

FROM CHARACTERISTIC FUNCTIONS TO IMPLIED VOLATILITY EXPANSIONS

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Abstract

For any strictly positive martingale $S = e^X$ for which X has a characteristic function, we provide an expansion for the implied volatility. This expansion is explicit in the sense that it involves no integrals, but only polynomials in the log-strike. We illustrate the versatility of our expansion by computing the approximate implied volatility smile in three well-known martingale models: one finite activity exponential Lévy model, Merton (1976), one infinite activity exponential Lévy model (variance gamma), and one stochastic volatility model, Heston (1993). Finally, we illustrate how our expansion can be used to perform a model-free calibration of the empirically observed implied volatility surface.

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1. Introduction

While it is rare to find a martingale model for which the transition density is available in closed form (the Black–Scholes model being a notable exception), there is a veritable zoo of models for which the characteristic function is available explicitly (e.g. exponential Lévy models and affine models [8]). The existence of an analytically tractable characteristic function allows for (vanilla) option prices to be computed quickly using (generalised) Fourier transforms [24], [25]. Every model contains unobservable parameters, which are usually calibrated to market data. This calibration procedure is typically performed using implied volatilities rather than option prices, the former being dimensionless. For a given model, one therefore has to compute (by finite difference, Monte Carlo or numerical integration) option prices first and then the corresponding implied volatilities by some root-finding algorithm. Both steps require sophisticated numerical tools and occasionally somewhat of an artistic touch. These are computationally expensive and render calibration a long and intensive task.

Over the past decade, many authors have focused on obtaining closed-form approximations for both option prices and implied volatilities, partly in order to speed up this calibration process. Perturbation methods have been used by Lorig *et al.* [27] (see also [9], [10], [12], [18]) to obtain such approximations for diffusion-type models. In extreme regions—where numerical schemes become less efficient—asymptotic expansions of densities and of implied volatilities have been obtained in [3], [6], [15], [19], and [33] in the small-maturity case (both for diffusions and jump models) and in [22] for the large-time behaviour of affine stochastic volatility models. Lee [23]

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pioneered the study of the tails of implied volatility, and more recent (model-dependent and model-free) results have appeared in [2], [6], [11], and [16].

The goal of this paper is to derive an approximation for the implied volatility in any model whose characteristic function is available in closed form. This approximation contains no special function and does not require any numerical integration. It can therefore be used efficiently to accelerate the aforementioned calibration issue. The methodology follows and extends the previous works [21], [26] and is related to some extent to the works by Takahashi and Toda [32]. Indeed, by writing the characteristic function as a perturbation around the Black–Scholes characteristic function, our expansion has the form of a Black–Scholes price perturbed by some additional quantity (which we shall make precise later), which can then be turned into an expansion for the corresponding implied volatility.

The rest of the paper proceeds as follows. In Section 2 we provide a brief review of the characteristic function approach to option pricing and introduce some notation needed later in the paper. Section 3 contains the main results, namely a series expansion for the implied volatility. More precisely, in Section 3.1 we show that whenever the characteristic function is available in closed form, the European call price can be written as a regular perturbation around the Black–Scholes price. A similar result then holds for the implied volatility, as detailed in Sections 3.2 and 3.3. In Section 4 we numerically test our results and provide practical details about this implementation.

2. Notation and preliminary results

We consider here a given probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$; all the processes studied will be \mathcal{F} -adapted. In particular $S = e^X$ will denote the stock price process, namely a \mathcal{F} -adapted martingale under the risk-neutral probability measure \mathbb{P} . The dynamics of X may depend on some auxiliary process $Y \in \mathbb{R}^m$ for $m \geq 1$, say some stochastic volatility. The starting point $(X_0, Y_0) = (x, y)$ is assumed to be nonrandom. For simplicity and notational convenience, we shall assume that $m = 1$ and that the risk-free interest rate is 0.

2.1. Pricing via Fourier transforms

Let h be the payoff function of a European call option on $S = e^X$ with strike e^ζ : $h(z) \equiv (e^z - e^\zeta)^+$, and denote \widehat{h} its (generalised) Fourier transform

$$\widehat{h}(\lambda) := \int_{\mathbb{R}} e^{-i\lambda z} h(z) dz = \frac{-e^\zeta - i\zeta\lambda}{i\lambda + \lambda^2} \quad \text{for } \text{Im}(\lambda) < -1.$$

The results obtained below for option prices remain valid for put options with payoff $h(z) \equiv (e^\zeta - e^z)^+$, but we shall chiefly consider European call option prices unless otherwise stated. For any $t \geq 0$, define the moment explosions $p^*(t) := \sup\{p \geq 0: \mathbb{E}_x(e^{pX_t}) < \infty\}$ and $q^*(t) := \sup\{q \geq 0: \mathbb{E}_x(e^{-qX_t}) < \infty\}$. Since S is a martingale, we have $p^*(t) \geq 1$ and $q^*(t) \geq 0$. We shall further make the stronger assumption.

Assumption 1. For any $t \geq 0$, $p^*(t) > 1$, and $q^*(t) > 0$.

This assumption holds for most models in practice, and allows us to write the value of a call option as

$$u(t, x) := \mathbb{E}_x h(X_t) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{h}(\lambda) \mathbb{E}_x(e^{i\lambda X_t}) d\lambda_r \quad \text{for all } t \geq 0 \tag{1}$$

with $\text{Im}(\lambda) \in (-p^*(t), -1)$, where we write $\lambda = \lambda_r + i\lambda_i$ ($\lambda_r, \lambda_i \in \mathbb{R}$) for a complex number. Of course the function u also depends on y , the starting point of Y , but we shall omit it in

the notation for clarity. In this paper we consider models for which the characteristic function $\mathbb{C} \ni \lambda \mapsto \mathbb{E}_x(e^{i\lambda X_t})$ admits the representation

$$\log \mathbb{E}_x(e^{i\lambda X_t}) = i\lambda x + \phi(t, \lambda) \tag{2}$$

for some analytic function $\phi: \mathbb{R}_+ \times \mathbb{C} \rightarrow \mathbb{C}$, satisfying $\phi(t, -i) = 0$ for all $t \geq 0$ (martingale property). From (1), this implies that the price of a call option may be written as (see also [24] or [25])

$$u(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{h}(\lambda) e^{i\lambda x + \phi(t, \lambda)} d\lambda_r.$$

Several well-known models fit within this class

(i) Lévy models:

$$\phi(t, \lambda) = t \left(i\mu\lambda - \frac{1}{2}a^2\lambda^2 + \int_{\mathbb{R}} v(dz)(e^{i\lambda z} - 1 - i\lambda z) \right).$$

(ii) Additive models:

$$\phi(t, \lambda) = i\mu(t)\lambda - \frac{1}{2}a^2(t)\lambda^2 + \int_{\mathbb{R}} v(t, dz)(e^{i\lambda z} - 1 - i\lambda z).$$

(iii) Affine models:

$$\phi(t, \lambda) = C(t, \lambda) + yD(t, \lambda).$$

In these models, (μ, a^2, v) is a Lévy triplet, $(\mu(t), a^2(t), v(t))$ are the spot characteristics of an additive process, the function C is fully characterised by $(d/dt)C = D$, and the function D satisfies a Riccati equation. For precise details on Lévy and affine processes, we refer the interested reader to the monograph by Sato [31] and the groundbreaking paper by Duffie *et al.* [7].

2.2. Black–Scholes and implied volatility

Option prices are commonly quoted in units of implied volatility (rather than in units of currency), firstly because the latter is dimensionless and secondly, because the shape and behaviour of the implied volatility provide more information than option prices. However, the implied volatility is scarcely available in closed form and has to be computed numerically via inversion of the Black–Scholes formula. We derive here a closed-form expansion for the implied volatility for models whose characteristic function is of the form (2). We begin our analysis by defining the Black–Scholes price and the implied volatility.

Definition 1. The Black–Scholes price $u^{BS}: \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is given by

$$u^{BS}(t, x, \sigma_0) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\phi_0(\lambda; \sigma_0)} \widehat{h}(\lambda) e^{i\lambda x} d\lambda_r,$$

where $\phi_0(\lambda; \sigma_0) := -(\sigma_0^2/2)(\lambda^2 + i\lambda)$.

Remark 1. Note that $\phi_0(\cdot; \sigma_0)$ is the Lévy exponent of a Brownian motion with volatility σ_0 and drift $-\frac{1}{2}\sigma_0^2$, so this definition is the Fourier representation of the usual Black–Scholes price, more typically written as

$$u^{BS}(t, x, \sigma_0) = e^x \mathcal{N}(d_+(x)) - e^k \mathcal{N}(d_-(x)) \quad \text{with } d_{\pm}(x) := \frac{x - \zeta \pm (1/2)\sigma_0^2 t}{\sigma_0 \sqrt{t}}, \tag{3}$$

where \mathcal{N} is the cumulative distribution function of a standard normal random variable.

Definition 2. For any maturity t , starting point x and log-strike ζ , the implied volatility is defined as the unique nonnegative real solution σ to the equation $u^{\text{BS}}(t, x, \sigma) = u$, where u is the (observed or computed) call option price with same maturity and strike.

Remark 2. For any $t > 0$, the existence and uniqueness of the implied volatility can be deduced using the general arbitrage bounds for call prices and the monotonicity of u^{BS} (see [10, Section 2.1, Remark (i)]).

