vanish at infinity in this case. We have no information when α is not a quadratic irrational as to the ergodicity of ν under T_α.

Singular measures with Fourier-Stieltjes coefficients vanishing at infinity are most easily obtained by using a Riesz product construction. Let (n_k) be a sequence of positive integers which is lacunary in the sense that $n_{k+1}n_k^{-1} \geq 3$ for all k, and let (a_k) be a sequence of real numbers with $0 \leq a_k \leq 1$. Then if

$$P_N(t) = \prod_{k=1}^{N}(1 + a_k \cos 2\pi n_k t)$$

we have $\int_0^1 P_n(t)\,dt = 1$ and $P_n(t) \geq 0$ for all t. The measure $\mu^{(N)}$ given by $d\mu^{(N)}(t) = P_N(t)\,dt$ is then a probability measure. Moreover, the sequence $(\mu^{(N)})$ converges in the weak*topology to a probability measure μ. This is called a *Riesz product measure* (see [8, Chapter V, §7]). We write

$$d\mu(t) = \prod_{k=1}^{\infty}(1 + a_k \cos 2\pi n_k t) \cdot dt,$$

though, in general, μ will be singular. It is a theorem of Zygmund that μ is absolutely continuous if and only if $\sum_{k=1}^{\infty} a_k^2 < \infty$ (loc cit.). Its Fourier Stieltjes coefficients can be written explicitly

$$\hat{\mu}(\varepsilon_1 n_1 + \varepsilon_2 n_2 + \cdots + \varepsilon_k n_k) = a_1^{|\varepsilon_1|} \cdot a_2^{|\varepsilon_2|} \cdots a_k^{|\varepsilon_k|}$$

provided $\varepsilon_i \in \{-1, 0, 1\}$ $(i = 1, 2, \ldots, k)$ and

$$\hat{\mu}(r) = 0 \quad \text{otherwise.}$$

It follows that $\hat{\mu}(r) \to 0$ as $|r| \to \infty$ if and only if $a_k \to 0$ as $k \to \infty$.

The ergodic properties of certain Riesz products have been investigated by Gavin Brown in [2]. He showed that if n_k divides n_{k+1} for all k then μ is quasi-invariant and ergodic for the action of the subgroup D of $\mathbf{T}$ generated by the set $\{n_k^{-1} \colon k = 1, 2, 3, \ldots\}$. His technique appears to rely heavily on the rigid arithmetical properties enjoyed by the n_k's and on the existence of appropriate finite subgroups of D. Of course, there are no such subgroups of $D_\alpha = \{j\alpha \colon j \in \mathbf{Z}\}$. However, by an appropriate modification of Brown's argument we are able to produce Riesz products with the desired properties.

We will show that these measures μ are uniquely defined as quasi-invariant probability measures by the Radon-Nikodym derivative $\frac{d\mu\circ T}{d\mu}$. Krieger [6] has shown that Radon-Nikodym derivatives of measures with this property form a dense G_δ in all such Radon-Nikodym derivatives.

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2. Construction

We fix, in advance, a sequence (a_k) of real numbers between 0 and ρ where $0 < \rho < 1$. The sequence (n_k) will be defined inductively along with a sequence (λ_k) of probability measures with supports in D_α. Then we shall take μ to be the Riesz product

$$d\mu(t) = \prod_{k=1}^{\infty}(1 + a_k \cos 2\pi n_k t) \cdot dt. \tag{1}$$

To facilitate the argument we make the following definitions

$$P_N(t) = \prod_{k=1}^{N}(1 + a_k \cos 2\pi n_k t); \tag{2}$$

$$d\mu_N(t) = \prod_{R=N+1}^{\infty}(1 + a_k \cos 2\pi n_k t) \cdot dt; \tag{3}$$

$$\Omega_N = \{\varepsilon_1 n_1 + \varepsilon_2 n_2 + \cdots + \varepsilon_N n_N : \varepsilon_i \in \{-1, 0, 1\}\}; \tag{4}$$

$$T_N = \sup\{r : r \in \Omega_N\}; \tag{5}$$

$$b_N = \sup\{|j| : j\alpha \in \operatorname{supp} \lambda_N\}. \tag{6}$$

It will be convenient also to write $e_k(t) = \exp 2\pi i k t$. Observe that $\mu = P_N \cdot \mu_N$, and

$$P_N(t) = \sum_{r \in \Omega_N} P_N^{\wedge}(r) e_r(t) = \sum_{r \in \Omega_N} \hat{\mu}(r) e_r(t). \tag{7}$$

