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THE STRICTLY EFFICIENT SUBGRADIENT OF SET-VALUED OPTIMISATION

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The subgradient, under strict efficiency, of a set-valued mapping is developed, and the existence of the subgradient is proved. Optimality conditions in terms of Lagrange multipliers for a strictly efficient point are established in the general case and in the case with ic-cone-convexlike data.

1. INTRODUCTION

In recent years, set-valued optimisation problems have received particular attentions from mathematics. For instance, Gong [4] has studied the connectedness of efficient solution sets, Tanino [5] has studied sensitivity analysis, Cheng and Fu [1] have studied density, Corley [3] established optimality conditions in terms of Lagrange, Kuhn,and Tucker with convex data. Lin [6], Taa [7] have generalised the Moreau–Rockafeller type theorem to set-valued maps and established some optimality conditions. In this paper, we first establish the definition of the strict subdifferential of a set-valued mapping, we prove the existence of strictly efficient subgradient and establish a characterisation of this subdifferential by scalarisation. Finally, the optimality conditions of set-valued optimisation are presented with a strictly efficient subgradient.

2. PRELIMINARIES AND DEFINITIONS

Throughout this paper, let X, $Y$and Z be real topological vector spaces, each with zero element θ; X^*, Y^* and Z^* be the dual spaces of X, Y and Z, respectively and let $D \subset Y$ and $E \subset Z$ are pointed convex cones, $F : X \to 2^Y$ and $G : X \to 2^Z$ are set-valued functions. The domain, the graph and epigraph of F are denoted by $\operatorname{dom} F, \operatorname{gr} F, \operatorname{epi}(F)$, respectively, in other words,

$$\operatorname{dom} F := \{x \in X : F(x) \neq \emptyset\},$$
$$\operatorname{gr} F := \{(x, y) \in X \times Y : y \in F(x)\},$$
$$\operatorname{epi}(F) := \{(x, y) \in X \times Y : y \in F(x) + D\}.$$

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The polar cone D^* of D is

$$D^* = \{f \in Y^* : f(d) \geqslant 0, \quad \forall d \in D\}.$$

The set of strictly positive function in D^* is denoted by $D^\sharp$, that is

$$D^\sharp = \{f \in D^* : f(y) > 0, \quad \forall y \in D\backslash\{0\}\}.$$

For a set $A \subset Y$, we write

$$\operatorname{cone} A = \{\lambda a : \lambda \geqslant 0, a \in A\}.$$

The closure and interior of set D are denoted by $\operatorname{cl} D$ and $\operatorname{int} D$, respectively. A convex subset B of a cone D is a base of D if $0 \notin \operatorname{cl} B$ and $D = \operatorname{cone} B$. It is easy to show that $\operatorname{int} D^* \neq \emptyset$ if and only if D has a bounded base. Write

$$B^{st} = \{\varphi \in Y^* : \exists t > 0 \text{ such that } \varphi(b) \geqslant t, \quad \forall b \in B\}.$$

Let B be a base of D, then $0 \notin \operatorname{cl} B$. By the separation theorem of convex sets, there is $0 \neq \varphi \in Y^*$, such that

$$t = \inf\{\varphi(b) : b \in B\} > 0.$$

Let

$$V_B = \left\{y \in Y : |\varphi(y)| < \frac{t}{2}\right\}.$$

Then V_B is an open convex circled neighbourhood of zero in Y. The notation V_B will be used through this paper. If V is a nonempty subset of X, then

$$F(V) = \bigcup_{x \in V} F(x).$$

DEFINITION 2.1: ([**1, 2**]) Let M be a nonempty subset of Y, and B be a base of D. $\overline{y} \in M$ is called a strictly efficient point of M with respect to B; $\overline{y} \in FE(M, B)$; if there is a neighbourhood U of 0 such that

$$\operatorname{cl}[\operatorname{cone}(M - \overline{y})] \cap (U - B) = \emptyset. \tag{2.1}$$

REMARK 2.1. ([**2**]) With respect to the definition of strictly efficient points, the equality (2.1) is equivalent to

$$\operatorname{cone}(M - \overline{y}) \cap (U - B) = \emptyset. \tag{2.2}$$

Moreover, if necessary, the neighbourhood U of zero can be chose to be open, convex or balanced.