For any $t \geq 0, x \in \mathbb{R}$, the function $u^{\text{BS}}(t, x, \cdot)$ is analytic on $(0, \infty)$, and, hence, for any $\sigma_0 > 0$ and $\delta \in \mathbb{R}$ such that $\sigma_0 + \delta > 0$, the function $u^{\text{BS}}(t, x, \cdot)$ at the point $\sigma_0 + \delta$ is given by its Taylor series:

$$u^{\text{BS}}(t, x, \sigma_0 + \delta) = \sum_{n=0}^{\infty} \frac{\delta^n}{n!} \partial_{\sigma}^n u^{\text{BS}}(t, x, \sigma_0), \tag{4}$$

where $\partial_{\sigma}^n u^{\text{BS}}(t, x, \sigma_0) = (1/2\pi) \int_{\mathbb{R}} (\partial_{\sigma}^n e^{t\phi_0(\lambda; \sigma)})|_{\sigma=\sigma_0} \widehat{h}(\lambda) e^{i\lambda x} d\lambda_r$. The interchange of the derivative and integral operators is justified by Fubini’s theorem. If one observes the option price u then the following proposition provides a way to compute the corresponding implied volatility.

Proposition 1. For any $t > 0, x \in \mathbb{R}$, let $u : (0, \infty) \rightarrow (0, \infty)$ be defined (as a function of σ) by $u^{\text{BS}}(t, x, \sigma) = u$, and let σ_0 be some strictly positive real number. Then the following expansion holds:

$$\sigma = \sigma_0 + \sum_{n=1}^{\infty} \frac{b_n}{n!} (u - u^{\text{BS}}(t, x, \sigma_0))^n$$

$$\text{with } b_n := \lim_{\sigma \rightarrow \sigma_0} \partial_{\sigma}^{n-1} \left(\frac{\sigma - \sigma_0}{u^{\text{BS}}(t, x, \sigma) - u^{\text{BS}}(t, x, \sigma_0)} \right)^n. \tag{5}$$

Proof. Since the function $u^{\text{BS}}(t, x, \cdot)$ is strictly increasing on $(0, \infty)$, analytic in a neighbourhood of σ_0 and $\partial_{\sigma} u^{\text{BS}}(\cdot, \cdot, \sigma_0) \neq 0$, the proposition follows from the Lagrange inversion theorem [1, Equation 3.6.6].

From Proposition 1, for every fixed $t > 0, x \in \mathbb{R}, \sigma_0 > 0$, there exists some radius of convergence $R > 0$ (depending on (t, x, ζ)) such that $|u - u^{\text{BS}}(t, x, \sigma_0)| < R$ implies that σ (defined implicitly through the equation $u^{\text{BS}}(t, x, \sigma) = u$) is fully characterised by (5). This result, however, seems to be only of theoretical interest. Once the option value u is known, computing the implied volatility by inverting the Black–Scholes formula is a simple numerical exercise. Moreover, computing the implied volatility using (5) is not numerically efficient since the option price u requires the computation of a (possibly highly oscillatory) Fourier integral. One may wish to use (5) to deduce some properties of the implied volatility, but then the proposition would benefit from precise error bounds when truncating the infinite sum. The rest of this paper focuses on developing a similar expansion, without the need for the (potentially computer-intensive) implementation of the value function u .

3. Implied volatility expansions

3.1. Call prices as perturbations around the Black–Scholes formula

For any $\varepsilon \in (0, 1]$ and $\sigma_0 > 0$ define the function $\phi^{\varepsilon}(\cdot, \cdot; \sigma_0) : \mathbb{R}_+ \times \mathbb{C} \rightarrow \mathbb{C}$ by

$$\phi^{\varepsilon}(t, \lambda; \sigma_0) := t\phi_0(\lambda; \sigma_0) + \varepsilon\phi_1(t, \lambda; \sigma_0),$$

where $e^{t\phi_0}$ is the Black–Scholes characteristic function from Definition 1 and

$$\phi_1(t, \lambda; \sigma_0) := \phi(t, \lambda) - t\phi_0(\lambda; \sigma_0). \tag{6}$$

Recall from Bochner [28, Theorem 4.2.2] that a complex-valued function f is a characteristic function if and only if it is nonnegative definite and $f(0) = 1$. Therefore, e^{ϕ^ε} is a well-defined characteristic function for any $t \geq 0$, and we can associate to it a (unique up to indistinguishability) stochastic process $(X_t^{\varepsilon, \sigma_0})_{t \geq 0}$, starting at $X_0^{\varepsilon, \sigma_0} = x$, which is a true martingale. The price u^ε of a call option written on $X^{\varepsilon, \sigma_0}$ thus reads

$$u^\varepsilon(t, x, \sigma_0) := \frac{1}{2\pi} \int_{\mathbb{R}} d\lambda_r e^{\phi^\varepsilon(t, \lambda; \sigma_0)} \widehat{h}(\lambda) e^{i\lambda x}. \tag{7}$$

Let σ^ε denote the implied volatility corresponding to the option price $u^\varepsilon(t, x, \sigma_0)$. Since $\phi^\varepsilon|_{\varepsilon=1} = \phi$ and $u^\varepsilon|_{\varepsilon=1} = u$, the implied volatility corresponding to the option price u is given by $\sigma = \sigma^\varepsilon|_{\varepsilon=1}$. We now seek an expression for σ^ε . The first step is to show that u^ε can be written as a power series in ε , whose first term corresponds to the Black–Scholes call price with volatility σ_0 . To this end, we first expand $e^{\phi^\varepsilon(t, \lambda; \sigma_0)}$ as

$$\exp(\phi^\varepsilon(t, \lambda; \sigma_0)) = e^{t\phi_0(\lambda; \sigma_0)} \sum_{n=0}^{\infty} \frac{1}{n!} \varepsilon^n \phi_1^n(t, \lambda; \sigma_0),$$

and deduce a series representation for u^ε in (7),

$$u^\varepsilon(t, x, \sigma_0) = \sum_{n=0}^{\infty} \varepsilon^n u_n(t, x, \sigma_0) \quad \text{for any } n \geq 0 \tag{8}$$

with $u_n(t, x, \sigma_0) := (1/2\pi n!) \int_{\mathbb{R}} d\lambda_r e^{t\phi_0(\lambda; \sigma_0)} \phi_1^n(t, \lambda; \sigma_0) \widehat{h}(\lambda) e^{i\lambda x}$, where the application of Fubini’s theorem is justified since $\int_{\mathbb{R}} |e^{t\phi^\varepsilon(\lambda)} \widehat{h}(\lambda) e^{i\lambda x}| d\lambda_r$ is finite. In particular note that $u_0 \equiv u^{\text{BS}}$.

3.2. Series expansion for implied volatility

From (8), it is clear that u^ε is an analytic function of ε (we have explicitly provided its power series representation). Since the composition of two analytic functions is also analytic [5, Section 24, p. 74], the expansion (4) implies that $\sigma^\varepsilon = [u^{\text{BS}}]^{-1}(u^\varepsilon)$ is an analytic function and, therefore, has a power series expansion in ε , which we write as $\sigma^\varepsilon := \sigma_0 + \delta^\varepsilon$, where $\delta^\varepsilon = \sum_{k \geq 1} \varepsilon^k \sigma_k$. The following proposition provides an expansion formula for the coefficients σ_k .

Proposition 2. *Fix $\sigma_0 > 0$, $k \geq 1$, and let R denote the radius of convergence of the expansion (5). If $|u(t, x) - u^{\text{BS}}(t, x, \sigma_0)| < R$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, then the following expansion holds:*

$$\sigma_k = \frac{1}{\partial_\sigma u^{\text{BS}}(t, x, \sigma_0)} \left(u_k - \sum_{n=2}^{\infty} \frac{1}{n!} \left(\sum_{j_1 + \dots + j_n = k} \prod_{i=1}^n \sigma_{j_i} \right) \partial_\sigma^n u^{\text{BS}}(t, x, \sigma_0) \right). \tag{9}$$

The right-hand side involves only σ_j for $j \leq k - 1$, so that the sequence can be determined recursively.

Proof. Let us fix some $t \geq 0$ and $x \in \mathbb{R}$. Taylor expanding $u^{BS}(t, x, \sigma^\varepsilon)$ around the point σ_0 , we obtain

$$\begin{aligned} u^{BS}(t, x, \sigma^\varepsilon) &= u^{BS}(t, x, \sigma_0 + \delta^\varepsilon) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (\delta^\varepsilon \partial_\sigma)^n u^{BS}(t, x, \sigma_0) \\ &= u^{BS}(t, x, \sigma_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\sum_{k=1}^{\infty} \varepsilon^k \sigma_k \right)^n \partial_\sigma^n u^{BS}(t, x, \sigma_0) \\ &= u^{BS}(t, x, \sigma_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\sum_{k=1}^{\infty} \left(\sum_{j_1+\dots+j_n=k} \prod_{i=1}^n \sigma_{j_i} \right) \varepsilon^k \right] \partial_\sigma^n u^{BS}(t, x, \sigma_0) \\ &= u^{BS}(t, x, \sigma_0) + \sum_{k=1}^{\infty} \varepsilon^k \left[\sum_{n=1}^{\infty} \frac{1}{n!} \left(\sum_{j_1+\dots+j_n=k} \prod_{i=1}^n \sigma_{j_i} \right) \partial_\sigma^n \right] u^{BS}(t, x, \sigma_0) \\ &= u^{BS}(t, x, \sigma_0) + \sum_{k=1}^{\infty} \varepsilon^k \left[\sigma_k \partial_\sigma + \sum_{n=2}^{\infty} \frac{1}{n!} \left(\sum_{j_1+\dots+j_n=k} \prod_{i=1}^n \sigma_{j_i} \right) \partial_\sigma^n \right] u^{BS}(t, x, \sigma_0). \end{aligned}$$

In order to recover the implied volatility from Definition 2, we need to equate the Black–Scholes call price above and the option value u^ε in (8), and collect terms of identical powers of ε .

