ON THE SEMIGROUP OF DIFFERENTIABLE MAPPINGS (II)

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1. In **[2]**, K. D. Magill, Jr. has proved that every automorphism of the semigroup (with respect to composition) of all real-valued differentiable functions of a real variable is inner. The purpose of this paper is to generalize this fact to arbitrary finite-dimensional real Banach spaces.

Let E be a real Banach space and $\mathscr{D}$ be the semigroup of all (Fréchet-) differentiable mappings of E into itself. We shall prove the following theorem.

THEOREM. *If E is finite-dimensional, every automorphism ϕ of $\mathscr{D}$ is inner, i.e., there exists a bijection $h \in \mathscr{D}$ such that $h^{-1} \in \mathscr{D}$ and*

$\phi(f) = hfh^{-1}$ *for any* $f \in \mathscr{D}$. (1)

For the proof, we shall only need the following two properties of the finite-dimensional spaces:

E is separable. (2)

Sequential weak convergence implies strong convergence. (3)

In fact, in the following, we shall prove that, *if E is separable, then for any automorphism ϕ there exists a bijection h such that h and h^{-1} are weakly Fréchet-differentiable and the relation* (1) *holds.*

NOTATIONS. The notations and terminologies used in this paper are almost the same as those in **[4]** and **[5]**. In particular, we shall frequently use the following notations.

The set of all real numbers is denoted by $\mathscr{R}$. By $\{\varepsilon_n\} \in (c_0)$ we mean that $\{\varepsilon_n\} \subset \mathscr{R}$ and $\lim_{n\to\infty} \varepsilon_n = 0$.

The conjugate space of E is denoted by $\bar{E}$. For $a \in E$ and $\bar{a} \in \bar{E}$, the mapping $a \otimes \bar{a}$ is defined by

$$(a \otimes \bar{a})(x) = \langle x, \bar{a} \rangle a,$$

where $\langle x, \bar{a} \rangle$ denotes the value of $\bar{a}$ at x.

The constant mapping whose single value is a is denoted by c_a, i.e., $c_a(x) = a$ for every $x \in E$.

2. Proof of the theorem. Let E be a separable real Banach space and $\mathscr{D}$ be the semigroup (with respect to composition) of all Fréchet-differentiable mappings of E into itself. Let ϕ be an automorphism of this semigroup $\mathscr{D}$. The method which Magill **[2]** has used for one-dimensional spaces can be applied here to show that *there exists a bijection h of E such that the relation* (1) *holds.* For the details of the proof of this fact, we refer to **[4**, Theorem 1, p. 456] and **[5**, p. 505], where it is also shown that *we can assume that*

$$h(0) = 0.$$

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This bijection h is determined uniquely by the automorphism ϕ. If we start with ϕ^{-1} instead of ϕ, then we obtain h^{-1}. Therefore, we may make use of the fact that any statement proved for h is also true for h^{-1}.

In [5, p. 506], we have shown that, for any $a \in E$ and $\bar{a} \in \bar{E}$, the function, of $\xi \in \mathscr{R}$,

$$\langle h(\xi a), \bar{a} \rangle$$

is continuous. Therefore, if the space is one-dimensional, h is a homeomorphism of $\mathscr{R}$ and hence, being monotone and continuous, it is differentiable except for a countable number of points. Magill [2] has used this fact and, using the semigroup property, has proved that h is differentiable everywhere. When the space is not one-dimensional, we cannot use this method. In fact, if a Banach space E has the dimension not less than two, there exists a homeomorphism which is not differentiable at any point even in the sense of Gâteaux. We owe the following example to Dr S. Swierczkowski and Prof. I. Mycielski: We take non-zero $a \in E$ and $\bar{a} \in \bar{E}$ such that $\langle a, \bar{a} \rangle = 0$. This is possible since the dimension of E is not less than two. Let $\alpha : \mathscr{R} \to \mathscr{R}$ be any continuous function which is nowhere differentiable. Then the mapping

$$f(x) = x + \alpha(\langle x, \bar{a} \rangle)a$$

is a homeomorphism of E onto E and is not Gâteaux-differentiable at any point.

We now start the proof of the fact that, *if E is separable, h is weakly Fréchet-differentiable at every point*. The proof will be divided into seven steps.

