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Isomorphism of the Groups of Vassiliev Invariants of Legendrian and Pseudo-Legendrian Knots

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Abstract. The study of the Vassiliev invariants of Legendrian knots was started by D. Fuchs and S. Tabachnikov who showed that the groups of $\mathbb{C}$-valued Vassiliev invariants of Legendrian and of framed knots in the standard contact $\mathbb{R}^3$ are canonically isomorphic. Recently we constructed the first examples of contact 3-manifolds where Vassiliev invariants of Legendrian and of framed knots are different. Moreover in these examples Vassiliev invariants of Legendrian knots distinguish Legendrian knots that are isotopic as framed knots and homotopic as Legendrian immersions. This raised the question what information about Legendrian knots can be captured using Vassiliev invariants. Here we answer this question by showing that for any contact 3-manifold with a cooriented contact structure the groups of Vassiliev invariants of Legendrian knots and of knots that are nowhere tangent to a vector field that coorients the contact structure are canonically isomorphic.

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1. Introduction

In this section we describe the main results of the paper. (In case any of the terminology appears to be new to the reader, the corresponding definitions are given in the next section.)

If a contact structure on a 3-manifold M is cooriented, then every Legendrian knot (i.e. a knot that is everywhere tangent to the contact distribution) has a natural framing (a continuous normal vector field). Hence, when studying Legendrian knots in such contact manifolds the main question is to distinguish those of them that realize isotopic framed knots.

On the other hand a cooriented contact structure C on a manifold M gives rise to a nondegenerate vector field V_C in TM that coorients the contact structure. Clearly if two Legendrian knots K_0 and K_1 are isotopic as Legendrian knots, then they are also isotopic in the category of knots that are everywhere nontangent to V_C.

This observation leads to the following definition. Let (M, C) be a contact manifold with a cooriented contact structure, and let V_C be the nondegenerate vector field on TM that coorients the contact structure. A knot K_0 in M is said to be *pseudo-Legendrian* if it is everywhere nontangent to V_C. Two pseudo-Legendrian knots K_0 and K_1 (in (M, C, V_C)) are *pseudo-Legendrian isotopic* if there exists an isotopy $I: [0,1] \times S^1 \to M$ such that $I\big|_{0\times S^1} = K_0$, $I\big|_{1\times S^1} = K_1$, and $\forall t \in [0,1]$ the knot $K_t = I\big|_{t\times S^1}$ is pseudo-Legendrian (with respect to V_C).

Clearly if K_0 and K_1 are Legendrian isotopic Legendrian knots, then they are also pseudo-Legendrian isotopic (with respect to any nondegenerate vector field V_C that coorients C).

Vassiliev invariants proved to be an extremely useful tool in the study of framed knots, and the conjecture is that they are sufficient to distinguish all the isotopy classes of framed knots. Vassiliev invariants can also be easily defined in the categories of Legendrian and of pseudo-Legendrian knots.

The study of the groups of Vassiliev invariants of Legendrian knots was initiated by the work [4] of D. Fuchs and S. Tabachnikov where it was proved that the groups of $\mathbb{C}$-valued Vassiliev invariants of Legendrian and of framed knots in the standard contact $\mathbb{R}^3$ are canonically isomorphic. Later the similar result was proved by J. Hill [8] for the groups of $\mathbb{C}$-valued Vassiliev invariants of Legendrian and of framed knots in the spherical cotangent bundle $ST^*\mathbb{R}^2$ of $\mathbb{R}^2$ with the standard contact structure. The proofs of these isomorphisms were based on the existence of the universal $\mathbb{C}$-valued Vassiliev invariant for these spaces, also known as the Kontsevich integral [9]. (For $ST^*\mathbb{R}^2$ such a universal invariant was first constructed by V. Goryunov [6].) Unfortunately the Kontsevich integral exists only for a rather limited collection of 3-manifolds. (Recently Andersen, Mattes, and Reshetikhin [1] constructed such an invariant for manifolds that are $\mathbb{R}^1$-fibered over an oriented surface F with $\partial F \neq \emptyset$.) For this reason the question whether the groups of Vassiliev invariants of Legendrian and of framed knots are always isomorphic was open for some time.

Recently, the author used different technique to prove [12, 13] that for any Abelian group $\mathcal{A}$ the groups of $\mathcal{A}$-valued Vassiliev invariants of Legendrian and of framed knots are canonically isomorphic for a large class of contact 3-manifolds with a cooriented contact structure. This class of contact 3-manifolds (M, C) includes all contact manifolds with a tight contact structure, all contact manifolds that are closed and admit a metric of negative sectional curvature, and all contact manifolds such that the Euler class of the contact bundle is in the torsion of $H^2(M, \mathbb{Z})$.

On the other hand [12, 13], the author constructed the first known examples of contact manifolds where the groups of Vassiliev invariants of Legendrian and of framed knots are not canonically isomorphic. In these examples Vassiliev invariants of Legendrian knots can be successfully used to distinguish Legendrian knots that realize isotopic framed knots and that are homotopic as Legendrian immersions of S^1. Namely, such examples were constructed for $M = S^1 \times S^2$ and for any M that is an orientable total space of an S^1-bundle over a nonorientable surface of genus bigger than one. This brought up a question what information about Legendrian knots can be captured with the help of Vassiliev invariants of Legendrian knots.

Here we answer this question* by proving the following Theorem, that says that the groups of Vassiliev invariants of Legendrian and of pseudo-Legendrian knots are always canonically isomorphic.

Let $\mathcal{A}$ be an Abelian group, let (M, C) be a contact 3-manifold with a cooriented contact structure, and let V_C be a nondegenerate vector field that coorients C. Let $\mathcal{L}$ be a connected component of the space of Legendrian immersions of S^1 and let $\mathcal{L}_p$ be a connected component of the space of pseudo-Legendrian immersions of S^1 (with respect to V_C) that contains $\mathcal{L}$. (Such a component always exists since a path in $\mathcal{L}$ corresponds to a path in $\mathcal{L}_p$.)

THEOREM 1. *The groups of $\mathcal{A}$-valued Vassiliev invariants of Legendrian knots from $\mathcal{L}$ and of pseudo-Legendrian knots from $\mathcal{L}_p$ are canonically isomorphic.*

See Theorem 2.2.3. In particular, if K_1 and K_2 are two Legendrian knots that are homotopic as Legendrian immersions and that realize isotopic framed knots, and x is a Vassiliev invariant of Legendrian knots such that $x(K_1) \neq x(K_2)$, then K_1 and K_2 are not isotopic as pseudo-Legendrian knots. This means that the only information about a Legendrian knot that can be captured using Vassiliev invariants of Legendrian knots is the pseudo-Legendrian isotopy class of the Legendrian knot.

2. Main Results

2.1. CONVENTIONS AND DEFINITIONS

We work in the smooth category.

In this paper $\mathcal{A}$ is an Abelian group (not necessarily torsion free), and M is a connected oriented three-dimensional Riemannian manifold (not necessarily compact).

A *contact structure* on a three-dimensional manifold M is a smooth field $\{C_x \subset T_xM | x \in M\}$ of tangent two-dimensional planes, locally defined as a kernel of a differential 1-form α with nonvanishing $\alpha \wedge d\alpha$. A manifold with a contact structure possesses the canonical orientation determined by the volume form $\alpha \wedge d\alpha$. The standard contact structure in $\mathbb{R}^3 = (x, y, z)$ is the kernel of the 1-form $\alpha = y\,dx - dz$.