(i) $\mathcal{O}(1)$:

$$u_0(t, x, \sigma_0) = u^{BS}(t, x, \sigma_0),$$

(ii) $\mathcal{O}(\varepsilon^k)$:

$$\begin{aligned} u_k(t, x, \sigma_0) &= \sigma_k \partial_\sigma u^{BS}(t, x, \sigma_0) \\ &+ \sum_{n=2}^{\infty} \frac{1}{n!} \left(\sum_{j_1+\dots+j_n=k} \prod_{i=1}^n \sigma_{j_i} \right) \partial_\sigma^n u^{BS}(t, x, \sigma_0), \quad k \geq 1. \end{aligned}$$

Solving the above equations for the sequence $(\sigma_k)_{k \geq 0}$, we find $\sigma_0 = \sigma_0$ at the 0th order and for any $k \geq 1$, the $\mathcal{O}(\varepsilon^k)$ order is given by (9).

Remark 3. Explicitly, up to $\mathcal{O}(\varepsilon^3)$, we have

(i) $\mathcal{O}(\varepsilon)$:

$$\sigma_1 = \frac{1}{\partial_\sigma u^{BS}} u_1,$$

(ii) $\mathcal{O}(\varepsilon^2)$:

$$\sigma_2 = \frac{1}{\partial_\sigma u^{BS}} \left(u_2 - \frac{1}{2} \sigma_1^2 \partial_\sigma^2 u^{BS} \right),$$

(iii) $\mathcal{O}(\varepsilon^3)$:

$$\sigma_3 = \frac{1}{\partial_\sigma u^{BS}} \left(u_3 - (\sigma_2 \sigma_1 \partial_\sigma^2 + \frac{1}{3!} \sigma_1^3 \partial_\sigma^3) u^{BS} \right),$$

where all the functions u_1, \dots, u_5 and u^{BS} are evaluated at (t, x, σ_0) .

Remark 4. Having served its purpose, we now dial ε up to 1. The implied volatility is then given by $\sigma = \sum_{k \geq 0} \sigma_k$, where σ_0 is a fixed positive constant and where the sequence $(\sigma_k)_{k \geq 1}$ is given by (9).

3.3. Simplification of the expressions for σ_k

The expression for the coefficients σ_k ($k \geq 1$) in (9) is not straightforward to apply; one needs to compute first the Fourier integrals u_j ($j \leq k$) via (8), then all the terms of the form $\partial_\sigma^j u^{\text{BS}}$ for $j \leq k$. We provide now a more explicit approximation—without integrals or special functions—for σ_k . The key to this simplification is that all the terms $\partial_\sigma^i u^{\text{BS}}$ and u_i for $i \in \mathbb{N}$ in (9) can be expressed in terms of derivatives of u^{BS} with respect to x , the starting point of the log-stock price process. Indeed, the classical Black–Scholes relation between the delta, the gamma, and the vega for call options, $\partial_\sigma u^{\text{BS}}(t, x, \sigma)|_{\sigma=\sigma_0} = t\sigma_0(\partial_x^2 - \partial_x)u^{\text{BS}}(t, x, \sigma_0)$, implies that the derivative $\partial_\sigma^k u^{\text{BS}}$ can be expressed as a sum of terms of the form $a_{ki} \partial_x^{ki} (\partial_x^2 - \partial_x)u^{\text{BS}}$. We shall also use the equality $p(\lambda)e^{i\lambda x} = p(-i\partial_x)e^{i\lambda x}$, which holds for any polynomial p (and for any analytic function—simply take p to be its power series). We start with the following theorem, which provides an approximation for the coefficients u_n in (8) as a differential operator acting on u^{BS} .

Theorem 1. Fix $t \geq 0$ and $\sigma_0 > 0$. If the power series $\phi_1(t, \lambda; \sigma_0) = \sum_{k \geq 1} a_k(t; \sigma_0)(i\lambda)^k$ holds in a complex neighbourhood of the origin, then for any integer $m \geq 2$, u_n defined in (8) can be written as $u_n(t, x, \sigma_0) = u_n^{(m)}(t, x, \sigma_0) + \varepsilon_n^{(m)}(t, x, \sigma_0)$, where

$$u_n^{(m)}(t, x, \sigma_0) = \frac{1}{n!} \left(\sum_{k=2}^m a_k(t, \sigma_0)(\partial_x^k - \partial_x) \right)^n u^{\text{BS}}(t, x, \sigma_0) \tag{10}$$

with $\varepsilon_n^{(m)}$ containing only derivatives (with respect to x) of u^{BS} of order higher than n .

Remark 5. Note that the power series for ϕ_1 in Theorem 1 starts at $k = 1$, which follows from the fact that the process $\exp(X^{\varepsilon, \sigma_0})$ is conservative. This expansion holds as soon as all the moments of X_t exist and $\lim_{k \uparrow \infty} |\lambda|^k \mathbb{E}(|X_t|^k)/k! = 0$ for small enough $|\lambda|$, which is valid under Assumption 1.

Proof. Assume that the power series for $\phi_1(t, \cdot; \sigma_0)$ holds around the origin, where the coefficients are

$$a_k(t; \sigma_0) = \frac{(-i)^k}{k!} \partial_\lambda^k \phi_1(t, \lambda; \sigma_0)|_{\lambda=0}. \tag{11}$$

The martingale condition implies that $\phi_1(t, -i; \sigma_0) = \sum_{k \geq 1} a_k(t; \sigma_0) = 0$, and, hence,

$$\begin{aligned} \phi_1(t, \lambda; \sigma_0) &= \sum_{k=1}^\infty a_k(t; \sigma_0)(i\lambda)^k \\ &= i\lambda a_1(t; \sigma_0) + \sum_{k=2}^\infty a_k(t; \sigma_0)[(i\lambda)^k - i\lambda] + i\lambda \sum_{k=2}^\infty a_k(t; \sigma_0) \\ &= \sum_{k=2}^\infty a_k(t; \sigma_0)[(i\lambda)^k - i\lambda]. \end{aligned}$$

Now let $\phi_1^{(m)}(t, \cdot; \sigma_0) : \mathbb{C} \rightarrow \mathbb{C}$ be the truncation of this series at the m th order, i.e.

$$\phi_1^{(m)}(t, \lambda; \sigma_0) := \sum_{k=2}^m a_k(t; \sigma_0)[(i\lambda)^k - i\lambda], \tag{12}$$

and define the operator δ acting on $\phi_1^{(m)}$ by

$$\delta\phi_1^{(m)}(t, \lambda; \sigma_0) := \phi_1(t, \lambda; \sigma_0) - \phi_1^{(m)}(t, \lambda; \sigma_0) = \frac{\lambda^{m+1}}{2\pi i} \int_{\partial\Gamma} \frac{\phi_1(t, z; \sigma_0) dz}{z^{m+1}(z - \lambda)} + i\lambda \sum_{k=1}^m a_k,$$

where Γ is a closed set within the radius of convergence of ϕ_1 , and the integral is nothing else than the remainder of the series expansion around the point $\lambda = 0$. Hence, for any $n \geq 1$, u_n in (8) can be written as

$$\begin{aligned} u_n(t, x, \sigma_0) &= \frac{1}{n!} \frac{1}{2\pi} \int_{\mathbb{R}} d\lambda_r e^{t\phi_0(t, \lambda, \sigma_0)} \widehat{h}(\lambda) (\phi_1(t, \lambda; \sigma_0))^n e^{i\lambda x} \\ &= \frac{1}{n!} \frac{1}{2\pi} \int_{\mathbb{R}} d\lambda_r e^{t\phi_0(\lambda; \sigma_0)} \widehat{h}(\lambda) (\phi_1^{(m)}(t, \lambda; \sigma_0) + \delta\phi_1^{(m)}(t, \lambda; \sigma_0))^n e^{i\lambda x} \\ &= \frac{1}{n!} (\phi_1^{(m)}(t, -i\partial_x; \sigma_0))^n \frac{1}{2\pi} \int_{\mathbb{R}} d\lambda_r e^{t\phi_0(\lambda; \sigma_0)} \widehat{h}(\lambda) e^{i\lambda x} + \varepsilon_n^{(m)}(t, x, \sigma_0) \\ &= \frac{1}{n!} (\phi_1^{(m)}(t, -i\partial_x; \sigma_0))^n u^{\text{BS}}(t, x, \sigma_0) + \varepsilon_n^{(m)}(\sigma_0) \\ &= u_n^{(m)}(t, x, \sigma_0) + \varepsilon_n^{(m)}(t, x, \sigma_0), \end{aligned}$$

where

$$\begin{aligned} u_n^{(m)}(t, x, \sigma_0) &:= \frac{1}{n!} (\phi_1^{(m)}(t, -i\partial_x; \sigma_0))^n u^{\text{BS}}(t, x, \sigma_0), \\ \varepsilon_n^{(m)}(t, x, \sigma_0) &:= \frac{1}{n!} \sum_{k=1}^n \binom{n}{k} (\phi_1^{(m)}(t, -i\partial_x; \sigma_0))^{n-k} (\delta\phi_1^{(m)}(t, -i\partial_x; \sigma_0))^k u^{\text{BS}}(t, x, \sigma_0). \end{aligned}$$