STEP 1. *Let $\lambda(\xi)$ be a real-valued function defined on $\mathscr{R}$. If*

(i) *$\lambda(\xi)$ is continuous,*

(ii) *there exists $\{\varepsilon_n\} \in (c_0)$ such that $\lim_{n\to\infty} \varepsilon_n^{-1}[\lambda(\xi \pm \varepsilon_n \eta) - \lambda(\xi)] = 0$ for any ξ and η,*

(iii) *$\lambda(0) = 0$,*

then $\lambda \equiv 0$.

Proof. Take any η and consider the function

$$\mu(\xi) = \lambda(\xi\eta) - \xi\lambda(\eta).$$

This function is continuous and has $\mu(1) = \mu(0) = 0$; so there exists $\xi_0 \in (0, 1)$ at which $\mu(\xi)$ takes its relative maximum (or minimum) value, i.e.,

$$\mu(\xi) \leqq \mu(\xi_0) \quad \text{if} \quad \xi \quad \text{is near to} \quad \xi_0.$$

Hence, for large n,

$$\mu(\xi_0 \pm \varepsilon_n) \leqq \mu(\xi_0),$$

which implies that

$$\lambda(\xi_0 \eta \pm \varepsilon_n \eta) - (\xi_0 \pm \varepsilon_n)\lambda(\eta) \leqq \lambda(\xi_0 \eta) - \xi_0 \lambda(\eta),$$

so that

$$\lambda(\xi_0 \eta + \varepsilon_n \eta) - \lambda(\xi_0 \eta) \leqq \varepsilon_n \lambda(\eta) \leqq -[\lambda(\xi_0 \eta - \varepsilon_n \eta) - \lambda(\xi_0 \eta)],$$

and, therefore, it follows from (ii) that $\lambda(\eta) = 0$. Since η is arbitrary, λ is identically zero.

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STEP 2. *For any nonzero $a \in E$ and any $\{\varepsilon_n\} \in (c_0)$, the sequence $\{\varepsilon_n^{-1} h(\varepsilon_n a)\}$ does not converge weakly to zero.*

Proof. Let us assume that there are nonzero $a \in E$ and $\{\varepsilon_n\} \in (c_0)$ such that

$$\lim_{n\to\infty} \langle \varepsilon_n^{-1} h(\varepsilon_n a), \bar{x} \rangle = 0 \quad \text{for any} \quad \bar{x} \in \bar{E}. \tag{4}$$

For any $\xi \in \mathscr{R}$, we always have the following equation:

$$\varepsilon_n^{-1}[h(\xi a \pm \varepsilon_n a) - h(\xi a)] = \phi(\xi c_a \pm 1)'(0)(\varepsilon_n^{-1} h(\varepsilon_n a)) + \varepsilon_n^{-1} r(\phi(\xi c_a \pm 1), 0, h(\varepsilon_n a)).$$

Since $\phi(\xi c_a \pm 1)'(0)$ is a continuous linear mapping, it follows from (4) that

$$\lim_{n\to\infty} \langle \phi(\xi c_a \pm 1)'(0)(\varepsilon_n^{-1} h(\varepsilon_n a)), \bar{x} \rangle = 0 \quad \text{for every} \quad \bar{x} \in \bar{E}.$$

On the other hand, we have

$$\lim_{n\to\infty} \varepsilon_n^{-1} r(\phi(\xi c_a \pm 1), 0, h(\varepsilon_n a)) = 0.$$

In fact, (4) implies that the sequence $\{\varepsilon_n^{-1} h(\varepsilon_n a)\}$ is bounded and hence

$$\lim_{n\to\infty} \| h(\varepsilon_n a) \| = 0.$$

Therefore, since $\phi(\xi c_a \pm 1) \in \mathscr{D}$,

$$\| \varepsilon_n^{-1} r(\phi(\xi c_a \pm 1), 0, h(\varepsilon_n a)) \| \leq (\| \varepsilon_n^{-1} h(\varepsilon_n a) \|)(\| h(\varepsilon_n a) \|^{-1} \| r(\phi(\xi c_a \pm 1), 0, h(\varepsilon_n a)) \|) \to 0$$

when $n \to \infty$. Thus

$$\lim_{n\to\infty} \langle \varepsilon_n^{-1}[h(\xi a \pm \varepsilon_n a) - h(\xi a)], \bar{x} \rangle = 0 \quad \text{for every} \quad \bar{x} \in \bar{E}. \tag{5}$$

Now consider the functions of $\xi \in \mathscr{R}$:

$$\lambda_{\bar{x}}(\xi) = \langle h(\xi a), \bar{x} \rangle.$$

By [5, p. 506] they are continuous, and the above formula (5) shows that each $\lambda_{\bar{x}}$ satisfies the conditions of the previous Step 1. Therefore $\lambda_{\bar{x}} \equiv 0$ for every $\bar{x} \in \bar{E}$, which means that

$$h(\xi a) = 0 \quad \text{for every} \quad \xi \in \mathscr{R}.$$

This is a contradiction.