A contact structure is *cooriented* if the two-dimensional planes defining the contact structure are continuously cooriented (transversally oriented). A contact structure is

*In their work, R. Benedetti and C. Petronio [2], p. 34, conjectured the fact that is very similar to the one shown in Theorem 1, but their definition of pseudo-Legendrian isotopy is different. Namely, let $\mathcal{V}$ be the space of nondegenerate vector fields on TM that are homotopic (as nondegenerate vector fields) to a vector field that coorients C. They call a pseudo-Legendrian knot in (M, C) a pair (K, V) that consists of $V \in \mathcal{V}$ and of a knot K that is everywhere nontangent to V. They say that pseudo-Legendrian knots (K_0, V_0) and (K_1, V_1) are pseudo-Legendrian isotopic if there exists a homotopy of vector fields $I_V: [0, 1] \to \mathcal{V}$ with $I_V(0) = V_0$, $I_V(1) = V_1$, and an isotopy $I: [0, 1] \times S^1 \to M$ with $I|_{0\times S^1} = K_0$, $I|_{1\times S^1} = K_1$ such that $\forall t \in [0, 1]$ $K_t = I|_{t\times S^1}$ is nowhere tangent to $I_V(t) \in \mathcal{V}$. We were not able to prove their conjecture and, moreover, we believe that it is possible to construct an example showing that the groups of Vassiliev invariants of Legendrian knots and of knots that are pseudo-Legendrian with respect to their definition are different.

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oriented if the two-dimensional planes defining the contact structure are continuously oriented. Since every contact manifold has a natural orientation we see that every cooriented contact structure is naturally oriented and every oriented contact structure is naturally cooriented.

A contact structure is *parallelizable* (*parallelized*) if the two-dimensional vector bundle $\{C_x\}$ over M is trivializable (trivialized). Since every contact manifold has a canonical orientation, one can see that every parallelized contact structure is naturally cooriented.

A *curve* in M is an immersion of S^1 into M. (All curves have the natural orientation induced by the orientation of S^1.) A *framed curve* in M is a curve equipped with a continuous unit normal vector field.

A *Legendrian curve* in a contact manifold (M, C) is a curve in M that is everywhere tangent to C. If the contact structure on M is cooriented, then every Legendrian curve has a natural framing given by the unit normals to the planes of the contact structure that point in the direction specified by the coorientation.

To a Legendrian curve K_l in a contact manifold (M, C) with a parallelized contact structure one can associate an integer that is the number of revolutions of the direction of the velocity vector of K_l (with respect to the chosen frames in C) under traversing K_l according to the orientation. This integer is called the *Maslov number* of K_l. The set of Maslov numbers enumerates the set of the connected components of the space of Legendrian curves in $\mathbb{R}^3$ (cf. 2.3.1).

For a contact manifold (M, C) with a cooriented contact structure fix a nondegenerate vector field V_C that coorients the contact structure. A *pseudo-Legendrian curve* in (M, C, V_C) is a curve that is nowhere tangent to V_C. Clearly every Legendrian curve in (M, C) *realizes a pseudo-Legendrian curve*. (This means that if L is a Legendrian curve in (M, C), then it is also a pseudo-Legendrian curve in (M, C, V_C).)

A *knot* (*framed knot*) in M is an embedding (framed embedding) of S^1 into M. In a similar way we define Legendrian knots, and pseudo-Legendrian knots in a contact manifold (M, C) with a cooriented contact structure.

A *singular* (*framed*) knot with n double points is a curve (framed curve) in M whose only singularities are n transverse double points. An *isotopy* of a singular (framed) knot with n double points is a path in the space of singular (framed) knots with n double points under which the preimages of the double points on S^1 change continuously. In a similar way we define singular Legendrian and pseudo-Legendrian knots and the notion of isotopy of singular Legendrian knots and of singular pseudo-Legendrian knots.

An $\mathcal{A}$-valued framed (resp. Legendrian, resp. pseudo-Legendrian) knot invariant is an $\mathcal{A}$-valued function on the set of the isotopy classes of framed (resp. Legendrian, resp. pseudo-Legendrian) knots.

A transverse double point t of a singular knot can be resolved in two essentially different ways. We say that a resolution of a double point is positive (resp. negative) if the tangent vector to the first strand, the tangent vector to the second strand, and the vector from the second strand to the first form the positive 3-frame. (This does not depend on the order of the strands). If the singular knot is Legendrian (resp. pseudo-Legendrian), then these resolution can be made in the category of Legendrian (resp. pseudo-Legendrian) knots.

A singular framed (resp. Legendrian, resp. pseudo-Legendrian) knot K with $(n+1)$ transverse double points admits 2^{n+1} possible resolutions of the double points. The sign of the resolution is put to be $+$ if the number of negatively resolved double points is even, and it is put to be $-$ otherwise. Let x be an $\mathcal{A}$-valued invariant of framed (resp. Legendrian, resp. pseudo-Legendrian) knots. The invariant x is said to be of *finite order* (or *Vassiliev invariant*) if there exists a nonnegative integer n such that for any singular knot K_s with $(n+1)$ transverse double points the sum (with appropriate signs) of the values of x on the nonsingular knots obtained by the 2^{n+1} resolutions of the double points is zero. An invariant is said to be of order not greater than n (of order $\leqslant n$) if n can be chosen as the integer in the definition above. The group of $\mathcal{A}$-valued finite order invariants has an increasing filtration by the subgroups of the invariants of order $\leqslant n$.

2.2. ISOMORPHISM BETWEEN THE GROUPS OF ORDER $\leqslant n$ INVARIANTS OF LEGENDRIAN AND OF PSEUDO-LEGENDRIAN KNOTS

Let (M, C) be a contact manifold with a cooriented contact structure, let V_C be a nondegenerate vector field that coorients the contact structure, and let $\mathcal{L}$ be a connected component of the space of Legendrian curves in (M, C). (The description of the set of connected components of the space of Legendrian curves is given in 2.3.1.)

Put $\mathcal{L}_p$ to be the connected component of the space of pseudo-Legendrian curves in (M, C, V_C) that contains $\mathcal{L}$. (Such a component exists because a Legendrian curve L in (M, C) is pseudo-Legendrian in (M, C, V_C). Moreover, as it is shown in Proposition 3.2.16 every component of the space of pseudo-Legendrian curves in (M, C, V_C) contains a unique component of the space of Legendrian curves in (M, C).)

Let $V_n^{\mathcal{L}}$ (resp. $V_n^{\mathcal{L}_p}$) be the group of $\mathcal{A}$-valued order $\leqslant n$ invariants of Legendrian (resp. pseudo-Legendrian) knots from $\mathcal{L}$ (resp. from $\mathcal{L}_p$). Clearly every invariant $y \in V_n^{\mathcal{L}_p}$ restricted to the category of Legendrian knots in $\mathcal{L}$ is an element $\phi(y) \in V_n^{\mathcal{L}}$. This gives a homomorphism $\phi\colon V_n^{\mathcal{L}_p} \to V_n^{\mathcal{L}}$.

We prove the following Theorems.

THEOREM 2.2.1. *$x(K_1) = x(K_2)$, for every $x \in V_n^{\mathcal{L}}$ and for every Legendrian knots $K_1, K_2 \in \mathcal{L}$ such that K_1 and K_2 are pseudo-Legendrian isotopic knots in (M, C, V_C).*

For the Proof of Theorem 2.2.1 see Section 3.1.

THEOREM 2.2.2. *The following two statements* I *and* II *are equivalent.*

(I) $\phi\colon V_n^{\mathcal{L}_p} \to V_n^{\mathcal{L}}$ *is an isomorphism.*
(II) $x(K_1) = x(K_2)$ *for every* $x \in V_n^{\mathcal{L}}$ *and for every Legendrian knots* $K_1, K_2 \in \mathcal{L}$ *such that* K_1 *and* K_2 *are pseudo-Legendrian isotopic knots in* (M, C, V_C).