Let X_0 be a nonempty subset of X. Now we consider the following set-valued map optimisation problem:

$$\text{(VP)} \min_{x \in X_0} \quad F(x)$$
$$\text{such that} \quad G(x) \cap (-E) \neq \emptyset,$$

$F : X_0 \to 2^Y$, $G : X_0 \to 2^Z$ are set-valued maps. The set of feasible solution of (VP) is denoted by C, that is

$$C = \{x \in x_0 : G(x) \cap (-E) \neq \emptyset\}.$$

DEFINITION 2.2: Let $Q \subset X$ be a convex set. The set-valued map F is said to be D-convex on Q if for any $x_1, x_2 \in Q, \lambda \in [0,1]$,

$$\lambda F(x_1) + (1-\lambda)F(x_2) \subset F(\lambda x_1 + (1-\lambda)x_2) + D.$$

DEFINITION 2.3: A set-valued map F from X into Y is said to be D-nearly subconvexlike on $Q \subset X$ if

$$\operatorname{cl}[F(Q) + \operatorname{int} D]$$

is convex. It is proved in [7] that if F is D-convex on Q then F is D-nearly subconvexlike on Q if D has nonempty interior.

DEFINITION 2.4: The set-valued map $F : X \to 2^Y$ is called ic $-$ D-convexlike if $\operatorname{int}\operatorname{cone}(F(X) + D)$ is convex and

$$F(X) + D \subset \operatorname{cl}\operatorname{int}\operatorname{cone}(F(X) + D).$$

It is obvious that if F is D-nearly subconvexlike, then F is ic $-$ D-convexlike on C if D has a nonempty interior [8].

3. SUBDIFFERENTIALS OF SET-VALUED MAPPING

DEFINITION 3.1: Let F be a set-valued map from $C \subset X$ into Y, $\overline{x} \in C$ and $\overline{y} \in F(\overline{x})$. A linear operator $T \in L(X,Y)$ is said to be a weak subgradient for $\overline{y}$ of F at $\overline{x}$ if

$$\overline{y} - T\overline{x} \in W \min \bigcup_{x \in C} (F(x) - T(x)).$$

The set of all weak subgradients for $\overline{y}$ of F at $\overline{x}$ is called the weak subdifferential for $\overline{y}$ of F at $\overline{x}$ is denoted by $\partial_w F(\overline{x}, \overline{y})$.

DEFINITION 3.2: Let F be a set-valued map from $C \subset X$ into Y, $\overline{x} \in C$ and $\overline{y} \in F(\overline{x})$. A linear operator $T \in L(X,Y)$ is said to be a strict subgradient for $\overline{y}$ of F at $\overline{x}$ if

$$\overline{y} - T\overline{x} \in FE\Big(\bigcup_{x \in C} (F(x) - T(x)), B\Big).$$

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The set of all strict subgradients for $\overline{y}$ of F at $\overline{x}$ is called the strict subdifferential for $\overline{y}$ of F at $\overline{x}$ and is denoted by $\partial_{FE}F(\overline{x},\overline{y})$.

DEFINITION 3.3: ([7]) The set-valued map F from $C \subset X$ into Y is said to be connected at $x_0 \in C$, if there exists a continuous function from C into Y such that $f(x) \in F(x)$ for all x in some neighbourhood of x_0.

LEMMA 3.1. ([7]) *Let F_1 and F_2 be two set-valued maps from the set*

$$X_0 := \{x \in X : F_1(x) \neq \emptyset \ \text{ and } \ F_2(x) \neq \emptyset\}$$

into Y, and F_1 and F_2 be D-convex on X_0. If F_1 is connected at some $x_0 \in \operatorname{int} X_0$, then

$$\operatorname{int}(\operatorname{epi}(F_1)) \cap \operatorname{epi}(F_2) \neq \emptyset.$$

THEOREM 3.1. *Let F be a D-convex set-valued map from C into Y. Then $\partial_{FE}F(\overline{x},\overline{y}) \neq \emptyset$, if $\overline{y} \in F(\overline{x}), \overline{y} \in FE(F(\overline{x}), B)$, F is connected at $\overline{x} \in \operatorname{int} C$.*

PROOF: Since $\overline{y} \in FE(F(\overline{x}), B)$, there exists some open convex circled neighbourhood U of zero in Y such that

$$\operatorname{cl}\operatorname{cone}(F(\overline{x}) - \overline{y}) \cap (U - B) = \emptyset. \tag{3.1}$$