STEP 3. *For any $a \in E$ and any $\{\varepsilon_n\} \in (c_0)$, the sequence $\{\varepsilon_n^{-1} h^{-1}(\varepsilon_n a)\}$ is bounded.*

Proof. Let us assume that there are nonzero $a \in E$ and $\{\varepsilon_n\} \in (c_0)$ such that the sequence $\{\varepsilon_n^{-1} h^{-1}(\varepsilon_n a)\}$ is not bounded. Taking a subsequence of $\{\varepsilon_n\}$ if necessary, we can assume that

$$\lim_{n\to\infty} \| \varepsilon_n^{-1} h^{-1}(\varepsilon_n a) \| = +\infty.$$

Then there exists $\bar{a} \in \bar{E}$ such that, taking a subsequence if necessary,

$$\lim_{n\to\infty} \langle \varepsilon_n^{-1} h^{-1}(\varepsilon_n a), \bar{a} \rangle = \infty. \tag{6}$$

For these $a \in E$ and $\bar{a} \in \bar{E}$, since $a \otimes \bar{a} \in \mathscr{D}$, $\phi(a \otimes \bar{a})$ is in $\mathscr{D}$ and

$$\begin{aligned}\phi(a\otimes\bar{a})'(0)(a) &= \lim_{n\to\infty} \varepsilon_n^{-1}\phi(a\otimes\bar{a})(\varepsilon_n a)\\ &= \lim_{n\to\infty} (\varepsilon_n^{-1}\langle h^{-1}(\varepsilon_n a), \bar{a}\rangle)(\langle h^{-1}(\varepsilon_n a), \bar{a}\rangle)^{-1}h[\langle h^{-1}(\varepsilon_n a), \bar{a}\rangle a],\end{aligned}$$

and hence, from (6), it follows that

$$\lim_{n\to\infty} \delta_n^{-1}h(\delta_n a) = 0 \quad \text{for} \quad \delta_n = \langle h^{-1}(\varepsilon_n a), \bar{a}\rangle,$$

where, since $\langle h^{-1}(\xi a), \bar{a}\rangle$ is continuous with respect to ξ, $\{\delta_n\} \in (c_0)$. This contradicts the fact proved in the previous Step 2.

STEP 4. *For any nonzero $a \in E$, there exists $\bar{a} \in \bar{E}$ such that $\phi(a\otimes\bar{a})'(0)(a) \neq 0$.*

Proof. Let us suppose that

$$\phi(a\otimes\bar{x})'(0)(a) = 0 \quad \text{for every} \quad \bar{x} \in \bar{E}.$$

Then, fqr any $\{\varepsilon_n\} \in (c_0)$, we have

$$\begin{aligned}0 &= \lim_{n\to\infty} \varepsilon_n^{-1}\phi(a\otimes\bar{x})(\varepsilon_n a)\\ &= \lim_{n\to\infty} (\varepsilon_n^{-1}\langle h^{-1}(\varepsilon_n a), \bar{x}\rangle)(\langle h^{-1}(\varepsilon_n a), \bar{x}\rangle)^{-1}h[\langle h^{-1}(\varepsilon_n a), \bar{x}\rangle a]\end{aligned}$$

for every $\bar{x} \in \bar{E}$. Now let $\bar{x}$ be fixed temporarily; we shall show that there are positive numbers α and β such that

$$\alpha \leqq \| \delta_n^{-1}h(\delta_n a) \| \leqq \beta \quad \text{for} \quad n = 1, 2, \ldots,$$

where $\delta_n = \langle h^{-1}(\varepsilon_n a), \bar{x}\rangle$. Since $\{\delta_n\} \in (c_0)$, the existence of such β follows from Step 3. If such α does not exist, then there is a subsequence $\{\delta_{n_k}\}$ of $\{\delta_n\}$ such that

$$\lim_{k\to\infty} \| \delta_{n_k}^{-1}h(\delta_{n_k} a) \| = 0,$$

which is impossible because of the fact proved in Step 2. Thus such α and β exist.