From decomposition (12), we have $\phi_1^{(m)}(t, -i\partial_x; \sigma_0) = \sum_{k=2}^m a_k(t; \sigma_0)(\partial_x^k - \partial_x)$, where the coefficients a_k are defined in (11). We can now compute

$$\begin{aligned} u_n^{(m)}(t, x, \sigma_0) &= \frac{1}{n!} (\phi_1^{(m)}(t, -i\partial_x; \sigma_0))^n u^{\text{BS}}(t, x, \sigma_0) \\ &= \frac{1}{n!} \left(\sum_{k=2}^m a_k(t; \sigma_0)(\partial_x^k - \partial_x) \right)^n u^{\text{BS}}(t, x, \sigma_0), \end{aligned}$$

which is precisely the expression given in (10). Regarding $\varepsilon_n^{(m)}$, since for any $z \in \Gamma$, there exists $M > 0$ such that $|\phi_1(t, z; \sigma_0)/(z - \lambda)| \leq M$, we have $|\delta\phi_1^{(m)}(t, \lambda; \sigma_0)| \leq M(|\lambda|/R)^{m+1} + |\lambda| |\sum_{k=1}^m a_k|$, where R denotes the radius of convergence of ϕ_1 . The sequence $(a_k)_{k \geq 1}$ is (eventually) decreasing and the sum tends to 0 as m tends to ∞ , so the sum can be made arbitrarily small. We then obtain

$$\begin{aligned} |\varepsilon_n^{(m)}(t, x, \sigma_0)| &\leq \frac{1}{n!} \sum_{k=1}^n \binom{n}{k} |\phi_1^{(m)}(t, -i\partial_x; \sigma_0)|^{n-k} \\ &\quad \times \left(M \left(\frac{|\lambda|}{R} \right)^{m+1} + |\lambda| \left| \sum_{k=1}^m a_k \right| \right)^k u^{\text{BS}}(t, x, \sigma_0) \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{n!} \sum_{k=1}^n \binom{n}{k} \left| \sum_{j=2}^m a_j(t; \sigma_0) ((i\lambda)^j - i\lambda) \right. \\ &\quad \times \left. \left| M \left(\frac{|\lambda|}{R} \right)^{m+1} + |\lambda| \left| \sum_{k=1}^m a_k \right| \right|^k u^{\text{BS}}(t, x, \sigma_0). \end{aligned}$$

One can then readily check that the sum behaves as $\mathcal{O}(\lambda^n)$ as λ tends to 0. Therefore, $\varepsilon_n^{(m)}$ contains derivatives (with respect to x) of u^{BS} of at least order n .

Expression (10) motivates the following definition.

Definition 3. For any integers $n \geq 0, m \geq 2$, and for fixed $\sigma_0 > 0$, we define the (n, m) th order approximation of the implied volatility as

$$\sigma^{(n,m)} := \sigma_0 + \sum_{k=1}^n \sigma_k^{(m)}, \tag{13}$$

where for any $k = 1, \dots, n, \sigma_k^{(m)}$ is defined as

$$\sigma_k^{(m)} := \frac{1}{\partial_\sigma u^{\text{BS}}(t, x, \sigma_0)} \left(u_k^{(m)} - \sum_{n=2}^\infty \frac{1}{n!} \left(\sum_{j_1+\dots+j_n=k} \prod_{i=1}^n \sigma_{j_i}^{(m)} \right) \partial_\sigma^n u^{\text{BS}}(t, x, \sigma_0) \right). \tag{14}$$

Note that $\sigma_k^{(m)}$ is obtained from σ_k by replacing u_k in (9) by its m th order approximation $u_k^{(m)}$. The following theorem, proved in Appendix A, is the main result of this paper and provides an explicit expression (not involving the derivatives of the Black–Scholes formula) for the (n, m) th order approximation of the implied volatility.

Theorem 2. Fix $(t, x, \zeta, \sigma_0) \in (0, \infty) \times \mathbb{R} \times \mathbb{R} \times (0, \infty)$, and define

$$y_0 := \frac{1}{\sigma_0 \sqrt{2t}} \left(x - \zeta - \frac{1}{2} \sigma_0^2 t \right). \tag{15}$$

For $\phi(t, \lambda), \phi_0(\lambda, \sigma_0), \phi_1(t, \lambda; \sigma_0)$, and $a_k(t; \sigma_0)$ defined in (2), Definition 1, (6), and (11), the (n, m) th approximation (13) holds with $\sigma_k^{(m)} = U_k^{(m)} - \Sigma_k^{(m)}$ and

$$\begin{aligned} U_n^{(m)} &:= \frac{1}{n! t \sigma_0} \sum_{k=2n}^{nm} \sum_{\substack{j_1+\dots+j_n=k \\ 2 \leq j_1, \dots, j_n \leq m}} \left(\prod_{i=1}^n a_{j_i} \right) \sum_{k_1=2}^{j_1} \dots \sum_{k_n=2}^{j_n} \binom{n-1}{m} (-1)^{n-1-m} \\ &\quad \times \left(-\frac{1}{\sigma_0 \sqrt{2t}} \right)^{-n-1+m+\sum_{j=1}^n k_j} H_{-n-1+m+\sum_{j=1}^n k_j}(y_0), \\ \Sigma_k^{(m)} &:= \sum_{n=2}^\infty \frac{1}{n!} \left(\sum_{j_1+\dots+j_n=k} \prod_{i=1}^n \sigma_{j_i}^{(m)} \right) \sum_{q=0}^{\lfloor n/2 \rfloor} \sum_{p=0}^{n-q-1} \binom{n-q-1}{p} c_{n,n-2q} \sigma_0^{-(q+p)} t^{n-q-1} \\ &\quad \times (\sqrt{2t})^{1-p-n+q} H_{p+n-q-1}(y_0). \end{aligned}$$

Here, $H_n(y) \equiv (-1)^n e^{y^2} \partial_y^n e^{-y^2}$ is the n th hermite polynomial, and the coefficients $(c_{n,n-2k})$ are defined recursively by $c_{n,n} = 1$ and $c_{n,n-2q} = (n - 2q + 1)c_{n-1,n-2q+1} + c_{n-1,n-2q-1}$ for any integer $q \in \{1, 2, \dots, \lfloor n/2 \rfloor\}$.

Example 1. To illustrate how Theorem 2 works in practice, we compute $\sigma_2^{(3)}$ explicitly. Fix (t, x, ζ, σ_0) and write $a_k = a_k(t; \sigma_0)$. Equation (10) implies that

$$\begin{aligned} u_1^{(3)}(t, x, \sigma_0) &= \{a_3\partial_x + (a_2 + a_3)\}(\partial_x^2 - \partial_x)u^{BS}(t, x, \sigma_0), \\ u_2^{(3)}(t, x, \sigma_0) &= \frac{1}{2}\{a_3^2\partial_x^4 + (2a_2a_3 + a_3^2)\partial_x^3 + (a_2^2 - a_3^2)\partial_x^2 + (-a_2^2 - a_3^2 - 2a_2a_3)\partial_x\} \\ &\quad \times (\partial_x^2 - \partial_x)u^{BS}(t, x, \sigma_0). \end{aligned}$$

Next, using Proposition 4, we have

$$\begin{aligned} \frac{u_1^{(3)}}{\partial_\sigma u^{BS}}(t, x, \sigma_0) &= \frac{1}{t\sigma_0} \left\{ a_3 \left(\frac{-1}{\sigma_0\sqrt{2t}} \right) H_1(y_0) + (a_2 + a_3) \right\}, \tag{16} \\ \frac{u_2^{(3)}}{\partial_\sigma u^{BS}}(t, x, \sigma_0) &= \frac{1}{t\sigma_0} \left\{ a_3^2 \left(\frac{-1}{\sigma_0\sqrt{2t}} \right)^4 H_4(y_0) + (2a_2a_3 + a_3^2) \left(\frac{-1}{\sigma_0\sqrt{2t}} \right)^3 H_3(y_0) \right. \\ &\quad + (a_2^2 - a_3^2) \left(\frac{-1}{\sigma_0\sqrt{2t}} \right)^2 H_2(y_0) \\ &\quad \left. + (-a_2^2 - a_3^2 - 2a_2a_3) \left(\frac{-1}{\sigma_0\sqrt{2t}} \right) H_1(y_0) \right\}, \tag{17} \end{aligned}$$

with y_0 defined in (15). From Proposition 3, we then have $\partial_\sigma^2 u^{BS} = (\sigma_0^2 \mathcal{L}^2 - \mathcal{L})u^{BS}$, where $\mathcal{L} = t(\partial_x^2 - \partial_x)$. Therefore, recalling that $\partial_\sigma u^{BS} = \sigma_0 \mathcal{L}u^{BS}$, we obtain

$$\begin{aligned} \frac{\partial_\sigma^2 u^{BS}}{\partial_\sigma u^{BS}}(t, x, \sigma_0) &= \frac{1}{t\sigma_0} \left(\frac{t^2 \sigma_0^2 \partial_x^2 (\partial_x^2 - \partial_x) u^{BS}}{(\partial_x^2 - \partial_x) u^{BS}} - \frac{t^2 \sigma_0^2 \partial_x (\partial_x^2 - \partial_x) u^{BS}}{(\partial_x^2 - \partial_x) u^{BS}} + \frac{t (\partial_x^2 - \partial_x) u^{BS}}{(\partial_x^2 - \partial_x) u^{BS}} \right) \\ &\quad \times (t, x, \sigma_0) \\ &= \frac{1}{t\sigma_0} \left[t^2 \sigma_0^2 \left(\frac{-1}{\sigma_0\sqrt{2t}} \right)^2 H_2(y_0) - t^2 \sigma_0^2 \left(\frac{-1}{\sigma_0\sqrt{2t}} \right) H_1(y_0) + t \right] \\ &= \frac{(\zeta - x)^2}{t\sigma_0^3} - \frac{t\sigma_0}{4}. \tag{18} \end{aligned}$$

Finally, from (13) and (14), we have

$$\sigma^{(2,3)} = \sigma_0 + \sigma_1^{(3)} + \sigma_2^{(3)}, \quad \sigma_1^{(3)} = \frac{u_1^{(3)}}{\partial_\sigma u^{BS}}, \quad \sigma_2^{(3)} = \frac{u_2^{(3)}}{\partial_\sigma u^{BS}} - \frac{1}{2}(\sigma_1^{(3)})^2 \frac{\partial_\sigma^2 u^{BS}}{\partial_\sigma u^{BS}}. \tag{19}$$

The explicit expression for $\sigma^{(2,3)}$ can be obtained by inserting (16), (17), and (18) into (19).