We define

$$A = \{(x,y) \in C \times Y : y \in F(x) + \operatorname{cone}(B - U)\}.$$

Since F is D-convex, then it is $(\operatorname{cone}(B - U))$-convex, since $D \subset \operatorname{cone}(B - U)$. It is easy to show A is convex set. Using Lemma 3.1 we know that $\operatorname{int} A \neq \emptyset$, since $\operatorname{epi} F \subset A, \operatorname{int}\operatorname{epi} F \neq \emptyset$. We wish to show that $(\overline{x},\overline{y}) \notin \operatorname{int} A$. Suppose that $(\overline{x},\overline{y}) \in \operatorname{int} A$, then there exists $\widetilde{U} \in N(0_Y)$ such that $(\overline{x}, \overline{y} + \widetilde{U}) \subset A$. Since $\operatorname{cone}(B - U)$ is a cone, then there exists $-d \in \operatorname{cone}(B - U) \setminus \{0\}$ such that $d \in \widetilde{U}$. Then

$$\overline{y} + d \in F(\overline{x}) + \operatorname{cone}(B - U).$$

Then there exist $y_1 \in F(\overline{x}), d_1 \in \operatorname{cone}(B - U)$, such that,

$$\begin{aligned} &\overline{y} + d = y_1 + d_1, \\ &y_1 - \overline{y} = d - d_1 \in -\operatorname{cone}(B - U) \setminus \{0\} \subset \operatorname{cone}(U - B) \setminus \{0\}. \end{aligned}$$

This contradicts (3.1), and shows that $(\overline{x},\overline{y}) \notin \operatorname{int} A$. Hence there exists nonzero $(f,g) \in X^* \times Y^*$, such that

$$f(x) + g(y) \geqslant f(\overline{x}) + g(\overline{y}), \forall x \in C, y \in F(x) + \operatorname{cone}(B - U). \tag{3.2}$$

We now show that $g \neq 0$. Suppose that $g = 0$; then $f(x - \overline{x}) \geqslant 0$ for any $x \in C$. Since $\overline{x} \in \operatorname{int} C$, this leads to a contradiction. Hence $g \neq 0$. On the other hand, in (3.2) taking $x = \overline{x}, y = \overline{y} + d, \forall d \in \operatorname{cone}(B - U)$, we get

$$g(d) \geqslant 0, \quad \forall d \in \operatorname{cone}(B - U).$$

Since $g \neq 0$, there exists $u \in U$, such that $g(u) = t > 0$, then

$$g(b) \geqslant g(u) = t, \quad \forall b \in B.$$

That is

$$g \in B^{st}.$$

Taking $b \in B$, setting $y_0 = b/\big(g(b)\big)$, we get $g(y_0) = 1$. Define a linear operator

$$T : X \to Y, \quad T(x) = -f(x)y_0. \tag{3.3}$$

Set $U = \big\{y \in Y : g(y) < t/2\big\}$, then U is a neighbourhood of zero, and

$$g(u - b) < \frac{t}{2} - t < 0, \quad \forall u \in U, b \in B. \tag{3.4}$$

Now we prove T is a strict subgradient for $\overline{y}$ of F at $\overline{x}$, that is

$$\operatorname{cone}\Big(\bigcup_{x \in C}(F(x) - T(x)) - (\overline{y} - T(\overline{x}))\Big) \cap (U - B) = \emptyset.$$

If not, there exist $r > 0, x_1 \in C, y_1 \in F(x_1)$, such that

$$r\Big(y_1 - T(x_1) - (\overline{y} - T(\overline{x}))\Big) \in U - B. \tag{3.5}$$

Using (3.4) and (3.5), we get

$$rg\Big(y_1 - T(x_1) - (\overline{y} - T(\overline{x}))\Big) < 0. \tag{3.6}$$

On the other hand, using (3.3) and (3.2) we have

$$rg\Big(y_1 - T(x_1) - (\overline{y} - T(\overline{x}))\Big) = r\Big(g(y_1) + f(x_1) - \big(f(\overline{x}) + g(\overline{y})\big)\Big) \geqslant 0.$$