Then, since

$$\beta^{-1} \| \varepsilon_n^{-1}\phi(a\otimes\bar{a})(\varepsilon_n a) \| \leqq | \varepsilon_n^{-1}\langle h^{-1}(\varepsilon_n a), \bar{x}\rangle | \leqq \alpha^{-1} \| \varepsilon_n^{-1}\phi(a\otimes\bar{a})(\varepsilon_n a) \|,$$

we have

$$\lim_{n\to\infty} \varepsilon_n^{-1}\langle h^{-1}(\varepsilon_n a), \bar{x}\rangle = 0,$$

and this is true for any $\bar{x} \in \bar{E}$. However, by Step 2, this is impossible.

We note that, if $\bar{a} \in \bar{E}$ satisfies $\phi(a\otimes\bar{a})'(0)(a) \neq 0$, then $-\bar{a}$ also satisfies this condition. This can be proved as follows. Since $-(a\otimes\bar{a}) = a\otimes(-\bar{a})$, we have

$$\phi(a\otimes(-\bar{a}))'(0) = \phi((-1)(a\otimes\bar{a}))'(0) = \phi(-1)'(0)\phi(a\otimes\bar{a})'(0).$$

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Moreover, $\phi(-1)'(0)$ is a bijection because

$$(\phi(-1)'(0))^2 = \phi(1)'(0) = 1.$$

Therefore $\phi(a\otimes\bar{a})'(0)(a) \neq 0$ is equivalent to $\phi(a\otimes(-\bar{a}))'(0)(a) \neq 0$.

STEP 5. *For any* $a \in E$ *and* $\{\varepsilon_n\} \in (c_0)$, *there is a subsequence* $\{\varepsilon_{n_k}\}$ *such that the sequence* $\{\varepsilon_{n_k}^{-1} h(\varepsilon_{n_k} a)\}$ *is convergent.*

Proof. We can assume that a is nonzero, and, by Step 4, we can take $\bar{a} \in \bar{E}$ such that $\phi(a\otimes\bar{a})'(0)(a) \neq 0$. Again we depend on the following equation:

$$\begin{aligned} 0 &\neq \phi(a\otimes\bar{a})'(0)(a) \\ &= \lim_{\delta\to 0} (\delta^{-1}\langle h^{-1}(\delta a), \bar{a}\rangle)(\langle h^{-1}(\delta a), \bar{a}\rangle)^{-1} h(\langle h^{-1}(\delta a), \bar{a}\rangle a). \end{aligned} \tag{7}$$

It is clear from this equation that the function $\langle h^{-1}(\delta a), \bar{a}\rangle$ of $\delta \in \mathscr{R}$ takes nonzero values in any small neighbourhood of zero, because, if this is not the case, $\phi(a\otimes\bar{a})'(0)(a)$ has to be zero. Therefore, since this function is continuous, there exists $\{\delta_n\} \in (c_0)$ such that

$$\langle h^{-1}(\delta_n a), \bar{a}\rangle = \varepsilon_n \quad \text{or} \quad -\varepsilon_n$$

is true for large n. Hence, taking a subsequence of $\{\varepsilon_n\}$ and replacing $\bar{a}$ by $-\bar{a}$ if necessary, we can assume that

$$\langle h^{-1}(\delta_n a), \bar{a}\rangle = \varepsilon_n \quad \text{for large} \quad n.$$

Then, from (7) it follows that

$$0 \neq \phi(a\otimes\bar{a})'(0)(a) = \lim_{n\to\infty} (\delta_n^{-1}\langle h^{-1}(\delta_n a), \bar{a}\rangle)(\varepsilon_n^{-1} h(\varepsilon_n a)).$$

On the other hand, the fact proved in Step 3 implies that the sequence $\{\delta_n^{-1}\langle h^{-1}(\delta_n a), \bar{a}\rangle\}$ is bounded, and hence that there is a subsequence $\{\delta_{n_k}\}$ of $\{\delta_n\}$ such that the limit

$$\lim_{k\to\infty} \delta_{n_k}^{-1}\langle h^{-1}(\delta_{n_k} a), \bar{a}\rangle = \alpha$$

exists. Since, again by Step 3, the sequence $\{\varepsilon_{n_k}^{-1} h(\varepsilon_{n_k} a)\}$ is bounded, the fact that $\phi(a\otimes\bar{a})'(0)(a) \neq 0$ implies that the limit α can not be zero. Therefore, we arrive at the conclusion that the following limit exists:

$$\lim_{k\to\infty} \varepsilon_{n_k}^{-1} h(\varepsilon_{n_k} a) = \alpha^{-1}\phi(a\otimes\bar{a})'(0)(a).$$

STEP 6. *The limit* $\lim_{\varepsilon\to 0} \varepsilon^{-1} h(\varepsilon a)$ *exists for any* $a \in E$.