4. Numerical implementation: discussions and examples

We now focus on the practical implementation of the results above, namely Theorem 2. In Section 4.1 we propose a smoothing procedure to further enhance the applicability of our methodology. In Sections 4.2 and 4.3 we implement our implied volatility expansion in two exponential Lévy models (Merton and variance gamma) and one stochastic volatility model (Heston).

4.1. Smoothing with the stochastic volatility inspired parameterisation

Option data is often noisy and limited by the number of strikes at which options are liquidly traded. In [13], Gatheral introduced the following *stochastic volatility inspired* (SVI) parameterisation:

$$\sigma_t^{\text{SVI}}(\zeta) = \left\{ \frac{a}{t} + \frac{b}{t}(\rho(\zeta - x - m) + \sqrt{(\zeta - x - m)^2 + \xi^2}) \right\}^{1/2} \tag{20}$$

for any maturity $t > 0$, where $a, b \geq 0, \xi > 0, m \in \mathbb{R}$, and $\rho \in [-1, 1]$. By fitting the SVI parameterisation to noisy option data, one is able to create a smooth implied volatility smile, which then can be used to interpolate implied volatility between strikes and extrapolate implied volatility to strikes that are not traded. The density $p^\sigma(t, x)$ corresponding to a given implied volatility parameterisation $\sigma(t, \zeta)$ can be computed via the Breeden–Litzenberger formula [4]: $p^\sigma(t, \zeta) = \partial_K^2 u^{\text{BS}}(t, x, \sigma(t, \log K); \log K)|_{K=e^\zeta}$. An implied volatility smile $\zeta \mapsto \sigma(t, \zeta)$ is said to be free of butterfly arbitrage if the corresponding density is nonnegative: $p(t, \cdot) \geq 0$. Let $p_t^{\text{SVI}}(\zeta)$ be the implied volatility smile corresponding to a given SVI parameterisation (20). In general, SVI parameterisation (20) is not guaranteed to be free of butterfly arbitrage. However, for a given set of SVI parameters (a, b, ρ, m, ξ) , one can easily verify that the corresponding density is nonnegative, and, therefore, free of butterfly arbitrage. This and recent arbitrage-free SVI parameterisations have recently been studied in [14] and [17], and we refer the interested reader to these papers for more details. As we shall see in the examples considered in Section 4, for finite (n, m) , the approximate implied volatility $\sigma^{(n,m)}$ derived in Section 2.2 has a tendency to oscillate around the true implied volatility (see Figures 1, 2, and 3). Taking $\sigma^{(n,m)}$ to be the true implied volatility could lead to arbitrage opportunities. In order to prevent this, we propose to smooth the implied volatility approximation $\sigma^{(n,m)}$ by fitting the SVI parameterisation to it. That is, given a model for the underlying X and a time to maturity t , we first compute the approximate implied volatility $\sigma^{(n,m)}$ as a function of log-strike ζ , and then fit an arbitrage-free SVI parameterisation σ_t^{SVI} to $\sigma^{(n,m)}$ over some range of strikes, usually chosen to be a symmetric interval around $\zeta = x$.

4.2. Exponential Lévy models

If X is a Lévy process with Lévy triplet (μ, a^2, ν) , its characteristic function reads

$$\phi(t, \lambda) = t \left(i\mu\lambda + \frac{1}{2}a^2(i\lambda)^2 + \int_{\mathbb{R}} \nu(dz)(e^{i\lambda z} - 1 - \mathbf{1}_{\{|z|<1\}}i\lambda z) \right),$$

where the drift μ is constrained by the martingale condition $\phi(t, -i) = 0$: $\mu = -\frac{1}{2}a^2 - \int_{\mathbb{R}} \nu(dz)(e^z - 1 - \mathbf{1}_{\{|z|<1\}}z)$. From the expansion $(e^{i\lambda z} - 1) = \sum_{n \geq 1} (1/n!)(i\lambda z)^n$, we have

$$\begin{aligned} \phi_1(t, \lambda; \sigma_0) &\equiv \phi(t, \lambda) - t\phi_0(\lambda; \sigma_0) \\ &\equiv t \left(\mu + I_1 + \frac{1}{2}\sigma_0^2 \right) i\lambda + \frac{t}{2}(a^2 - \sigma_0^2)(i\lambda)^2 + t \sum_{n=2}^{\infty} \frac{1}{n!} I_n (i\lambda)^n \end{aligned}$$

with $I_1 := \int_{|z| \geq 1} \nu(dz)z$ and $I_n := \int_{\mathbb{R}} \nu(dz)z^n$ for any $n \geq 2$. The existence of I_n is equivalent to the finiteness of the n th moment of X by [31, Theorem 25.3], which is clearly satisfied under Assumption 1. Hence, the coefficients $a_n(t; \sigma_0)$ in (11) are

$$a_2(t; \sigma_0) = \frac{t}{2}(a^2 - \sigma_0^2 + I_2), \quad a_n(t) = \frac{t}{n!} I_n, \quad n \geq 3. \tag{21}$$

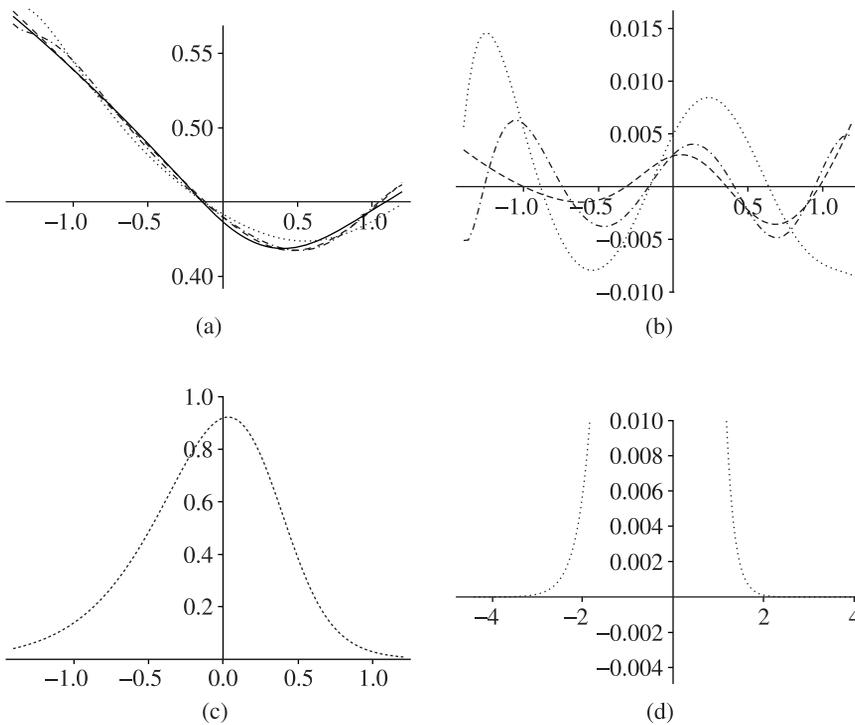


FIGURE 1: Numerics for the Merton model. (a) and (b) denote the implied volatility and its error, respectively. The plots denote the true smile (solid line), the approximations $\sigma^{(2,7)}$ (dots) and $\sigma^{(3,7)}$ (dots-dashed), and the SVI smoothing of $\sigma^{(3,7)}$ (dashed). The plots in (c) and (d) correspond to the density and its tails of the SVI smoothing.

We examine two exponential Lévy models in detail—the Merton model [30] and the variance gamma model [29]—whose Lévy measures are given by

(i) Merton:

$$\nu(dz) = \frac{\alpha}{\sqrt{2\pi s^2}} \exp\left(-\frac{(z - m)^2}{2s^2}\right) dz,$$

(ii) Variance gamma:

$$\nu(dz) = \alpha \left(\frac{e^{Gz}}{-z} \mathbf{1}_{\{z < 0\}} + \frac{e^{-Mz}}{z} \mathbf{1}_{\{z > 0\}} \right) dz,$$

where $\alpha, s, G, M > 0, m \in \mathbb{R}$. The Merton model is a finite-activity Lévy process ($\nu(\mathbb{R}) < \infty$), whereas the variance gamma model has infinite activity ($\nu(\mathbb{R}) = \infty$). For infinite activity Lévy processes, one typically takes the diffusion component to be 0, namely $a = 0$. We now examine the accuracy of the implied volatility expansion for these models: the Merton model in Figure 1 and the variance gamma in Figure 2. For each of these two sets of plots, we fix some parameters, and draw the implied volatility approximations $\sigma^{(n,m)}$ with $m = 7$ for the Merton model and $m = 8$ for the variance gamma model, and $n \in \{1, 2, 3\}$. We also plot the SVI smoothing of $\sigma^{(3,m)}$ as well as the true implied volatility. The true option price is computed by a quadrature of the inverse Fourier transform representation (1) and the true implied volatility