This is a contradiction. Thus, $T \in \partial_{FE}F(\overline{x}, \overline{y})$. ∎

THEOREM 3.2. *Let F be a D-convex set-valued function from X into Y and $\overline{y} \in F(\overline{x})$. Then $T \in \partial_{FE}F(\overline{x}, \overline{y})$ if and only if there exists $f \in B^{st}$ such that*

$$f\big(y - \overline{y} - T(x - \overline{x})\big) \geqslant 0, \quad \forall x \in X, \quad y \in F(x). \tag{3.7}$$

PROOF: Since $f \in B^{st}$, there exists $t > 0$ such that $f(b) \geqslant t$, for any $b \in B$. Set

$$V = \big\{y \in Y : f(y) < t\big\}.$$

Then V is a neighbourhood of zero. Since f is continuous at zero, there exists an open convex circled neighbourhood U of zero such that $U \subset V \cap V_B$, we have

$$U - B \subset \big\{y \in Y : f(y) < 0\big\}. \tag{3.8}$$

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Then $T \in \partial_{FE}F(\overline{x},\overline{y})$. Indeed, if there exists

$$y \in \operatorname{cone}\Big(\bigcup_{x\in X}(F(x)-T(x)) - (\overline{y}-T(\overline{x}))\Big)\cap(U-B),$$

then, there exist $r>0, x_1 \in X, y_1 \in F(x_1)$, such that

$$r\Big(y_1 - T(x_1) - (\overline{y}-T(\overline{x}_1))\Big) \in U-B.$$

By (3.8),

$$f\Big(y_1 - T(x_1) - (\overline{y}-T(\overline{x}_1))\Big) < 0.$$

But by (3.7),

$$f\Big(y_1 - T(x_1) - (\overline{y}-T(\overline{x}_1))\Big) \geqslant 0.$$

This is a contradiction. Thus, $T \in \partial_{FE}F(\overline{x},\overline{y})$.

Now let $T \in \partial_{FE}F(\overline{x},\overline{y})$. By the definition, there exists an open convex circled neighbourhood U of zero with $U \subset V_B$ such that

$$\operatorname{cone}\Big(\bigcup_{x\in X}(F(x)-T(x)) - (\overline{y}-T(\overline{x}))\Big)\cap(U-B)=\emptyset. \tag{3.9}$$

It is clear that

$$\operatorname{cone}\Big(\bigcup_{x\in X}(F(x)-T(x)) + D - (\overline{y}-T(\overline{x}))\Big)\cap(U-B)=\emptyset. \tag{3.10}$$

If not, there exists $\lambda>0, x_1\in X, y_1\in F(x_1), d\in D\setminus\{0\}, u\in U, b\in B$, such that

$$\lambda\Big(y_1 - T(x_1) + d - (\overline{y}-T(\overline{x}_1))\Big) = u-b$$

Since B is a base of D, there exist $\lambda_1>0, b_1\in B$, such that $d=\lambda_1 b_1$. Then

$$\begin{aligned}\lambda\Big(y_1 - T(x_1) - (\overline{y}-T(\overline{x}_1))\Big) &= u-(b+\lambda\lambda_1 b_1)\\ &= (1+\lambda\lambda_1)\Big(\frac{u}{1+\lambda\lambda_1} - \Big(\frac{1}{1+\lambda\lambda_1}b + \frac{\lambda\lambda_1}{1+\lambda\lambda_1}b_1\Big)\Big).\end{aligned}$$

That is

$$\begin{aligned}\frac{\lambda}{1+\lambda\lambda_1}\Big(y_1 - T(x_1) - (\overline{y}-T(\overline{x}_1))\Big) &= \frac{u}{1+\lambda\lambda_1} - \Big(\frac{1}{1+\lambda\lambda_1}b + \frac{\lambda\lambda_1}{1+\lambda\lambda_1}b_1\Big)\\ &\in \operatorname{cone}\Big(\bigcup_{x\in X}(F(x)-T(x)) - (\overline{y}-T(\overline{x}))\Big)\cap(U-B).\end{aligned}$$

This is a contradiction. Thus (3.10) holds. Since F is D-convex and T is a linear operator, then $F-T$ is a D-convex map. It is clear that

$$\operatorname{cone}\Big(\bigcup_{x\in X}(F(x)-T(x)) + D - (\overline{y}-T(\overline{x}))\Big)$$