The Proof of Theorem 2.2.2 becomes obvious when the mapping from the Legendrian isotopy classes of Legendrian knots from $\mathcal{L}$ to the pseudo-Legendrian isotopy classes of pseudo-Legendrian knots from $\mathcal{L}_p$ is surjective. However the famous Bennequin inequality shows that this map is not surjective even when (M, C) is the standard contact $\mathbb{R}^3$.

The Proof of Theorem 2.2.2 is given in Section 3.3.

Combining Theorems 2.2.1 and 2.2.2 we get the following.

THEOREM 2.2.3. *The groups* $V_n^{\mathcal{L}}$ *and* $V_n^{\mathcal{L}_p}$ *of* $\mathcal{A}$*-valued Vassiliev invariants of Legendrian knots from* $\mathcal{L}$ *and of pseudo-Legendrian knots from* $\mathcal{L}_p$ *are canonically isomorphic.*

2.3. SOME IMPORTANT TECHNIQUES FOR WORKING WITH LEGENDRIAN AND PSEUDO-LEGENDRIAN KNOTS

2.3.1. *h-Principle for Legendrian Curves and the Connected Components of the Space of Legendrian Curves*

For (M, C) a contact manifold with a cooriented contact structure, we put CM to be the total space of the fiberwise spherization of the contact bundle, and we put $\mathrm{pr}\colon CM \to M$ to be the corresponding locally trivial S^1-fibration. The h-principle proved for the Legendrian curves by M. Gromov ([7], pp. 338–339) says that the space of Legendrian curves in (M, C) is weak homotopy equivalent to the space of free loops ΩCM in CM. The equivalence is given by mapping a point of a Legendrian curve to the point of CM corresponding to the direction of the velocity vector of the curve at this point. *In particular the h-principle implies that the set of the connected components of the space of Legendrian curves in* (M, C) *can be naturally identified with the set of conjugacy classes of elements of* $\pi_1(CM)$.

2.3.2. *Description of Legendrian and of pseudo-Legendrian Knots in* $\mathbb{R}^3$

The contact Darboux theorem says that every contact 3-manifold (M, C) is locally contactomorphic to $\mathbb{R}^3 = (x, y, z)$ with the standard contact structure that is the kernel of the 1-form $\alpha = y\mathrm{d}x - \mathrm{d}z$. A chart in which (M, C) is contactomorphic to the standard contact $\mathbb{R}^3$ is called *a Darboux chart*.

Legendrian knots in the standard contact $\mathbb{R}^3$ are conveniently presented by the projections into the plane (x, z). Identify a point $(x, y, z) \in \mathbb{R}^3$ with the point $(x, z) \in \mathbb{R}^2$ furnished with the fixed direction of an unoriented straight line through

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(x, z) with the slope y. Then the curve in $\mathbb{R}^3$ is a one parameter family of points with non-vertical directions in $\mathbb{R}^2$.

While a generic regular curve has a regular projection into the (x, z)-plane, the projection of a generic Legendrian curve into the (x, z)-plane has isolated critical points (since all the planes of the contact structure are parallel to the y-axis). Hence the projection of a generic Legendrian curve may have cusps. A curve in $\mathbb{R}^3$ is Legendrian if and only if the corresponding planar curve with cusps is everywhere tangent to the field of directions. In particular, this field is determined by the curve with cusps. This description of a Legendrian curve is often called *the front projection description of the Legendrian curve*.

A pseudo-Legendrian knot in $(\mathbb{R}^3, \ker(y\mathrm{d}x - \mathrm{d}z))$ can be depicted as follows. Let V_C be a unit vector field on $\mathbb{R}^3$ that coorients the contact structure. Choose a system of coordinates (x', y', z') in $\mathbb{R}^3$ so that V_C is parallel to the z'-axis and points in the same direction. Then a pseudo-Legendrian knot K can be depicted by the standard knot diagram in (x', y', z'). Since K is pseudo-Legendrian it means that the velocity vector of K at every point is not parallel to the z'-axis. A pseudo-Legendrian isotopy of a pseudo-Legendrian knot can be depicted by a sequence of second and third Reidemeister moves. (The first Reidemeister move does not occur, since during this move the velocity vector of one of the points on the kink becomes parallel to the z'-axis.)

3. Proofs

3.1. PROOF OF THEOREM 2.2.1

3.1.1. *Some Useful Facts Proved by D. Fuchs and S. Tabachnikov* ([4])

There are two types of cusps arising under the projection of the part of a Legendrian knot that is contained in a Darboux chart to the (x, z)-plane (see 2.3.2). They are formed by cusps for which the branch of the projection of the knot going away from the cusp is locally located respectively above or below the tangent line at the cusp point. (See Figures 1(b) and (c) respectively.) For a Legendrian knot K and $i, j \in \mathbb{N}$ we denote by $K^{-i,-j}$ the Legendrian knot that is the same as K everywhere except of a part contained in a Darboux chart; in the Darboux chart $K^{-i,-j}$ differs

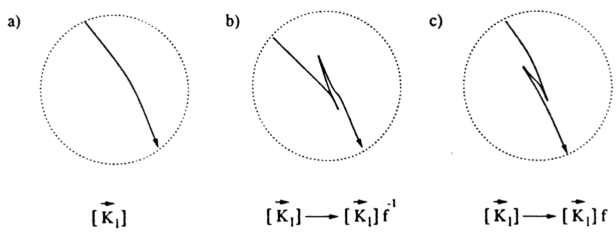

Figure 1.

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from K in the way described, see 2.3.2, by the addition of i cusp pairs of the first type and j cusp pairs of the second type to the front projection description of the part of K located in the Darboux chart. (It is possible to show that the Legendrian isotopy class of $K^{-i,-j}$ depends only on i, j and on the Legendrian isotopy class of K, and does not depend on the choice of the Darboux chart, but we will not use this fact.)

The following three facts were proved by Fuchs and Tabachnikov [4].

(1) Let K_1 and K_2 be Legendrian knots in the standard contact $\mathbb{R}^3$ that realize isotopic unframed knots. Then for any $n_1, n_2 \in \mathbb{N}$ large enough there exist $n_3, n_4 \in \mathbb{N}$ such that the Legendrian knot $K_1^{-n_1,-n_2}$ is Legendrian isotopic to $K_2^{-n_3,-n_4}$.

(2) If there exists $n \in \mathbb{N}$ such that Legendrian knots $K_1^{-n,-n}$ and $K_2^{-n,-n}$ are Legendrian isotopic, then every Vassiliev invariant of Legendrian knots takes equal values on K_1 and on K_2.

(3) The number n from the previous observation exists if the ambient contact manifold is $\mathbb{R}^3$ and the Legendrian knots K_1 and K_2 belong to the same component of the space of Legendrian curves and realize isotopic framed knots.

As it was later observed by Fuchs and Tabachnikov [5] the first two observation are true for any contact 3-manifold (since the proof of the corresponding facts is local). But the number n from the statement of the third observation does not exist in general. In the case of the ambient manifold being $\mathbb{R}^3$ Fuchs and Tabachnikov showed the existence of such n using the explicit calculation involving the Maslov class and the Bennequin invariant of Legendrian knots. However in order for the Bennequin invariant to be well-defined the knots have to be zero-homologous, and in order for the Maslov class to be well-defined the knots have to be zero-homologous or the contact structure has to be parallelizable.

Clearly to prove Theorem 2.2.1 it suffices to show that the number n from the third observation exists for any Legendrian knots K_1 and K_2 that realize isotopic pseudo-Legendrian knots in (M, C, V_C). (We assume that the contact structure C on M is cooriented.)

Let K_1 and K_2 be Legendrian knots that realize pseudo-Legendrian isotopic pseudo-Legendrian knots (in (M, C, V_C)), and let $n_1, n_2, n_3, n_4 \in \mathbb{N}$ be such that $K_1^{-n_1,-n_2}$ and $K_2^{-n_3,-n_4}$ are Legendrian isotopic.