Proof. We can assume that a is nonzero. We take an arbitrary $\bar{a} \in \bar{E}$ such that $\|\bar{a}\| = 1$, and first we shall show that the function of $\xi \in \mathscr{R}$:

$$\lambda(\xi) = \langle h(\xi a), \bar{a}\rangle$$

is differentiable almost everywhere. In fact, as in the proof of Step 2, for any $\{\varepsilon_n\}\in(c_0)$, we have

$$\begin{aligned}
&|\varepsilon_n^{-1}[\lambda(\xi+\varepsilon_n)-\lambda(\xi)]|\\
&\leqq \|\varepsilon_n^{-1}[h(\xi a+\varepsilon_n a)-h(\xi a)]\|\\
&= \|\phi(\xi c_a+1)'(0)(\varepsilon_n^{-1}h(\varepsilon_n a))+\varepsilon_n^{-1}r(\phi(\xi c_a+1), 0, h(\varepsilon_n a))\|\\
&\leqq \|\phi(\xi c_a+1)'(0)\|\,\|\varepsilon_n^{-1}h(\varepsilon_n a)\|+\|\varepsilon_n^{-1}h(\varepsilon_n a)\|\cdot\|h(\varepsilon_n a)\|^{-1}\|r(\phi(\xi c_a+1), 0, h(\varepsilon_n a))\|,
\end{aligned}$$

which means that none of the Dini derivatives of λ can take infinite value. Therefore, by [**3**, p. 271], the function λ is differentiable almost everywhere.

We now turn to the proof of the existence of the limit $\lim_{\varepsilon\to 0}\varepsilon^{-1}h(\varepsilon a)$. In view of the fact proved in the Step 5, what we have to show is the following: If there are $\{\delta_n\}\in(c_0)$ and $\{\varepsilon_n\}\in(c_0)$ such that

$$\lim_{n\to\infty}\varepsilon_n^{-1}h(\varepsilon_n a)=a_1 \quad\text{and}\quad \lim_{n\to\infty}\delta_n^{-1}h(\delta_n a)=a_2,$$

then $a_1=a_2$.

Since the space is separable, by [**1**, p. 124] we can take countable $\bar{a}_i\in\bar{E}$ such that the condition that $\langle x, \bar{a}_i\rangle=0$ for every i implies that $x=0$. We consider the following functions of $\xi\in\mathscr{R}$:

$$\lambda_i(\xi)=\langle h(\xi a), \bar{a}_i\rangle.$$

Since each λ_i is differentiable almost everywhere, there exists $\alpha\in\mathscr{R}$ where all λ_i are differentiable, i.e., the limit

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}[\lambda_i(\alpha+\varepsilon)-\lambda_i(\alpha)]$$

exists for every i. On the other hand, we have

$$\lim_{n\to\infty}\varepsilon_n^{-1}[h(\alpha a+\varepsilon_n a)-h(\alpha a)]=\phi(\alpha c_a+1)'(0)(a_1)$$

and

$$\lim_{n\to\infty}\delta_n^{-1}[h(\alpha a+\delta_n a)-h(\alpha a)]=\phi(\alpha c_a+1)'(0)(a_2).$$

Therefore

$$\langle\phi(\alpha c_a+1)'(0)(a_1), \bar{a}_i\rangle=\langle\phi(\alpha c_a+1)'(0)(a_2), \bar{a}_i\rangle$$

for every i, which implies that

$$\phi(\alpha c_a+1)'(0)(a_1)=\phi(\alpha c_a+1)'(0)(a_2).$$

Here $\phi(\alpha c_a+1)'(0)$ is injective, because, since

$$(1-\alpha c_a)(\alpha c_a+1)=1,$$

we have

$$\phi(1-\alpha c_a)'(h(\alpha a))\phi(\alpha c_a+1)'(0)=1.$$

We therefore have $a_1=a_2$.

STEP 7. *h is weakly Fréchet-differentiable.*

Proof. This statement means that for any $x \in E$ there exists a continuous linear mapping of E into itself, which we denote by $h'(x)$, such that

$$\operatorname*{weak-lim}_{\|y\| \to 0} \|y\|^{-1}[h(x+y)-h(x)-h'(x)(y)] = 0.$$

This can be proved in exactly the same manner as in the proofs of Steps 6, 7, 8, 9 and 10 of our previous paper [5].

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