Let (ε_N) be a sequence of real numbers with $0 < \varepsilon_N < 1$ and $\varepsilon_N \to 0$. We shall require (n_k) and (λ_k) to have the following properties,

(a) $|\lambda_N^{\wedge}(m)| < \frac{1}{2} 6^{-N} \varepsilon_N \qquad (0 < |m| \le 2T_N)$;

(b) for $|j| \le b_N$, $\delta(j\alpha) * \mu_n$ is equivalent to μ_N and

$$\|\delta(j\alpha) * \mu_N - \mu_N\| < \tfrac{1}{2}\varepsilon_N 6^{-N}.$$

Assume, for the moment, that this has been done. We shall show that (a) and (b) together yield the ergodicity of μ.

LEMMA 1. $\|P_N \cdot (\lambda_N * \mu) - \mu\| < \varepsilon_n$.

PROOF. Let $\nu_N = \lambda_N * \mu - \mu_N$. Then, by (7),

$$\begin{aligned}\nu_N &= \sum_{r \in \Omega_N} P_N^{\wedge}(r) \lambda_N * \mu(e_r \cdot \mu_N) - \mu_N \\ &= \sum_{r \in \Omega_N} P_N^{\wedge}(r) \sum_{|j| \le b_N} \beta_j \overline{e_r(j\alpha)} e_r \cdot \delta(j\alpha) * \mu_N - \mu_N\end{aligned}$$

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where $\lambda_N = \sum_{|j|\le b_N} \beta_j \delta(j\alpha)$. It follows that

$$\begin{aligned}
\|\nu_N\| &\le \left\| \sum_{r\in\Omega_N} P_N^\wedge(r) \sum_{|j|\le b_N} \beta_j \overline{e_r(j\alpha)} e_r \cdot \mu_N - \mu_N l \right\| \\
&\quad + \left\| \sum_{r\in\Omega_N} P_N^\wedge(r) \sum_{|j|\le b_N} \beta_j \overline{e_r(j\alpha)} e_r \cdot (\delta(j\alpha) * \mu_N - \mu_N) \right\| \\
&\le \left\| \sum_{r\in\Omega_N} P_N^\wedge(r) \lambda_N^\wedge(r) e_r \cdot \mu_N - \mu_N \right\| + 3^N \sup_{|j|\le b_N} \|\delta(j\alpha) * \mu_N - \mu_N\| \\
&\le \left\| \sum_{\substack{r\in\Omega_N \\ r\ne 0}} P_N^\wedge(r) \lambda_N^\wedge(r) e_r \cdot \mu_N \right\| + 2^{-N-1}\varepsilon_N
\end{aligned}$$

by property (b), and by (a) this is less than $2^{-N}\varepsilon_N$. Now $P_N \cdot (\lambda_N * \mu) - \mu = P_N \cdot \nu_N$ and $\sup_t |P_N(t)| \le 2^n$ so the result follows.

We are now able to state and prove our main result, subject of course, to having found (n_k) and (λ_k) satisfying (a) and (b).

THEOREM. *The measure μ of* (1) *is ergodic for the action of T_α.*

PROOF. Fix a Borel set E which is invariant for the action of T_α, and let $r \in \mathbf{Z}$. By Lemma 1, we have, for any N,

$$\left| \int_E \overline{e_r(t)}\, d\mu(t) - \int_E \overline{e_r(t)} P_N(t)\, d\lambda_N * \mu(t) \right| < \varepsilon_N. \tag{8}$$

Moreover, by (7),

$$\begin{aligned}
\int_E \overline{e_r(t)} P_N(t)\, d\lambda_N * \mu(t) &= \sum_{s\in\Omega_N} \hat\mu(s) \int_E e_{s-r}(t)\, d\lambda_N * \mu(t) \\
&= \sum_{s\in\Omega_N} \hat\mu(s) \lambda_N^\wedge(r-s) \int_E e_{s-r}(t)\, d\mu(t)
\end{aligned}$$

If N is sufficiently large, then $|r| < T_N$ and so $|r-s| < 2T_N$ for all $s \in \Omega_N$. Thus we have

$$\int_E \overline{e_r(t)} P_N(t)\, d\lambda_N * \mu(t) = \hat\mu(r)\mu(E) + \sum_{\substack{s\in\Omega_N \\ s\ne r}} \hat\mu(s) \lambda_N^\wedge(r-s) \int_E e_{s-r}(t)\, d\mu(t)$$

and the final term is less in absolute value than $2^{-(N+1)}\varepsilon_N$. Combining this with (8) and letting N tend to infinity we see that

$$\int_E \overline{e_r(t)}\, d\mu(t) = \hat\mu(r)\mu(E)$$

for all $r \in \mathbf{Z}$, and hence that $\mu(E)$ is 0 or 1.