We start by showing that if K_1 and K_2 are pseudo-Legendrian isotopic, then n_1, n_2, n_3, n_4 can be chosen so that $n_1 + n_2 = n_3 + n_4$, and that $n_1 - n_2 = n_3 - n_4$.

3.1.2. *Proof of the Fact that n_1, n_2, n_3, n_4 can be Chosen so that $n_1 + n_2 = n_3 + n_4$*

Let $I: S^1 \times [0,1] \to M$ be the pseudo-Legendrian isotopy that changes K_1 to K_2.

Analyzing the proof of Fuchs and Tabachnikov one verifies that for n_1, n_2 large enough the Legendrian isotopy $\mu: S^1 \times [0,1] \to M$ changing $K_1^{-n_1,-n_2}$ to $K_2^{-n_3,-n_4}$ can be chosen so that for every $t \in [0,1]$ the Legendrian knot $\mu^t = \mu|_{(S^1 \times t)}$ is

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contained in a thin tubular neighborhood T_t of $I_t = I\big|_{(S^1\times t)}$ and is isotopic (as an unframed knot) to μ^t inside T_t.

Every pseudo-Legendrian knot is naturally framed. Put $\bar{\mu}^t$ and $\bar{I}_t$ to be the framed knots corresponding respectively to μ^t and to I_t. For two framed knots $\bar{\mu}^t$ and $\bar{I}_t$ realizing unframed knots that are isotopic inside T_t there is a well-defined $\mathbb{Z}$-valued obstruction to be isotopic inside T_t in the category of framed knots.

This obstruction is the difference of the self-linking numbers of the inclusions of $\bar{\mu}^t$ and $\bar{I}_t$ into $\mathbb{R}^3$ induced by an identification of T_t with the standard solid torus in $\mathbb{R}^3$. (One verifies that for $\bar{\mu}^t$ and $\bar{I}_t$ that are isotopic as unframed knots inside T_t this difference does not depend on the choice of the identification of T_t with the standard solid torus in $\mathbb{R}^3$.)

From the formula for the Bennequin invariant stated in [4] one gets that the value of the obstruction for $K_1^{-n_1,-n_2}$ to be isotopic as a framed knot to K_1 inside T_0 is equal to n_1+n_2. Similarly the value of the obstruction for $K_2^{-n_3,-n_4}$ to be isotopic as a framed knot to K_2 inside T_1 is equal to n_3+n_4. Clearly the value of the obstruction for $\bar{\mu}^t$ to be isotopic to $\bar{I}_t$ inside T_t does not depend on t, and we get that $n_1+n_2=n_3+n_4$.

3.1.3. *Proof of the Fact that n_1, n_2, n_3, n_4 can be Chosen so that $n_1-n_2=n_3-n_4$*

Identify S^1 with $\{z\in\mathbb{C}\big|\ |z|=1\}$. Put $[1,i]$, $[i,-1]$, $[-1,-i]$, and $[-i,1]$ to be the four non-overlapping arcs of S^1 with the end points at $1,-1,i,-i\in S^1$.

Let $p\colon S^1\times S^1\to M$ be the mapping, such that

(1) $p\big|_{S^1\times[1,i]}$ is a homotopy of loops that changes K_1 to $K_1^{-n_1,-n_2}$ and happens in a thin tubular neighborhood of K_1;

(2) $p\big|_{S^1\times[i,-1]}$ is (up to a reparametrization) the Legendrian isotopy μ that changes $K_1^{-n_1,-n_2}$ to $K_2^{-n_3,-n_4}$;

(3) $p\big|_{S^1\times[-1,-i]}$ is a homotopy of loops that changes $K_2^{-n_3,-n_4}$ to K_2 and happens in a thin tubular neighborhood of K_2;

(4) $p\big|_{S^1\times[-i,1]}$ is (up to a reparametrization) the isotopy I^{-1} that changes K_2 to K_1.

Consider the oriented $\mathbb{R}^2$ bundle $\mathrm{pr}\colon\xi\to S^1\times S^1$ that is induced by $p\colon S^1\times S^1\to M$ from the contact bundle C on M. Since the isotopies I_K and μ were chosen so that they are C^0 close, we get that the image of the fundamental class of $S^1\times S^1$ under p_* is zero in $H_2(M)$. Thus the Euler class e_ξ of $\mathrm{pr}\colon\xi\to S^1\times S^1$ is $0\in\mathbb{Z}=H^2(S^1\times S^1)$.

Put $\mathrm{pr}_1\colon\xi_1\to S^1\times[1,i]$, $\mathrm{pr}_2\colon\xi_2\to S^1\times[i,-1]$, $\mathrm{pr}_3\colon\xi_3\to S^1\times[-1,-i]$, and $\mathrm{pr}_4\colon\xi_4\to S^1\times[-i,1]$ to be the restrictions of $\mathrm{pr}\colon\xi\to S^1\times S^1$.

The velocity vectors of Legendrian knots K_1, $K_1^{-n_1,-n_2}$, $K_2^{-n_3,-n_4}$, and K_2 give the nonzero section of $\mathrm{pr}:\xi\to S^1\times S^1$ over $\partial(S^1\times[1,i])$, $\partial(S^1\times[i,-1])$, $\partial(S^1\times[-1,-i])$, and $\partial(S^1\times[-i,1])$.

Put e_{ξ_1} to be the $\mathbb{Z}$-valued obstruction to extend the nonzero section of $\mathrm{pr}_1:\xi_1\to S^1\times[1,i]$ over $\partial(S^1\times[1,i])=S^1\times\{1\}\cup S^1\times\{i\}$ given by the velocity

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vectors of K_1 and of $K_1^{-n_1,-n_2}$ to a nonzero section of $\mathrm{pr}_1\colon \xi_1 \to S^1 \times [1, i]$ over $S^1 \times [1, i]$. (It is the relative Euler class that takes values in $\mathbb{Z} = H^2(S^1 \times [1, i], \partial(S^1 \times [1, i]))$.)

Similarly put $e_{\xi_2}, e_{\xi_3}, e_{\xi_4} \in \mathbb{Z}$ to be the obstructions to extend the nonzero sections of ξ_2, ξ_3, ξ_4 over $\partial(S^1 \times [i, -1]), \partial(S^1 \times [-1, -i])$, and $\partial(S^1 \times [-i, 1])$ that were described above to nonzero sections respectively over $S^1 \times [i, -1], S^1 \times [-1, -i]$, and $S^1 \times [-i, 1]$.

One verifies that

$$e_{\xi_1} + e_{\xi_2} + e_{\xi_3} + e_{\xi_4} = e_\xi = 0. \tag{1}$$

For a Legendrian knot K in the standard contact $\mathbb{R}^3$ put $m(K) \in \mathbb{Z}$ to be the Maslov class of K. In [4] it is shown that $m(K^{-i_1,-i_2}) = m(K) + (i_2 - i_1)$ for $i_1, i_2 \in \mathbb{Z}$ and for the Legendrian knots K and $K^{-i_1,-i_2}$ in the standard contact $\mathbb{R}^3$. A straightforward verification based on this equality shows that

$$e_{\xi_1} = n_2 - n_1 \quad \text{and} \quad e_{\xi_3} = n_3 - n_4. \tag{2}$$

The velocity vectors of the Legendrian knots $\mu(t)$ define the nonzero section of $\mathrm{pr}_2\colon \xi_2 \to S^1 \times [i, -1]$ that extends the nonzero section over $\partial(S^1 \times [i, -1])$ defined by the velocity vectors of $K_1^{-n_1,-n_2}$ and of $K_2^{-n_3,-n_4}$. Thus

$$e_{\xi_2} = 0. \tag{3}$$

Since $\forall t \in [0, 1]$ the knot $I_t = I\big|_{S^1 \times t}$ is nowhere tangent to V_C we get that for every point of I_t the projection to C along V_C of the velocity vector of I_t at this point is nonzero. Thus the isotopy I defines the nonzero section of ξ_4 over $S^1 \times [-i, 1]$ that is the extension of the nonzero section of ξ_4 over $\partial(S^1 \times [-i, 1])$ defined by the velocity vectors of K_1 and of K_2. Hence we get that

$$e_{\xi_4} = 0. \tag{4}$$

Combining together identities (1)–(4), we get that $0 = e_\xi = e_{\xi_1} + e_{\xi_2} + e_{\xi_3} + e_{\xi_4} = e_{\xi_1} + e_{\xi_3} = (n_2 - n_1) + (n_3 - n_4)$. Thus $n_1 - n_2 = n_3 - n_4$.