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is a convex set. Applying the separation theorem of convex sets, we can get an $f \in Y^*\backslash\{0\}$ such that

$$\lambda f\Big(y-T(x)+d-\big(\overline{y}-T(\overline{x})\big)\Big) > f(u)-f(b), \ \forall \lambda \geqslant 0, x \in X, y \in F(x), d \in D, u \in U, b \in B.$$

From this, we have

$$f\Big(y - T(x) - \big(\overline{y} - T(\overline{x})\big)\Big) \geqslant 0, \ \forall x \in X, y \in F(x), \tag{3.11}$$

and

$$f(b) > f(u), \quad \forall u \in U, b \in B.$$

Since $f \neq 0$, U is a neighbourhood of zero, then there exists $u_1 \in U$ such that

$$f(u_1) = t > 0.$$

That is

$$f(b) > t, \quad \forall b \in B.$$

Thus, $f \in B^{st}$. Combining with (3.11), this proof is completed. ∎

Theorem 3.3. *Let F_1 and F_2 be set-valued functions from the set*

$$V = \{v \in X : F_1(v) \neq \emptyset, F_2(v) \neq \emptyset\}$$

into 2^Y, V be convex, and F_1 and F_2 be D-convex on V. If F_1 is connected at some $x_0 \in \operatorname{int} V$, then for $\overline{x} \in V$ and $y_1 \in F_1(\overline{x}), y_2 \in F_2(\overline{x})$, we have

$$\partial_{FE}(F_1 + F_2)(\overline{x}, y_1 + y_2) \subset \partial_{FE} F_1(\overline{x}, y_1) + \partial_{FE} F_2(\overline{x}, y_2)$$

Proof: Let $T \in \partial_{FE}(F_1+F_2)(\overline{x}, y_1+y_2)$ and define $H_1(x) = F_1(x)-y_1-T(x-\overline{x})$ and $H_2(x) = F_2(x)-y_2$. Since $F_1, F_2 : V \to 2^Y$ are D-convex, it follows that H_1 and H_2 are D-convex set- valued functions and $\theta \in H_1(\overline{x}) \cap H_2(\overline{x})$. Because $T \in \partial_{FE}(F_1+F_2)(\overline{x}, y_1+y_2)$, it follows that

$$y_1 + y_2 - T(\overline{x}) \in FE\Big(\bigcup_{x \in V}\big(F_1(x) + F_2(x) - Tx\big), B\Big).$$

This implies that $0 \in FE\Big(\bigcup_{x \in V}\big(H_1(x) + H_2(x)\big), B\Big)$. We define

$$A = \{(x, y) \in V \times Y : y \in H_1(x) + \operatorname{cone}(B - U)\},$$
$$Q = \{(x, -y) \in V \times Y : y \in H_2(x) + \operatorname{cone}(B - U)\}.$$

Since H_1 and H_2 are D-convex, then H_1 and H_2 are $\operatorname{cone}(B - U)$-convex, it follows that A and Q are convex subsets of $V \times Y$. Because F_1 is connected at $x_0 \in \operatorname{int} V$, by Lemma

3.1, it is clear that $\operatorname{int} A \neq \emptyset$. We wish to show that $\operatorname{int} A \cap Q = \emptyset$. Suppose that $(x,y) \in \operatorname{int} A \cap Q$; then there exists

$$x \in V, y_1' \in H_1(x), d_1 \in \operatorname{int}\operatorname{cone}(B-U), y_2' \in H_2(x), d_2 \in \operatorname{cone}(B-U),$$

such that

$$y = y_1' + d_1, \quad -y = y_2' + d_2.$$

Thus $y_1' + y_2' = -(d_1+d_2) \in \operatorname{int}\operatorname{cone}(U-B)$. That is

$$\big(H_1(x) + H_2(x)\big) \cap \operatorname{int}\operatorname{cone}(U-B) \neq \emptyset.$$

It is clear that

$$\operatorname{cone}\big(H_1(x) + H_2(x)\big) \cap (U-B) \neq \emptyset.$$

This contradicts $0 \in FE\Big(\bigcup_{x\in V}(H_1(x)+H_2(x)), B\Big)$. Thus $\operatorname{int} A \cap Q = \emptyset$. Hence there exists nonzero $(f,g) \in X^* \times Y^*$ and $\alpha \in R$ such that

$$f(x) + g(y) \geqslant \alpha \geqslant f(x^1) + g(y^1), \ \forall (x,y) \in A, \quad (x^1,y^1) \in Q. \tag{3.12}$$