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Now we turn to the definition of (n_k) and (λ_k) to satisfy (a) and (b). Assume that $\lambda_1, \lambda_2, \ldots, \lambda_k$ and $n_1, n_2, \ldots, n_k$ have been defined to satisfy (a) and (b). Choose $n_{k+1} \geq 3n_k$ such that

$$0 < \{n_{k+1}\alpha\} < \frac{(1-\rho)}{24\pi} b_k^{-1} 6^{-k} \min\left(\frac{\varepsilon_1}{2^k} \frac{\varepsilon_2}{2^{k-1}}, \ldots, \frac{\varepsilon_k}{2}\right). \tag{9}$$

Define Ω_{k+1} and P_{k+1} according to (2) and (4) and choose λ_{k+1} to be a probability measure with finite support contained in D_α which is a weak * approximation to Lebesgue measure, sufficiently close for (a) to hold.

We shall use the following lemma to achieve (b).

LEMMA 2. *For* $|j| \leq b_N$, *the infinite product*

$$Q_j^N(t) = \prod_{k=N+1}^{\infty} \left(\frac{1 + \cos 2\pi n_k(t - j\alpha)}{1 + \cos 2\pi n_k t}\right) \tag{10}$$

converges uniformly and

$$|Q_j^N(t) - 1| < \tfrac{1}{2} 6^{-N} \varepsilon_N.$$

PROOF. We observe by (9) that if $k > N$,

$$\begin{aligned} \left|\left(\frac{1 + a_k \cos 2n_k(t - j\alpha)}{1 + a_k \cos 2n_k t}\right) - 1\right| &\leq \frac{1}{1-\rho} |\cos 2\pi n_k(t - j\alpha) - \cos 2\pi n_k t| \\ &\leq \frac{2}{1-\rho} |\sin 2\pi n_k j\alpha| \leq \frac{4\pi}{1-\rho} j\{n_k\alpha\} \\ &\leq \frac{1}{6} 6^{-N} \varepsilon_N 2^{-k+N-1} \end{aligned}$$

Now the result follows from the general inequality

$$\left|\prod_{k=1}^{\infty}(1 + c_k) - 1\right| \leq \exp\left(\sum_{k=1}^{\infty} |c_k|\right) - 1, \quad \text{and} \quad \varepsilon_N < 1.$$

PROPOSITION. *The measure* μ *is quasi-invariant for the action of* T_α. *Furthermore, for* $|j| \leq b_N$, $\delta(j\alpha) * \mu_N$ *is equivalent to* μ_N *and*

$$\|\delta(j\alpha) * \mu_N - \mu_N\| < \tfrac{1}{2} 6^{-N} \varepsilon_N.$$

PROOF. We prove first that $\delta(j\alpha) * \mu_N$ is equivalent to μ_N. To see this, note that, for $f \in C(\mathbf{T})$,

$$\int f(t)\, d\delta(j\alpha) * \mu_N(t) = \lim_{R\to\infty} \int_0^1 f(t) \prod_{k=N+1}^{R} (1 + a_k \cos 2\pi n_k(t - j\alpha))\, dt$$

$$= \lim_{R\to\infty} \int_0^1 f(t) \prod_{k=N+1}^{R} \left(\frac{1 + a_k \cos 2\pi n_k(t - j\alpha)}{1 + a_k \cos 2\pi n_k t} \right) \prod_{k=N+1}^{R} (1 + \cos 2\pi n_k t)\, dt$$

$$= \int_0^1 f(t) Q_j^N(t)\, d\mu(t)$$

by Lemma 2. It follows that

$$\frac{d(\delta(j\alpha) * \mu_N)}{d\mu_N} = Q_j^N(t),$$

and hence that

$$\|\delta(j\alpha) * \mu_N - \mu_N\| < \tfrac{1}{2} 6^{-N} \varepsilon_N$$

also by Lemma 2.

Finally taking $j = N = 1$ and noting that $(1 + a_k \cos 2\pi n_n(t - \alpha))(1 + a_k \cos 2\pi n_1 t)^{-1}$ is a continuous function bounded away from 0, we obtain the quasi-invariance of μ.