3.1.4. From the Identities $n_1 + n_2 = n_3 + n_4$ and $n_1 - n_2 = n_3 - n_4$ one gets that $n_1 = n_3$ and $n_2 = n_4$. Assume that $n_1 \geqslant n_2$. (The case where $n_2 > n_1$ is treated similarly.) Put $k = n_1 - n_2$. It is easy to show that since $K_1^{-n_1,-n_2}$ and $K_2^{-n_3,-n_4}$ are Legendrian isotopic, then $K_1^{-n_1,-n_2-k}$ and $K_2^{-n_3,-n_4-k}$ are also Legendrian isotopic. (Basically one can keep the k extra cusp pairs close together on a small piece of the projection of the part of the knot contained in a Darboux chart during the whole isotopy process.) But $K_1^{-n_1,-n_2-k}$ and $K_2^{-n_3,-n_4-k}$ are obtained from K_1 and

K_2 by the modification corresponding to the addition of $n_1 = n_2 + k = n_3 = n_4 + k$ pairs of cusps of each of the two types, and we can take n from the observation **2** to be $n_1 = n_2 + k = n_3 = n_4 + k$.

This shows that K_1 and K_2 can not be distinguished by the Vassiliev invariants of Legendrian knots provided that they realize isotopic pseudo-Legendrian knots. □

3.2. SOME PROPOSITION NEEDED FOR THE PROOF OF THEOREM 2.2.2

DEFINITION 3.2.1. There are four types of small kinks that can be added to a pseudo-Legendrian knot. They are kinks of types $(1,1),(1,-1),(-1,-1),(-1,1)$, shown in Figure 2. (Here the first number in the pair corresponds to the increment to the rotation number of a knot and the second to the increment to the selflinking number of the knot that occurs if one adds a kink to a pseudo-Legendrian knot in $\mathbb{R}^3$.)

Let K_p be a pseudo-Legendrian knot. For $i,j \in \mathbb{Z}$ the (i,j)-*stabilization* $K_p^{i,j}$ of K_p is a pseudo-Legendrian knot obtained from K_p by the addition of i kinks of type $(1,1)$, provided that $i \geqslant 0$, $|i|$ kinks of type $(-1,-1)$, provided that $i < 0$ and of j kinks of type $(-1,1)$, provided that $j \geqslant 0$ $|j|$ kinks of type $(1,-1)$, provided that $j < 0$.

Similarly one defines the (i,j)-*stabilization of a singular pseudo-Legendrian knot* with n transverse double points.

It is easy to verify that the pseudo-Legendrian isotopy type of the knot $K_p^{i,j}$ does not depend on the places on K_p where the kinks are added. (To get the same fact for singular pseudo-Legendrian knots one observes that it is possible to pull a kink

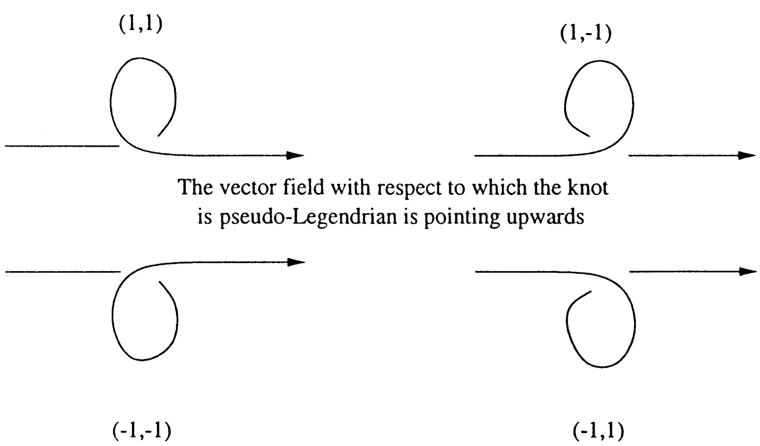

Figure 2. The four types of kinks.

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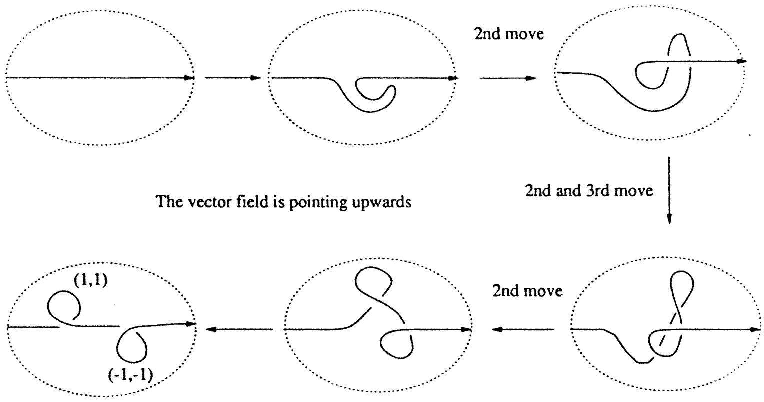

Figure 3. The creation of a (1,1) and (−1,−1) kinks by a pseudo-Legendrian isotopy.

through a transverse double point of a singular pseudo-Legendrian knot by a pseudo-Legendrian isotopy of the singular knot.)

As it is shown in Figure 3 the pair of kinks of types $(1, 1)$ and $(-1, -1)$ can be cancelled by a pseudo-Legendrian isotopy. Similar considerations show that a pair of kinks of types $(1, -1)$ and $(-1, 1)$ also can be cancelled by a pseudo-Legendrian isotopy.

Thus we get the following Proposition.

PROPOSITION 3.2.2. *For a pseudo-Legendrian knot K_p and for $i, j, k, l \in \mathbb{Z}$ the (i, j)-stabilization $(K_p^{k,l})^{i,j}$ of $K_p^{k,l}$ is pseudo-Legendrian isotopic to $K_p^{i+k,j+l}$.*

PROPOSITION 3.2.3. *Let K_1 and K_2 be pseudo-Legendrian (with respect to V_C) knots that are C^0-close isotopic to each other as unframed knots. Then there exist $i, j \in \mathbb{Z}$ such that K_1 is pseudo-Legendrian isotopic to $K_2^{i,j}$.*

3.2.4. *Proof.* Let T be a tubular neighborhood of K_1 inside which K_1 and K_2 are isotopic as unframed knots.

Identify $(T, V_C|_T)$ with the standard solid torus in $\mathbb{R}^3 = (x, y, z)$ such that both the axis of the torus and the vector field are parallel to the z-axis. Similar to 2.3.2 we can depict the pseudo-Legendrian knots K_1 and K_2 by their knot diagrams obtained by projection to an annulus A along the z-axis.

Since K_1 and K_2 are isotopic inside T as unframed knots, we get that the knot diagram of K_1 in A can be changed to a knot diagram of K_2 in A by a sequence of Reidemeister moves. The second and the third Reidemeister moves can be done in the pseudo-Legendrian category. The first Reidemeister move can not be done in the pseudo-Legendrian category since under it the velocity vector of a knot at one of the points becomes parallel to the z-axis, and hence the knot becomes tangent to V_C at one point.