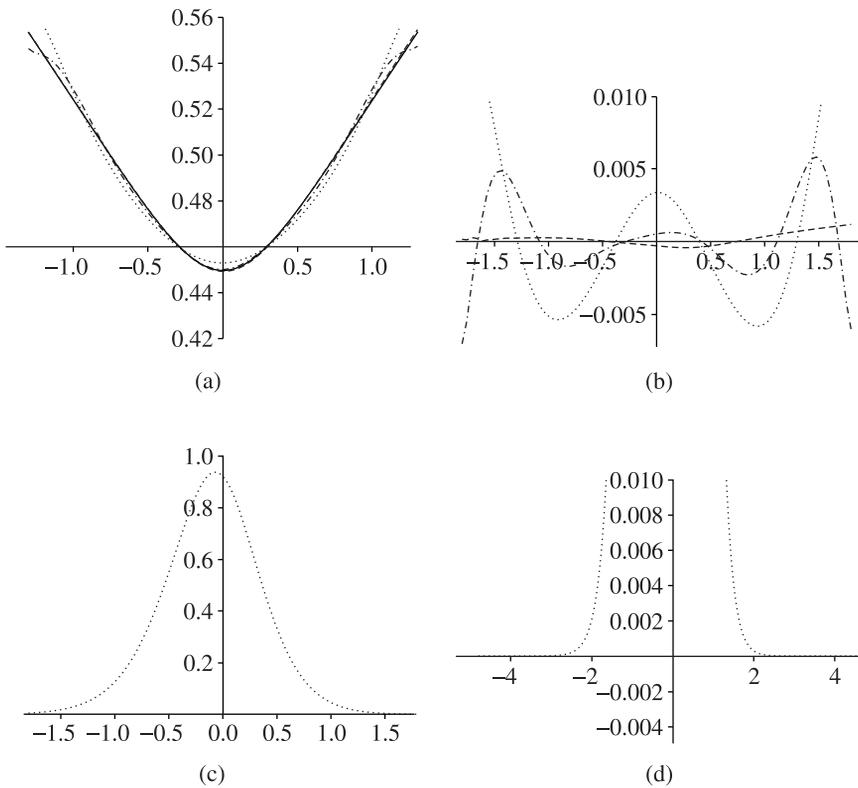


FIGURE 2: Numerics for the variance gamma model. (a) and (b) denote the implied volatility and its error, respectively. The plots denote the true implied volatility (solid line), approximations $\sigma^{(2,8)}$ (dots) and $\sigma^{(3,8)}$ (dots-dashed), and the SVI smoothing of $\sigma^{(3,8)}$ (dashed). The plots in (c) and (d) correspond to the density and its tails of the SVI smoothing.

is computed by numerical inversion of the Black–Scholes formula (we use a simple Newton–Raphson algorithm). We also plot the total errors between each approximation (and $\sigma^{(3,\cdot)}$ with SVI smoothing) and the true implied volatility. As discussed above, the implied volatility approximation oscillates around the true implied volatility σ . However, the relative error corresponding to $\sigma^{(3,m)}$ is less than 1% for nearly all log-moneyness to maturity ratios (LMMRs) satisfying $|\zeta - x|/t < 1.4$ for both models, which is well within the implied volatility bid–ask spread of S&P 500 options. Furthermore, the relative error of $\sigma^{(3,m)}$ with SVI smoothing is about $\frac{1}{2}\%$ for all $|\zeta - x|/t < 1.0$. As a no-arbitrage consistency check, we also plot the density corresponding to the SVI fit. The parameters for each model are as follows.

(i) Merton model:

$$\sigma_0 = 0.55, a = 0.25, m = -0.15, s = 0.3, \alpha = 1.5, t = 1, x = 0.$$

(ii) Variance gamma model:

$$\sigma_0 = 0.55, a = 0, M = 7, G = 6, \alpha = 4.5, t = 1, x = 0.$$

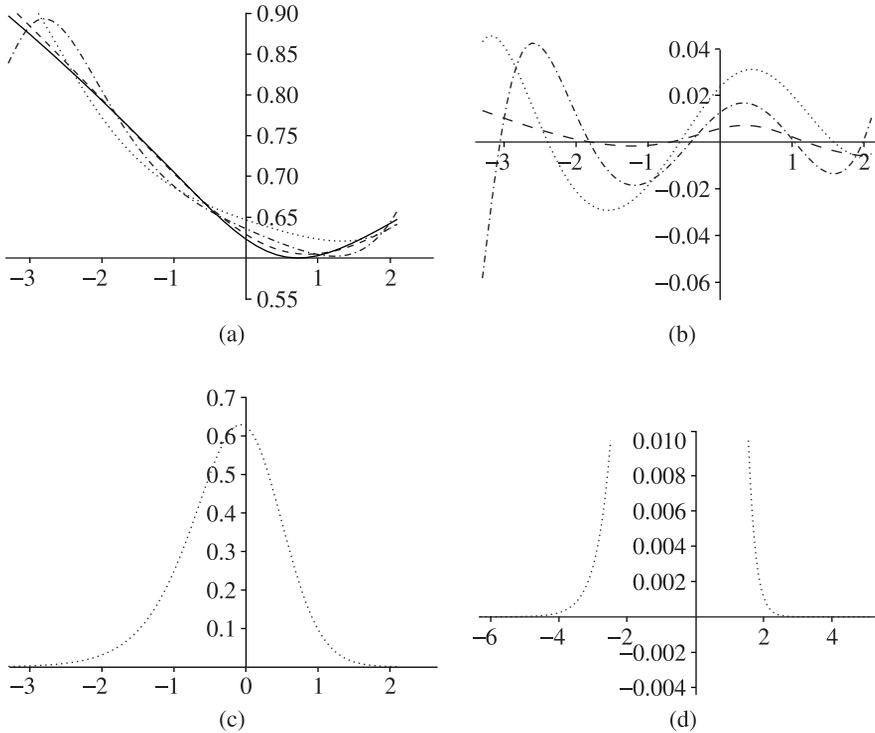


FIGURE 3: Numerics for the Heston model. (a) and (b) denote the implied volatility and its error, respectively. The plots correspond to the true implied volatility (solid line), approximations $\sigma^{(2,6)}$ (dots) and $\sigma^{(3,6)}$ (dots-dashed), and the SVI smoothing of $\sigma^{(3,6)}$ (dashed). The plots in (c) and (d) correspond to the density and its tails of the SVI smoothing.

4.3. The Heston model

In the Heston model [20], the risk-neutral dynamics of (X, Y) are given by

$$dX_t = -\frac{1}{2}Y_t dt + \sqrt{Y_t} dW_t, \quad dY_t = \kappa(\theta - Y_t) dt + \delta\sqrt{Y_t} dB_t, \quad d\langle W, B \rangle_t = \rho dt$$

with $(X_0, Y_0) = (x, y) \in \mathbb{R} \times (0, \infty)$, $\kappa, \theta, \delta > 0$, and where W and B are two standard Brownian motions with correlation $\rho \in [-1, 1]$. Its characteristic function is $\phi(t, \lambda, y) = C(t, \lambda) + yD(t, \lambda)$, where

$$C(t, \lambda) := \frac{\kappa\theta}{\delta^2} \left((\kappa - i\rho\delta\lambda - d(\lambda))t - 2 \log \left[\frac{1 - \gamma(\lambda)e^{-d(\lambda)t}}{1 - \gamma(\lambda)} \right] \right),$$

$$D(t, \lambda) := \frac{\kappa - i\rho\delta\lambda - d(\lambda)}{\delta^2} \frac{1 - e^{-d(\lambda)t}}{1 - \gamma(\lambda)e^{-d(\lambda)t}},$$

$$d(\lambda) := (\delta^2(\lambda^2 + i\lambda) + (\kappa - \rho i\lambda\delta)^2)^{1/2}, \quad \gamma(\lambda) := \frac{\kappa - i\rho\delta\lambda - d(\lambda)}{\kappa - i\rho\delta\lambda + d(\lambda)}.$$

Unlike the exponential Lévy setting, there is no simple general formula for the coefficients $a_n(t, \sigma_0)$ for $n \geq 2$. However, from (11), one can compute

$$\begin{aligned}
 a_2(t; \sigma_0) = & \frac{e^{2\kappa t}}{16\kappa^3} [4e^{\kappa t} [2(\theta - y)\kappa^2 + 2(y + y\kappa t - \theta(2 + \kappa t))\kappa\rho\delta + (\theta + (\theta - y)\kappa t)\delta^2] \\
 & - (2y - \theta)\delta^2] \\
 & + \frac{1}{16\kappa^3} [8\kappa^2(y + (\kappa t - 1)\theta) - 8(y + \theta(\kappa t - 2))\kappa\rho\delta - ((5 - 2\kappa t)\theta - 2y)\delta^2] \\
 & - \frac{\sigma_0^2 t}{2}.
 \end{aligned}$$

Higher-order terms ($3 \leq n \leq 6$) are easily computed using any mathematical software, and are omitted here for clarity. In Figure 3, we plot the function $\zeta \mapsto \sigma_n^{(m)}(\zeta)$ with $m = 6$ and $n \in \{1, 2, 3\}$ a calibrated SVI to $\sigma_3^{(6)}$ and the true implied volatility (computed exactly as for the Lévy models above). We also plot the relative errors between each approximation (and the SVI smoothing of $\sigma^{(3,6)}$) and the true implied volatility. Again, the approximation $\sigma^{(n,m)}$ oscillates around the true implied volatility, but the relative error of $\sigma^{(3,6)}$ is less than 2% for nearly all LMMRs satisfying $|\zeta - x|/t < 2.0$, and that of $\sigma^{(3,6)}$ with SVI smoothing is roughly 1% for all $|\zeta - x|/t < 2.0$. As before, we also plot the density corresponding to the calibrated SVI parameterisation as a no-arbitrage consistency check. We use the following set of parameters: $\sigma_0 = 0.95, \kappa = 1, \theta = 0.3, \delta = 0.7, \rho = -0.3, t = 1, x = 0, y = 0.5$.