The proof that μ is quasi-invariant and ergodic is complete. If in addition we assume that $\sum_{k=1}^{\infty} a_k^2 = \infty$ then μ is singular, and if $a_k \to 0$ as $k \to \infty$ then $\hat{\mu}$ vanishes at infinity. Thus a choice of $a_k = (1 + k)^{-1/2}$ will produce a quasi-invariant ergodic singular measure with Fourier-Stieltjes coefficients vanishing at infinity. A more austere choice of (a_k), say $a_k = (\log(k + 1))^{-1}$, will produce a measure every convolution power of which is singular, yet μ is still ergodic and quasi-invariant for the action of T_α and has Fourier-Stieltjes coefficients vanishing at infinity.

3. Uniqueness

Here we show that μ is the unique quasi-invariant probability measure τ for the action of $\mathbf{T}$ such that $(d\delta(j\alpha) * \tau)/d\tau = Q_j$ $(= Q_j^0)$. Observe that both sides of the equation are 1-cocycles, that is, they satisfy $\varphi_{j+k}(t) = \varphi_j(t - k)\varphi_k(t)$ for $j, k \in \mathbf{Z}$, $t \in \mathbf{T}$.

Suppose then that τ is such a measure and fix $r \in \mathbf{Z}$. Consider the integral

$$I = \int e_{-r}(t) P_M(t)\, d\lambda_M * \nu(t) \tag{11}$$

for suitably large M. On the one hand this is equal to

$$\sum_{s \in \Omega_M} \hat{\mu}(s) \lambda_M^{\wedge}(r - s) \hat{\nu}(r - s); \tag{12}$$

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on the other, writing $\lambda_M = \sum_{|j|\le b_M} \beta_j \delta(j\alpha)$, we obtain

$$\begin{aligned} I &= \sum_{|j|\le b_M} \beta_j \int e_{-r}(t) P_M(t)\, d\delta(j\alpha) * \nu(t) \\ &= \sum_{|j|\le b_M} \beta_j \int e_{-r}(t) P_M(t) Q_j(t)\, d\nu(t) \\ &= \sum_{|j|\le b_M} \beta_j \int e_{-r}(t) P_M(t - j\alpha) Q_j^M(t)\, d\nu(t) \end{aligned}$$

by (10). It follows from Lemma 2 that

(13) $$\begin{aligned} &\left| I - \sum_{|j|\le b_M} \beta_j \int e_{-r}(t) P_M(t-j\alpha)\, d\nu(t) \right| \\ &\qquad \le 2^M \sup_t |Q_j^M(t) - 1| \le \tfrac{1}{2} 3^{-M} \varepsilon_M. \end{aligned}$$

Now we observe that

$$\begin{aligned} J &= \sum_{|j|\le b_M} \beta_j \int e_{-r}(t) P_M(t - j\alpha)\, d\nu(t) \\ &= \sum_{|j|\le b_M} \beta_j \sum_{s\in\Omega_M} \hat{\mu}(s) \int e_{-r}(t) e_s(t - j\alpha)\, d\nu(t) \\ &= \sum_{s\in\Omega_M} \hat{\mu}(s)\hat{\nu}(r-s) \sum_{|j|\le b_M} \beta_j e_s(-j\alpha). \end{aligned}$$

Therefore,

(14) $$J = \sum_{s\in\Omega_M} \hat{\mu}(s)\hat{\nu}(r-s)\lambda_M^{\wedge}(s).$$

By (a), $|\lambda_M^{\wedge}(m)| \le \frac{1}{2} 6^{-M} \varepsilon_M$ provided $0 < |m| < 2T_M$, so that, by (14), if $|r| < T_M$,

(15) $$\begin{aligned} |J - \hat{\nu}(r)| &< \sum_{\substack{s\in\Omega_M \\ s\ne 0}} \hat{\mu}(s)\hat{\nu}(r-s)\lambda_M^{\wedge}(s) \\ &\le 2^{-(M+1)}\varepsilon_M, \end{aligned}$$

whereas, by (12)

(16) $$|I - \hat{\mu}(r)| \le 2^{-(M+1)}\varepsilon_M.$$

Now (13), (15) and (16) combine to give $|\hat{\nu}(r) - \hat{\mu}(r)| \le 2^{-M+1}\varepsilon_M$. Therefore $\hat{\mu}(r) = \hat{\nu}(r)$ for all $r \in \mathbf{Z}$ and so $\mu = \nu$.

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