Because $(\overline{x}, 0) \in A \cap Q$, it follows that $\alpha = f(\overline{x})$. Further, we may prove that $g \in B^{st}$, this way is similar to the proof of Theorem 3.1. Let $d_1 \in D \setminus \{0\}$ satisfying $g(d_1) = 1$, we define $T_1 : X \to Y$ by $T_1(x) = f(x)d_1$. Since

$$\big(x, y_1' - y_1 - T(x-\overline{x})\big) \in A, (x, y_2 - y_2') \in Q, \quad \forall x \in V, y_1' \in F_1(x), y_2' \in F_2(x).$$

From (3.12) we get

$$f(x) + g\big(y_1' - y_1 - T(x-\overline{x})\big) \geqslant f(\overline{x}) \geqslant f(x) + g(y_2 - y_2').$$

Since $f(x) = g\big(T_1(x)\big)$, we have

$$g\big(y_1' - y_1 - T(x-\overline{x})\big) \geqslant g\big(T_1(\overline{x}-x)\big) \geqslant g(y_2 - y_2').$$

That is

$$g\big(y_1' - y_1 - (T-T_1)(x-\overline{x})\big) \geqslant 0, \quad \forall x \in V, y_1' \in F_1(x),$$

and

$$g\big(y_2' - y_2 - T_1(x-\overline{x})\big) \geqslant 0, \quad \forall x \in V, y_2' \in F_2(x).$$

By Theorem 3.2, we have

$$T - T_1 \in \partial_{FE} F_1(\overline{x}, y_1), \quad T_1 \in \partial_{FE} F_2(\overline{x}, y_2).$$

Thus we complete the proof the theorem. ∎

4. Optimality conditions

In this section, we establish optimality conditions in terms of Lagrange and Fritz John, and under some conditions, we obtain the Lagrange-Kuhn-Tucker multipliers of the problem (VP).

Definition 4.1: $x_0 \in C$ is called a strictly efficient solution of (VP), if

$$F(x_0) \cap FE(F(C), B) \neq \emptyset;$$

(x_0, y_0) is called a strictly efficient element of (VP), if $x_0 \in C$ and $y_0 \in F(x_0) \cap FE(F(C), B)$.

For each $\beta \in [0, 1)$, let us consider a set-valued map $H_\beta : X \to Y \times Z$ whose domain is the set X,

$$H_\beta(x) = (F(x) - y_0) \times (G(x) - \beta z_0), \quad x \in X.$$

Let $K = D \times E$. From now on, we make the following assumption.

Assumption (A). There exists $\beta \in [0, 1)$ such that H_β is ic-K-convexlike.

Observe that in Assumption (A) no topological property is imposed on D and E, so the assumption can be used in studying proper efficiency in (VP) without requiring that $\operatorname{int} D \neq \emptyset$ and $\operatorname{int} E \neq \emptyset$.

Definition 4.2: We say that condition (CQ) holds if

$$\operatorname{cl}\operatorname{cone}(\operatorname{im} G + E) = Z.$$

Observe that, for any $\beta \geqslant 0$,

$$\operatorname{im}(G - \beta z_0) + E \subset \operatorname{im} G + \beta E + E \subset \operatorname{im} G + E.$$

Thus, (CQ) holds if

$$\operatorname{cl}\operatorname{cone}\left[\operatorname{im}(G - \beta z_0) + E\right] = Z, \text{for some } \ \beta \geqslant 0.$$

Remark 4.1. It is easy to see, if the generalised Slater condition $\operatorname{im} G \cap (-\operatorname{int} E) \neq \emptyset$ is satisfied, then condition (CQ) holds.