There are four types of first Reidemeister move. They are distinguished by the four possible types of kinks that appear under them, see Figure 2.

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As it is shown in Figure 3 one can create (or annihilate) a $\{(1,1);(-1,-1)\}$-pair of kinks by a pseudo-Legendrian isotopy. Similar considerations show that one can create (or annihilate) a $\{(1,-1);(-1,1)\}$-pair of kinks by a pseudo-Legendrian isotopy.

Using this observation we can imitate the isotopy of unframed knots that changes K_1 to K_2 by a pseudo-Legendrian isotopy. Namely, if for example the isotopy that changes K_1 to K_2 as an unframed knot involves the first Reidemeister that creates the $(1,1)$-kink, instead of this move we perform the pseudo-Legendrian isotopy that creates a $\{(1,1);(-1,-1)\}$-pair of kinks, make the $(-1,-1)$ kink very small and keep it very small during the later isotopy process.

In the end of this imitation process we get that K_1 is pseudo-Legendrian isotopic to a knot that looks exactly as K_2 except of a number of small extra kinks that are present on it.

Cancel by a pseudo-Legendrian isotopy the $\{(1,1),(-1,-1)\}$ pairs of extra kinks either till there are no extra kinks of type $(1,1)$ left or till there are no extra kinks of type $(-1,-1)$ left. (Observe that we can bring any pair of kinks to be close together by a pseudo-Legendrian isotopy. For this we make one of the kinks small and slide it along the knot to the desired position.)

Cancel by a pseudo-Legendrian isotopy the $\{(1,-1),(-1,1)\}$ pairs of extra kinks either till there are no extra kinks of type $(1,-1)$ left or till there are no extra kinks of type $(-1,1)$ left.

It is clear that as the result of this process we obtain the pseudo-Legendrian isotopy of K_1 to $K_2^{i,j}$, for some $i,j,\in\mathbb{Z}$. □

PROPOSITION 3.2.5. *Let K_1 and K_2 be pseudo-Legendrian knots, and $i,j\in\mathbb{Z}$ be such that K_1 and $K_2^{i,j}$ are pseudo-Legendrian isotopic. Then for any $k,l\in\mathbb{Z}$ the knots $K_1^{k,l}$ and $K_2^{i+k,j+l}$ are also pseudo-Legendrian isotopic.*

3.2.6. *Proof.* Make the $|k|$ and $|l|$ extra kinks used to obtain $K_1^{k,l}$ very small and concentrate them on a small piece of K_1. Keep them small and close together during the isotopy process that was connecting K_1 and $K_2^{i,j}$. As a result we get a pseudo-Legendrian isotopy between $K_1^{k,l}$ and $(K_2^{i,j})^{k,l}$. Finally Proposition 3.2.2 says that $(K_2^{i,j})^{k,l}$ is pseudo-Legendrian isotopic to $K_2^{i+k,j+l}$. □

PROPOSITION 3.2.7. *Let K be a pseudo-Legendrian knot, that is pseudo-Legendrian isotopic to a Legendrian knot (i.e. its pseudo-Legendrian isotopy type is realizable by a Legendrian knot). Then for any $i,j\in\mathbb{N}$ the pseudo-Legendrian isotopy class of $K^{-i,-j}$ is also realizable by a Legendrian knot.*

3.2.8. *Proof of Proposition 3.2.7.* One observes that if K_l is a Legendrian knot that is pseudo-Legendrian isotopic to K, then the Legendrian knot obtained by the modification shown in Figure 1b is pseudo-Legendrian isotopic to $K^{-1,0}$. Similarly the Legendrian knot obtained by the modification shown in Figure 1c is pseudo-Legendrian isotopic to $K^{0,-1}$. Performing the two modifications i and j times respectively we get the Legendrian knot that is pseudo-Legendrian isotopic to $K^{-i,-j}$. □

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PROPOSITION 3.2.9. *Every component of the space of pseudo-Legendrian curves contains at most one component of the space of Legendrian curves.*

In Fact every component of the space of pseudo-Legendrian curves contains exactly one component of the space of Legendrian curves, but to prove this statement, see Proposition 3.2.16 *we have to use Proposition* 3.2.14 *that is in turn based on Proposition* 3.2.9.

3.2.10. *Proof.* The h-principle, see 2.3.1, says that the connected components of the space of Legendrian curves in (M, C) are in one to one correspondence with the conjugacy classes of elements of $\pi_1(CM)$. Under this identification the connected component of the space of Legendrian curves that contains a Legendrian curve K_l corresponds to the conjugacy class of $\vec{K}_l \in \pi_1(CM, \vec{K}_l(1))$, where the loop $\vec{K}_l$ is obtained by mapping $t \in S^1$ to the point of CM that corresponds to the velocity vector of K_l at $K_l(t)$.

Let $K_{0,l}$ and $K_{1,l}$ be Legendrian curves that are in $\mathcal{L}_p$. Consider a pseudo-Legendrian homotopy $H\colon [0,1] \times S^1 \to M$ that connects $K_{1,l}$ and $K_{2,l}$. Observe that a pseudo-Legendrian curve K also defines a loop $\vec{K}$ in CM by mapping t in S^1 to the point of CM that corresponds to the projection of the velocity vector of K at $K(t)$ to a contact plane along V_C. (Since K is pseudo-Legendrian the projection is nonzero.)

Thus the family of pseudo-Legendrian curves $K_t = H\big|_{t\times S^1}$ defines the free homotopy of loops $\vec{K}_{1,l}$ and $\vec{K}_{2,l}$. Hence, by the h-principle $K_{1,l}$ and $K_{2,l}$ belong to the same component of the space of Legendrian curves.

Since $K_{1,l}$ and $K_{2,l}$ were arbitrary Legendrian curves from $\mathcal{L}_p$ we get that $\mathcal{L}_p$ contains at most one component of the space of Legendrian curves. □

PROPOSITION 3.2.11. *Let $\mathcal{L}_p$ be a connected component of the space of pseudo-Legendrian curves, and let $K \in \mathcal{L}_p$ be a pseudo-Legendrian knot. Then for any $i \in \mathbb{Z}$ the pseudo-Legendrian knots $K^{i,i}$ also belong to $\mathcal{L}_p$.*

3.2.12. *Proof.* The pseudo-Legendrian homotopies shown in Figures 3 and 4 imply that for any $m > 0$ the knot $K^{-m,-m}$ is in $\mathcal{L}_p$. Similar considerations show that $K^{m,m}$ is also in $\mathcal{L}_p$. □

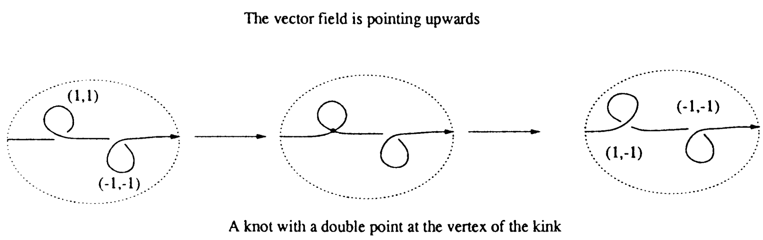

Figure 4. A pseudo-Legendrian homotopy changing the $\{(1,1), (-1,-1)\}$ pair of kinks to a $\{(1,-1), (-1,-1)\}$ pair of kinks.