4.4. Model-free calibration

As noted previously, the model-specific dependence of the approximate implied volatility expansion $\sigma^{(n,m)}$ is entirely captured by the coefficients $a_i(t, \sigma_0)$ for $2 \leq i \leq m$. This simple structure allows for a model-free calibration of the implied volatility surface. Assume one observes implied volatilities for maturities $(t_i)_{i=1, \dots, n_T}$ and $(k_j)_{j=1, \dots, n_K}$, where n_T and n_K are two integers. We shall assume for simplicity that the number of available strikes is the same for each maturity. We suggest the following procedure.

- (i) Let $\sigma_{i,j} := \sigma(t_i, k_j)$ be the quoted implied volatility for an option with maturity t_i and log-strike k_j .
- (ii) Let $\sigma_{i,j}^{(n,m)} := \sigma^{(n,m)}(t_i, k_j; \sigma_0)$ be the approximate implied volatility for an option with maturity t_i and log-strike k_j computed using approximation (13).
- (iii) At each maturity t_i , leave σ_0 and $a_q(t_i; \sigma_0)$ ($2 \leq q \leq m$) as free parameters. Fit $\sigma^{(n,m)}(t_i, \cdot)$ to the market's t_i -maturity implied volatility smile $\sigma(t_i, \cdot)$ by minimising $\sum_{j=1}^{n_K} |\sigma^{(n,m)}(t_i, k_j) - \sigma_{ij}^{(n,m)}|^2$.
- (iv) As an initial guess, use the largest quoted implied volatility at each maturity for σ_0 and $a_q(t_i, \sigma_0) = 0$.

Remark 6. With $n = 3$ and $m = 8$, step (iii) is instantaneous using the 'FitTo' function in Mathematica® or the 'lsqnonlin' function in Matlab®, for instance.

We test this procedure on S&P 500® (SPX) options from 4 January 2010 with $n = 3$ and $m = 8$. The results for three maturities ($t = 0.033, t = 0.70, t = 1.45$ years) are shown in Figure 4. The calibrated parameters are shown in Table 1.

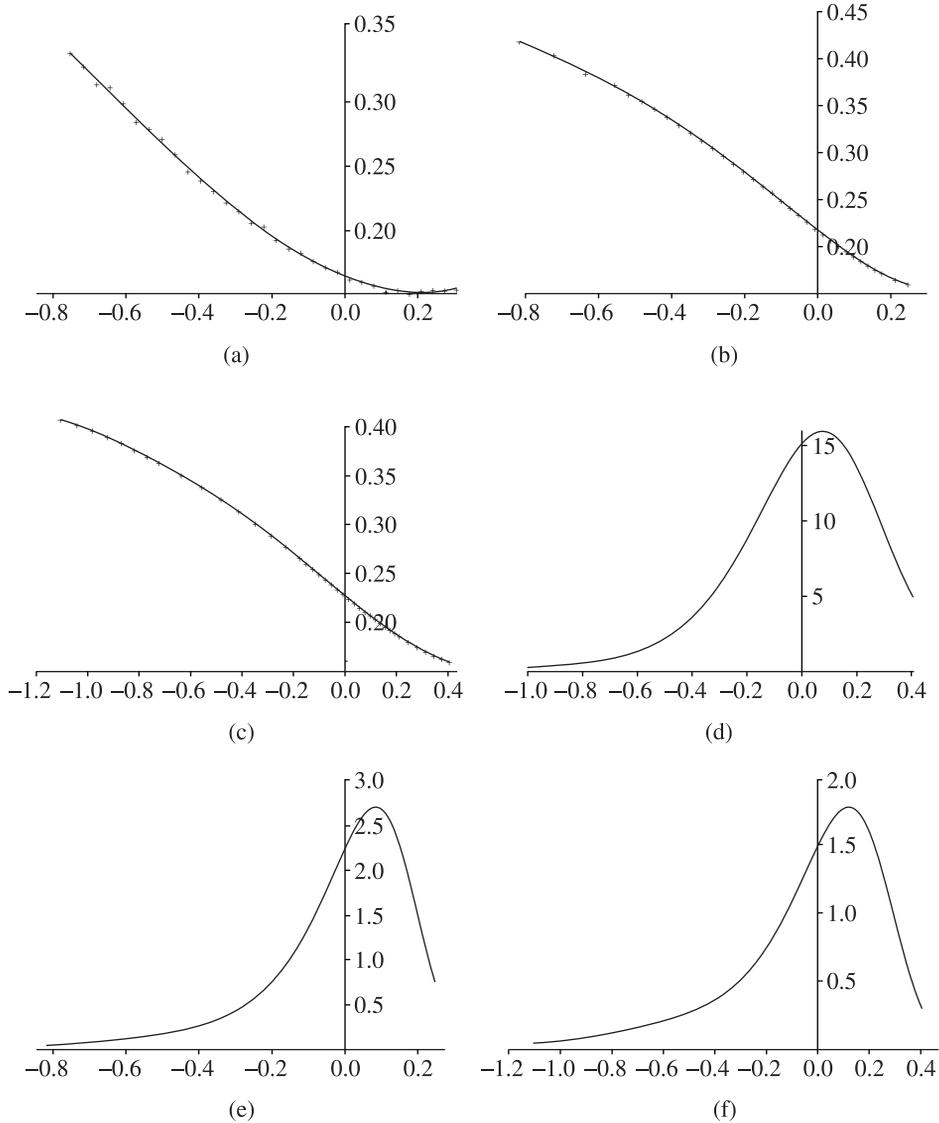


FIGURE 4: Model-free fit to SPX options from 4 January 2010 for (a) $t=0.033$ years, (b) $t=0.700$ years, (c) $t=1.450$ years, (d) $t=0.033$ years, (e) $t=0.700$ years, and (f) $t=1.450$ years. The horizontal axis represents the log-moneyness $(\zeta - x)$. The plots represent the densities of the fit.

Remark 7. If the stock price is an exponential Lévy model, then (21) implies that $\frac{1}{2}\sigma_0^2 + (1/t_i)a_2(t_i, \sigma_0) = \frac{1}{2}(a^2 + I_2)$ and $(1/t_i)a_q(t_i) = (1/q!)I_q$, for $3 \leq q \leq m$, should be constant. If this is not so, then exponential Lévy models are probably not the best dynamics to describe the underlying.

Remark 8. Our whole methodology is based on approximating the characteristic function of a process by a truncated version of its expansion with respect to some small parameter. In

TABLE 1: Calibrated parameters a_i are shorthand notation for $a_i(t; \sigma_0)$ for the three maturities (t) shown in Figure 4.

	Maturity (years)		
	$t = 0.033$	$t = 0.70$	$t = 1.45$
σ_0	0.382	0.659	0.436
a_2	-1.64×10^{-3}	-1.31×10^{-1}	-8.35×10^{-2}
a_3	-1.00×10^{-6}	-5.00×10^{-3}	-1.22×10^{-2}
a_4	1.64×10^{-6}	2.40×10^{-3}	1.68×10^{-3}
a_5	1.89×10^{-8}	4.24×10^{-4}	-6.26×10^{-5}
a_6	8.62×10^{-10}	6.96×10^{-5}	3.42×10^{-5}
a_7	5.22×10^{-12}	2.88×10^{-6}	-1.77×10^{-6}
a_8	1.40×10^{-13}	4.83×10^{-7}	3.96×10^{-7}

essence, this truncation tends to ignore the tail behaviour (high-order terms in the expansion) of the process, and, hence, even though the resulting volatility expansion is accurate around the money, there is no reason why it should be so in the tails. The latter, however, are usually not observable in practice, so this should be of lesser concern for practical implementation. This, in particular, means that should one plot the densities corresponding to the fit in Figure 4, the latter may become negative in the tails (hence, allowing for arbitrage opportunities). In our calibration example (see Figure 4), the density does remain nonnegative though. If, however, it was to become negative, one could perform an SVI fit, as explained in Section 4.1, or, even better, use the fully no-arbitrage SVI version developed in [14].

Appendix A. Proof of Theorem 2

From (14), we observe that $\sigma_k^{(m)}$ involves both $\partial_\sigma^n u^{BS} / \partial_\sigma u^{BS}$ and $u_k^{(m)} / \partial_\sigma u^{BS}$. Theorem 2 will follow directly from Definition 3 and Propositions 3 and 4, both of which provide explicit expressions for these quantities. In all the statements and results below, we shall consider $(t, x, \zeta, \sigma) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times (0, \infty)$, and define

$$y := \left(\frac{1}{\sigma\sqrt{2t}} \right) \left(x - \zeta - \frac{1}{2}\sigma^2t \right).$$

Furthermore, we recall that, for any $n \in \mathbb{N}$, H_n denotes the n th hermite polynomial from Theorem 2. We start with the following lemma.