Theorem 4.1. *If $F : X \to 2^Y$ is a set-valued map, then (x_0, y_0) is a strictly efficient element of (VP) if and only if $0_L \in \partial_{FE} F(x_0, y_0)$.*

Proof: Obvious from the definition of the strict subgradient. □

Lemma 4.1. ([9]) *Suppose D has a base, $x_0 \in C$, let Assumption (A) be satisfied, condition (CQ) hold. Then (x_0, y_0) is a strictly efficient element of problem (VP) if and only if there exist $s^* \in B^{st}, k^* \in E^*$ such that*

$$s^*(y) + k^*(z) \geqslant s^*(y_0), \quad \forall (y, z) \in \operatorname{im}(F \times G). \tag{4.1}$$

$$k^*(z_0^1) = 0, \quad \forall z_0^1 \in G(x_0) \cap (-E). \tag{4.2}$$

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THEOREM 4.2. *Suppose D has a base, $x_0 \in C$. Let Assumption (A) be satisfied, and condition (CQ) hold. Then (x_0, y_0) is a strictly efficient element of problem (VP) if and only if there exist $s^* \in B^{st}, k^* \in E^*$ such that*

$$k^*(z_0^1) = 0, \quad \forall z_0^1 \in G(x_0) \cap (-E).$$

and

$$0 \in \partial_w\big(s^*(F) + k^*(G)\big)\big(x_0, s^*(y_0)\big);$$

that is $\big(x_0, s^*(y_0)\big)$ is a weak efficient point of the following problem with respect to R^+

$$\min_{x \in C} s^*\big(F(x)\big) + k^*\big(G(x)\big),$$

where $R^+ = [0, +\infty)$.

PROOF: Necessity. From Lemma 4.1, we get

$$k^*(z_0^1) = 0, \quad \forall z_0^1 \in G(x_0) \cap (-E).$$

Hence

$$s^*(y_0) = s^*(y_0) + k^*(z_0^1) \in \bigcup_{x \in C} \Big[s^*\big(F(x)\big) + k^*\big(G(x)\big)\Big].$$

It follows from (4.1) that $\big(x_0, s^*(y_0) + k^*(z_0^1)\big)$ is a minimal element of the following problem with respect to R^+

$$\min_{x \in C} s^*\big(F(x)\big) + k^*\big(G(x)\big),$$

which is equivalent to

$$0 \in \partial_w\big(s^*(F) + k^*(G)\big)\big(x_0, s^*(y_0)\big),$$

thus the proof of necessity of the theorem is completed. □

SUFFICIENCY. Since

$$k^*(z_0^1) = 0, \quad \forall z_0^1 \in G(x_0) \cap (-E)$$

and

$$0 \in \partial_w\big(s^*(F) + k^*(G)\big)\big(x_0, s^*(y_0)\big),$$

hence (4.2) holds and $\big(x_0, s^*(y_0) + k^*(z_0^1)\big)$ is a minimal element of the following problem with respect to R^+

$$\min_{x \in C} s^*\big(F(x)\big) + k^*\big(G(x)\big),$$

which implies

$$s^*(y) + k^*(z) \geqslant s^*(y_0) + k^*(z_0^1) = s^*(y_0), \forall (y, z) \in \operatorname{im}(F \times G).$$

From Lemma 4.1 it follows that (x_0, y_0) is a strictly efficient element of problem (VP).

Lemma 4.2. ([9]) *Suppose D has a base, $x_0 \in C$. Let Assumption (A) be satisfied, and condition (CQ) hold. Then (x_0, y_0) is a strictly efficient element of problem (VP) if and only if there exists $\overline{T} \in L_+(Z, Y)$ such that $\overline{T}\big(G(x_0) \cap (-E)\big) = \{0_Y\}$ and (x_0, y_0) is a strictly efficient element of the following unconstrained optimisation problem.*

$$\text{(UVP)} \quad \min_{x \in X} \psi(x) = F(x) + \overline{T}\big(G(x)\big).$$

Theorem 4.3. *Suppose D has a base, Assumption (A) is satisfied and condition (CQ) holds. Then (x_0, y_0) is a strictly efficient element of (VP) if and only if there exists $\overline{T} \in L_+(Z, Y)$ such that $\overline{T}\big(G(x_0) \cap (-E)\big) = \{0_Y\}$ and*

$$0_L \in \partial_{FE}\big(F + \overline{T}(G)\big)(x_0, y_0),$$

that is (x_0, y_0) is a strictly efficient point of the following problem

$$\min_{x \in C}\big(F(x) + \overline{T}\big(G(x)\big)\big).$$

Proof: By Theorem 4.1 and Lemma 4.2, we can easily complete the proof of the theorem. ∎

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