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PROPOSITION 3.2.13. *The statements of Propositions* 3.2.2, 3.2.3, 3.2.5, 3.2.7*, and* 3.2.11 *are true if one substitutes everywhere in the statements of these Propositions pseudo-Legendrian knots by singular pseudo-Legendrian knots, and Legendrian knots by singular Legendrian knots.*

The Proof of this Proposition is straightforward. One has to observe that it is possible to pull a kink through a transverse double point of a singular pseudo-Legendrian knot by a pseudo-Legendrian isotopy of the singular knot.

LEMMA 3.2.14. *Let* (M, C) *be a contact manifold with a cooriented contact structure and let* V_C *be a vector field that coorients the contact structure. Let* $\mathcal{L}$ *be a connected component of the space of Legendrian curves in* (M, C) *and let* $\mathcal{L}_p$ *be the corresponding component of the space of pseudo-Legendrian* (*with respect to* V_C) *curves.*

(*a*) *Let* $K \in \mathcal{L}_p$ *be a pseudo-Legendrian knot, then there exists* $n \in \mathbb{N}$ *such that the knot* $K^{-n,-n}$ *is pseudo-Legendrian isotopic to a Legendrian knot from* $\mathcal{L}$.
(*b*) *Let* $K_s \in \mathcal{L}_p$ *be a singular pseudo-Legendrian knot* (*whose only singularities are i transverse double points*), *then there exists* $n \in \mathbb{N}$ *such that the knot* $K_s^{-n,-n}$ *is pseudo-Legendrian isotopic to a singular Legendrian knot from* $\mathcal{L}$.

3.2.15. *Proof*. We prove statement b of Lemma 3.2.14. The Proof of statement a of Lemma 3.2.14 is obtained as a particular case of the proof of b, when the number of double points of a singular knot is zero.

The result of W. L. Chow [3] and P. K. Rashevskii [11] says that every unframed knot K is isotopic to a Legendrian knot K_l (and this isotopy can be made C^0-small). Similar considerations show that every singular unframed knot K_{us} with n double points is isotopic to a singular Legendrian knot (and this isotopy can be made C^0-small).

Let K_{ls} be a singular Legendrian knot that is C^0 small isotopic to K_s as an unframed knot. By the version of Proposition 3.2.3 for singular knots, see 3.2.13, we get that there exist $i, j \in \mathbb{Z}$ such that $K_s^{i,j}$ is pseudo-Legendrian isotopic to K_{ls}.

The version of Proposition 3.2.7 for singular knots, see 3.2.13, says that the pseudo-Legendrian isotopy classes of singular knots $K_s^{i-1,j}$ and of $K_s^{i,j-1}$ are also realizable by singular Legendrian knots. By the versions of Propositions 3.2.2 and 3.2.5 for singular knots, see 3.2.13, we get that $K_s^{i-1,j}$ is pseudo-Legendrian isotopic to $K_{ls}^{-1,0}$ and $K_s^{i,j-1}$ is pseudo-Legendrian isotopic to $K_{ls}^{0,-1}$. Using these facts we get that there exist $m \in \mathbb{N}$ such that $K_s^{-m,-m}$ is pseudo-Legendrian isotopic to a singular Legendrian knot.

The version of Proposition 3.2.11 for singular knots, see 3.2.13, says that $K_s^{-m,-m}$ is in $\mathcal{L}_p$. Since by Propositions 3.2.9 there is at most one component of the space of Legendrian curves contained in $\mathcal{L}_p$, we get that $K_s^{-m,-m}$ is realizable by a singular Legendrian knot from $\mathcal{L}$. This finishes the proof of statement b of Lemma 3.2.14. □

PROPOSITION 3.2.16. *Every component of the space of pseudo-Legendrian curves contains exactly one component of the space of Legendrian curves.*

3.2.17. *Proof.* Let $\mathcal{L}_p$ be a connected component of the space of pseudo-Legendrian curves, and let $K \in \mathcal{L}_p$ be a pseudo-Legendrian knot. One verifies that the proof of Lemma 3.2.14 contains the proof of the fact that there exists $n \in \mathbb{N}$ such that $K^{-n,-n} \in \mathcal{L}_p$ is pseudo-Legendrian isotopic to a Legendrian knot. In particular this means that $\mathcal{L}_p$ contains at least one component of the space of Legendrian curves. Combining this with Proposition 3.2.9 we get the statement of the Proposition. □

3.3. PROOF OF THEOREM 2.2.2

The fact that statement I of Theorem 2.2.2 implies statement II is clear. Thus we have to show that statement II implies statement I. This is done by showing that there exists a homomorphism $\psi\colon V_n^{\mathcal{L}} \to V_n^{\mathcal{L}_p}$ such that $\phi \circ \psi = \mathrm{id}_{V_n^{\mathcal{L}}}$ and $\psi \circ \phi = \mathrm{id}_{V_n^{\mathcal{L}}p}$.

Let $x \in V_n^{\mathcal{L}}$ be an invariant. In order to construct $\psi(x) \in V_n^{\mathcal{L}_p}$ we have to specify the value of $\psi(x)$ on every pseudo-Legendrian knot $K \in \mathcal{L}_p$.

3.3.1. *Definition of $\psi(x)$*

If the pseudo-Legendrian isotopy class of the knot $K \in \mathcal{L}_p$ is realizable by a Legendrian knot $K_l \in \mathcal{L}$, then put $\psi(x)(K) = x(K_l)$. The value $\psi(x)(K)$ is well-defined because if $K_l' \in \mathcal{L}$ is another knot realizing K, then $x(K_l) = x(K_l')$ by statement I of Theorem 2.2.3.

Fix a pseudo-Legendrian knot K. We explain how to define the value of $\psi(x)$ on the pseudo-Legendrian isotopy classes of all the $K^{m,m}$, $m \in \mathbb{Z}$. (Proposition 3.2.11 says that all the $K^{m,m}$ also belong to $\mathcal{L}_p$.)

There are two cases: either 1) all the isotopy classes of $K^{m,m}$, $m \in \mathbb{Z}$, are realizable by Legendrian knots from $\mathcal{L}$ or 2) there exists $q \in \mathbb{Z}$ such that the class of $K^{q,q}$ is realizable by a Legendrian knot from $\mathcal{L}$ (see 3.2.14) and the class of $K^{q+1,q+1} \in \mathcal{L}_p$ is not realizable by a Legendrian knot from $\mathcal{L}$. (In this case the classes of $K^{q+2,q+2}$, $K^{q+3,q+3}$ etc. also are not realizable by Legendrian knots from $\mathcal{L}$, see 3.2.7.) In the case 1) the value of $\psi(x)$ is already defined on all the pseudo-Legendrian knots from $\mathcal{L}_p$ that are pseudo-Legendrian isotopic to $K^{m,m}$, for some $m \in \mathbb{Z}$.

In case (2), Propositions 3.2.5 and 3.2.7 imply that the pseudo-Legendrian isotopy classes of

$$K^{q+1,q+1}, K^{q+2,q+2}, K^{q+3,q+3}, \ldots \tag{5}$$

are all pairwise distinct. We put

$$\psi(x)(K^{q+1,q+1}) = \sum_{i=1}^{n+1}\left((-1)^{i+1}\frac{(n+1)!}{i!(n+1-i)!}\psi(x)(K^{q+1-i,q+1-i})\right). \tag{6}$$

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(Proposition 3.2.7 implies that the sum on the right-hand side is well-defined.) Similarly we put

$$\psi(x)(K^{q+2,q+2}) = \sum_{i=1}^{n+1}\left((-1)^{i+1}\frac{(n+1)!}{i!(n+1-i)!}\psi(x)(K^{q+2-i,q+2-i})\right),$$
$$\psi(x)(K^{q+3,q+3}) = \sum_{i=1}^{n+1}\left((-1)^{i+1}\frac{(n+1)!}{i!(n+1-i)!}\psi(x)(K^{q+3-i,q+3-i})\right), \text{ etc.} \tag{7}$$

Doing this for all the classes of (m,m)-stabilization equivalence for which case **2** holds we define the value of $\psi(x)$ on all the knots from $\mathcal{L}_p$.