Lemma 1. *For any integers $m \geq 0, n \geq 2$, we have*

$$\frac{\partial_x^m (\partial_x^n - \partial_x) u^{BS}}{(\partial_x^2 - \partial_x) u^{BS}}(t, x, \sigma) = \sum_{i=2}^n \left(-\frac{1}{\sigma\sqrt{2t}} \right)^{m+i-2} H_{m+i-2}(y). \tag{22}$$

Proof. From the Black–Scholes call price formula (3), we clearly obtain

$$(\partial_x^2 - \partial_x) u^{BS}(t, x, \sigma) = \frac{1}{\sigma\sqrt{t}} \exp\left(x - \frac{1}{2}d_+^2(x) \right),$$

where $d_{\pm}(x) := (1/\sigma\sqrt{t})(x - \zeta \pm \frac{1}{2}\sigma^2t)$. Now, for any integers $m \geq 0$ and $n \geq 2$, we have

$$\begin{aligned} \partial_x^m (\partial_x^n - \partial_x) u^{\text{BS}}(t, x, \sigma) &= \partial_x^m \sum_{i=2}^n (\partial_x^i - \partial_x^{i-1}) u^{\text{BS}}(t, x, \sigma) \\ &= \sum_{i=2}^n \partial_x^{m+i-2} (\partial_x^2 - \partial_x) u^{\text{BS}}(t, x, \sigma) \\ &= \frac{1}{\sigma\sqrt{t}} \sum_{i=2}^n \partial_x^{m+i-2} \exp\left(x - \frac{d_{\pm}^2(x)}{2}\right). \end{aligned}$$

The lemma then follows from the identity $x - \frac{1}{2}d_{\pm}^2(x) = -y^2 + \zeta$ and from

$$\begin{aligned} \frac{\partial_x^m (\partial_x^n - \partial_x) u^{\text{BS}}(t, x, \sigma)}{(\partial_x^2 - \partial_x) u^{\text{BS}}(t, x, \sigma)} &= \frac{\sum_{i=2}^n \partial_x^{m+i-2} \exp(x - d_{\pm}^2(x)/2)}{\exp(x - d_{\pm}^2(x)/2)} \\ &= \sum_{i=2}^n e^{y^2} \left(\frac{1}{\sigma\sqrt{2t}}\right)^{m+i-2} \partial_y^{m+i-2} (e^{-y^2}) \\ &= \sum_{i=2}^n \left(-\frac{1}{\sigma\sqrt{2t}}\right)^{m+i-2} H_{m+i-2}(y). \end{aligned}$$

Proposition 3. *The identity*

$$\begin{aligned} \frac{\partial_{\sigma}^n u^{\text{BS}}}{\partial_{\sigma} u^{\text{BS}}}(t, x, \sigma) &= \sum_{q=0}^{\lfloor n/2 \rfloor} \sum_{p=0}^{n-q-1} \binom{n-q-1}{p} c_{n,n-2q} \sigma^{n-2q-1} t^{n-q-1} \\ &\quad \times (\sigma\sqrt{2t})^{1-p-n+q} H_{p+n-q-1}(y) \end{aligned}$$

holds, where the coefficients $(c_{n,n-2k})$ are defined recursively by $c_{n,n} = 1$ and

$$c_{n,n-2q} = (n - 2q + 1)c_{n-1,n-2q+1} + c_{n-1,n-2q-1} \quad \text{for } q \in \left\{1, 2, \dots, \left\lfloor \frac{n}{2} \right\rfloor\right\}.$$

Proof. For any integers $j \geq 2$ and $k \geq 2$, we can write

$$\begin{aligned} (\partial_x^j - \partial_x)^k &= (\partial_x^j - \partial_x)^{k-1} (\partial_x^j - \partial_x) \\ &= \sum_{l=0}^{k-1} \binom{k-1}{l} (\partial_x^j)^l (-\partial_x)^{k-1-l} (\partial_x^j - \partial_x) \\ &= \sum_{l=0}^{k-1} \binom{k-1}{l} (-1)^{k-1-l} \partial_x^{n(j-1)+k-1} (\partial_x^j - \partial_x). \end{aligned}$$

Combining this with Lemma 1 and the identity $\partial_\sigma u^{BS} = t\sigma(\partial_x^2 - \partial_x)u^{BS}$, we obtain

$$\begin{aligned} \frac{(\partial_x^j - \partial_x)^k u^{BS}}{\partial_\sigma u^{BS}} &= \frac{1}{t\sigma} \sum_{l=0}^{k-1} \binom{k-1}{l} (-1)^{k-1-l} \frac{\partial_x^{l(j-1)+k-1} (\partial_x^j - \partial_x) u^{BS}}{(\partial_x^2 - \partial_x) u^{BS}} \\ &= \frac{1}{t\sigma} \sum_{l=0}^{k-1} \sum_{i=2}^j \binom{k-1}{l} (-1)^{k-1-l} \\ &\quad \times \left(-\frac{1}{\sigma\sqrt{2t}}\right)^{l(j-1)+k-1+i-2} H_{l(j-1)+k-1+i-2}(y). \end{aligned} \tag{23}$$

Let $\mathcal{L} := t(\partial_x^2 - \partial_x)$ (so $\partial_\sigma u^{BS} = \sigma \mathcal{L} u^{BS}$); for any $n \in \mathbb{N}$, the identity

$$\partial_\sigma^n u^{BS} = \sum_{q=0}^{\lfloor n/2 \rfloor} c_{n,n-2q} \sigma^{n-2q} \mathcal{L}^{n-q} u^{BS}$$

follows from a simple recursion with the coefficients $(c_{n,n-2k})$ defined as in the proposition. Therefore, using (23) with $j = 2$ and $k = n - q$,

$$\begin{aligned} \frac{\partial_\sigma^n u^{BS}}{\partial_\sigma u^{BS}} &= \sum_{q=0}^{\lfloor n/2 \rfloor} c_{n,n-2q} \sigma^{n-2q} \frac{\mathcal{L}^{n-q} u^{BS}}{\partial_\sigma u^{BS}} \\ &= \sum_{q=0}^{\lfloor n/2 \rfloor} c_{n,n-2q} \sigma^{n-2q} t^{n-q} \frac{(\partial_x^2 - \partial_x)^{n-q} u^{BS}}{\partial_\sigma u^{BS}} \\ &= \sum_{q=0}^{\lfloor n/2 \rfloor} \sum_{p=0}^{n-q-1} c_{n,n-2q} \sigma^{n-2q-1} t^{n-q-1} \binom{n-q-1}{p} (-1)^{n-q-1-p} \\ &\quad \times \left(-\frac{1}{\sigma\sqrt{2t}}\right)^{p+n-q-1} H_{p+n-q-1}(y). \end{aligned}$$

Proposition 4. *The following equality holds:*

$$\begin{aligned} \frac{u_n^{(m)}}{\partial_\sigma u^{BS}}(t, x, \sigma) &= \frac{1}{n!t\sigma} \sum_{k=2n}^{nm} \sum_{\substack{j_1+\dots+j_n=k \\ 2 \leq j_1, \dots, j_n \leq m}} \left(\prod_{i=1}^n a_{j_i}\right) \sum_{k_1=2}^{j_1} \dots \sum_{k_n=2}^{j_n} \binom{n-1}{m} (-1)^{n-1-m} \\ &\quad \times \left(-\frac{1}{\sigma\sqrt{2t}}\right)^{-n-1+m+\sum_{j=1}^n k_j} H_{-n-1+m+\sum_{j=1}^n k_j}(y). \end{aligned}$$

Proof. An expansion (and slight reorganisation of the terms) of (10) yields

$$u_n^{(m)} = \frac{1}{n!} \left(\sum_{k=2n}^{nm} \sum_{\substack{j_1+\dots+j_n=k \\ 2 \leq j_1, \dots, j_n \leq m}} \prod_{i=1}^n a_{j_i} (\partial_x^{j_i} - \partial_x) \right) u^{BS}. \tag{24}$$

Furthermore, we have

$$\begin{aligned}
 \prod_{i=1}^n a_{j_i} (\partial_x^{j_i} - \partial_x) &= \prod_{i=1}^n a_{j_i} \sum_{k_i=2}^{j_i} (\partial_x^{k_i} - \partial_x^{k_i-1}) \\
 &= \prod_{i=1}^n a_{j_i} \sum_{k_i=2}^{j_i} \partial_x^{k_i-2} (\partial_x^2 - \partial_x) \\
 &= \left(\prod_{i=1}^n a_{j_i} \right) \sum_{k_1=2}^{j_1} \dots \sum_{k_n=2}^{j_n} \partial_x^{k_1-2} \dots \partial_x^{k_n-2} (\partial_x^2 - \partial_x)^{n-1} (\partial_x^2 - \partial_x) \\
 &= \left(\prod_{i=1}^n a_{j_i} \right) \sum_{k_1=2}^{j_1} \dots \sum_{k_n=2}^{j_n} \partial_x^{k_1-2} \dots \partial_x^{k_n-2} \binom{n-1}{m} \partial_x^{n-1+m} (-1)^{n-1-m} \\
 &\quad \times (\partial_x^2 - \partial_x) \\
 &= \left(\prod_{i=1}^n a_{j_i} \right) \sum_{k_1=2}^{j_1} \dots \sum_{k_n=2}^{j_n} \binom{n-1}{m} (-1)^{n-1-m} \partial_x^{-n-1+m+\sum_{j=1}^n k_j} (\partial_x^2 - \partial_x).
 \end{aligned}
 \tag{25}$$

Using Lemma 1 and the equality $\partial_\sigma u^{\text{BS}}(t, x, \sigma) = t\sigma(\partial_x^2 - \partial_x)u^{\text{BS}}(t, x, \sigma)$, we have

$$\begin{aligned}
 &\frac{\partial_x^{-n-1+m+\sum_{j=1}^n k_j} (\partial_x^2 - \partial_x) u^{\text{BS}}}{\partial_\sigma u^{\text{BS}}}(t, x, \sigma) \\
 &= \frac{1}{t\sigma} \left(-\frac{1}{\sigma_0\sqrt{2t}} \right)^{-n-1+m+\sum_{j=1}^n k_j} H_{-n-1+m+\sum_{j=1}^n k_j}(y).
 \end{aligned}$$

Combining this with (24) and (25) concludes the proof of the proposition.

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