Below we show that $\psi(x)$ is an order $\leqslant n$ invariant of knots from $\mathcal{L}_p$. We start by proving the following Proposition.

PROPOSITION 3.3.2. *Let $K^{q+1,q+1}$ be a pseudo-Legendrian knot from $\mathcal{L}_p$, then $\psi(x)$ defined as above satisfies identity* (6).

3.3.3. *Proof.* If the pseudo-Legendrian isotopy class of $K^{q+1,q+1}$ is not realizable by a Legendrian knot from $\mathcal{L}$, then the statement of the proposition follows from the formulas (6), (7) we used to define $\psi(x)(K^{q+1,q+1})$.

If the pseudo-Legendrian isotopy class of $K^{q+1,q+1}$ is realizable by a Legendrian knot K_l, then consider a singular Legendrian knot K_{ls} with $(n+1)$ double points that are vertices of $(n+1)$ small kinks such that we get K_l if we resolve all the double points positively staying in the class of the Legendrian knots. (To create K_{ls} we perform the first half of the homotopy shown in Figure 5 in $n+1$ places on K_l.)

Let Σ be the set of the 2^{n+1} possible resolutions of the double points of K_{ls}. For $\sigma \in \Sigma$ put sign(σ) to be the sign of the resolution, and put K_{ls}^{σ} to be the nonsingular Legendrian knot obtained via the resolution σ. Since x is an order $\leqslant n$ invariant of Legendrian knots we get that

$$\begin{aligned}0 &= \sum_{\sigma\in\Sigma}\left(\operatorname{sign}(\sigma)x(K_{ls}^{\sigma})\right)\\ &= \psi(x)(K^{q+1,q+1}) + \sum_{i=1}^{n+1}(-1)^i\frac{(n+1)!}{i!(n+1-i)!}\psi(x)(K^{q+1-i,q+1-i}).\end{aligned} \tag{8}$$

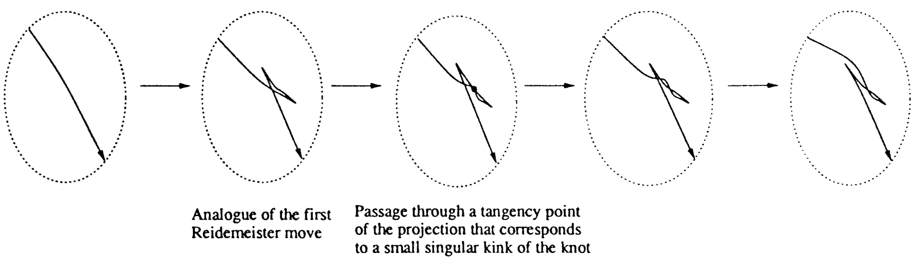

Figure 5.

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(A straightforward verification shows that if we resolve i double points of K_{ls} negatively, then we get the pseudo-Legendrian isotopy class of $K^{q+1-i,q+1-i}$.) This finishes the proof of the Proposition. □

3.3.4. Let $K_s \in \mathcal{L}_p$ be a singular pseudo-Legendrian knot with $(n+1)$ double points. Let Σ be the set of the 2^{n+1} possible resolutions of the double points of K_s. For $\sigma \in \Sigma$ put sign(σ) to be the sign of the resolution, and put K_s^σ to be the pseudo-Legendrian isotopy class of the knot obtained via the resolution σ.

In order to prove that $\psi(x)$ is an order $\leqslant n$ invariant of framed knots from $\mathcal{L}_p$, we have to show that

$$0 = \sum_{\sigma \in \Sigma} (\mathrm{sign}(\sigma)\psi(x)(K_s^\sigma)), \tag{9}$$

for every singular $K_s \in \mathcal{L}_p$ with $(n+1)$ double points.

First we observe that the fact whether identity (9) holds or not depends only on the pseudo-Legendrian isotopy class of the singular pseudo-Legendrian knot K_s with $(n+1)$ double points.

If the isotopy class of K_s is realizable by a singular Legendrian knot from $\mathcal{L}$, then identity (9) holds for K_s, since x is an order $\leqslant n$ invariant of Legendrian knots (and the value of $\psi(x)$ on a pseudo-Legendrian knot $K \in \mathcal{L}_p$ realizable by a Legendrian knot $K_l \in \mathcal{L}$ was put to be $x(K_l)$).

Lemma 3.2.14(b) and the version of 3.2.11 for singular knots, see 3.2.13, imply that there exists $r \in \mathbb{N}$ such that the isotopy class of the singular pseudo-Legendrian knot $K_s^{-r,-r} \in \mathcal{L}_p$ is realizable by a singular Legendrian knot from $\mathcal{L}$.

If all the isotopy classes of singular pseudo-Legendrian knots $K_s^{m,m} \in \mathcal{L}_p, m \in \mathbb{Z}$, are realizable by singular Legendrian knots from $\mathcal{L}$, then (9) automatically holds for K_s.

Otherwise Lemma 3.2.14.b and the version of Proposition 3.2.7 for singular knots (see 3.2.13) imply that there exists $q \in \mathbb{Z}$ such that $K_s^{q,q}$ is realizable by a singular Legendrian knot from $\mathcal{L}$ and such that $K_s^{q+1,q+1}, K_s^{q+2,q+2}, K_s^{q+3,q+3}, \ldots$ are not realizable by a singular Legendrian knot from $\mathcal{L}$.

The version of Proposition 3.2.7 for singular knots, see 3.2.13, says that $K_s^{q-i,q-i}$, $i > 0$, are realizable by singular Legendrian knots from $\mathcal{L}$ and hence identity (9) holds for $K_s^{q-i,q-i}$, $i \geqslant 0$. Using Proposition 3.3.2 and the fact that identity (9) holds for $K_s^{q-i,q-i}$, $i \geqslant 0$, we show that (9) holds for $K_s^{q+1,q+1}$. Namely,

$$\begin{aligned}
&\sum_{\sigma \in \Sigma} \mathrm{sign}(\sigma)\psi(x)((K_s^{q+1,q+1})^\sigma) \\
&\quad = \sum_{\sigma \in \Sigma}\left(\mathrm{sign}(\sigma)\sum_{i=1}^{n+1}(-1)^{i+1}\frac{(n+1)!}{i!(n+1-i)!}\psi(x)((K_s^{(q+1-i),(q+1-i)})^\sigma)\right) \\
&\quad = \sum_{i=1}^{n+1}\left((-1)^{i+1}\frac{(n+1)!}{i!(n+1-i)!} \times \left(\sum_{\sigma \in \Sigma}\mathrm{sign}(\sigma)\psi(x)((K_s^{(q+1-i),(q+1-i)})^\sigma)\right)\right) \\
&\quad = \sum_{i=1}^{n+1}\left((-1)^{i+1}\frac{(n+1)!}{i!(n+1-i)!} \times (0)\right) = 0.
\end{aligned} \tag{10}$$

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Similarly we show that (9) holds for $K_s^{q+2,q+2}, K_s^{q+3,q+3}$, etc $\cdots$.

3.3.5. Clearly ψ is a homomorphism and $\phi \circ \psi = \mathrm{id}_{V_n^{\mathcal{L}}}$.

Considering the values of $y \in V_n^{\mathcal{L}_p}$ on the 2^{n+1} possible resolutions of a singular pseudo-Legendrian knot with $n+1$ singular fragments shown in the middle part of Figure 4 we get that y should satisfy identity (6). Hence, $\psi \circ \phi = \mathrm{id}_{V_n^{\mathcal{L}_p}}$ and this finishes the proof of Theorem 2.2.3. □

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