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Cotangent Sums Related to the Riemann Hypothesis for Various Shifts of the Argument

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Abstract. One of the approaches to the Riemann Hypothesis is the Nyman–Beurling criterion. Cotan-gent sums play a significant role in this criterion. Here we investigate the values of these cotangent sums for various shifts of the argument.

1 Introduction

In several papers ([7–11]) the authors have investigated the distribution of the cotangent sums

$$c_0\left(\frac{r}{b}\right) := -\sum_{m=1}^{b-1} \frac{m}{b} \cot\left(\frac{\pi m r}{b}\right),$$

where $r, b \in \mathbb{N}$, $b \geq 2$, $1 \leq r \leq b$, and $(r, b) = 1$.

They could establish a link with the function $g(\alpha)$, defined as follows.

Definition 1.1 Let

$$g(\alpha) := \sum_{l=1}^{+\infty} \frac{1 - 2\{l\alpha\}}{l}, \quad \alpha \in (0,1), \quad \text{where } \{u\} := u - \lfloor u \rfloor,\ u \in \mathbb{R},$$

which is convergent for almost all α (see [5]).

Definition 1.2 For $z \in \mathbb{R}$, let

$$F(z) := \text{meas}\{\alpha \in (0,1) : g(\alpha) \leq z\},$$

where "meas" denotes the Lebesgue measure.

Let μ be the uniquely defined positive measure on $\mathbb{R}$ with the following property. For $\alpha < \beta \in \mathbb{R}$, we have $\mu([\alpha, \beta]) = F(\beta) - F(\alpha)$. We set

$$C_0(\mathbb{R}) := \Big\{f \in C(\mathbb{R}) : \forall \epsilon > 0,\ \exists \text{ a compact set } \mathcal{K} \subset \mathbb{R}, \\ \text{such that } |f(x)| < \epsilon, \forall x \notin \mathcal{K}\Big\}.$$

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The second author established the following result in his thesis [13] (see also [8, Theorem 1.2]). Let A_0, A_1 be fixed constants, such that $1/2 < A_0 < A_1 < 1$. For all $f \in C_0(\mathbb{R})$, we have

$$\lim_{b\to+\infty} \frac{1}{(A_1 - A_0)\phi(b)} \sum_{\substack{(r,b)=1 \\ A_0 b\le r\le A_1 b}} f\left(\frac{1}{b} c_0\left(\frac{r}{b}\right)\right) = \int f\, d\mu, \tag{1.1}$$

where $\phi(\cdot)$ denotes the Euler phi-function.

Later, S. Bettin [3] replaced the inequality $1/2 < A_0 < A_1 < 1$ by $0 < A_0 < A_1 \le 1$. In [12] the authors considered the distribution of the values of c_0 for rational numbers with primes as numerator and a fixed prime as denominator and proved a result analogous to (1.1) ([12, Theorem 1.3]). Namely, they proved that for all $f \in C_0(\mathbb{R})$, the following holds true:

$$\lim_{\substack{q\to+\infty \\ q \text{ prime}}} \frac{\log q}{(A_1 - A_0)q} \sum_{\substack{A_0 q\le p\le A_1 q \\ p \text{ prime}}} f\left(\frac{1}{q} c_0\left(\frac{p}{q}\right)\right) = \int f\, d\mu.$$

The cotangent sum $c_0(r/b)$ has gained importance in the Nyman–Beurling criterion for the Riemann Hypothesis through its relation with the Vasyunin sum, which is defined by

$$V\left(\frac{r}{b}\right) := \sum_{m=1}^{b-1} \left\{\frac{mr}{b}\right\} \cot\left(\frac{\pi m r}{b}\right).$$

In this paper, we simultaneously consider the values of c_0 for various shifts of the argument. We consider general numerators as well as prime numerators. For simplicity, the denominator will be a fixed prime q. Our main results are the following.

Theorem 1.3 *Let A_0, A_1 be fixed constants, such that $1/2 < A_0 < A_1 < 1$. Let $a_1, a_2, \dots, a_L$ be distinct non-negative integers. Let $f_1, f_2, \dots, f_L \in C_0(\mathbb{R})$ $(L \in \mathbb{N})$. Then we have*

(i)

$$\lim_{\substack{q\to+\infty \\ q \text{ prime}}} \frac{1}{\phi(q)} \sum_{A_0 q\le r\le A_1 q} \frac{1}{(A_1 - A_0)^L} \prod_{l=1}^{L} f_l\left(c_0\left(\frac{r + a_l}{q}\right)\right) = \prod_{l=1}^{L} \left(\int f_l\, d\mu\right),$$

(ii)

$$\lim_{\substack{q\to+\infty \\ q \text{ prime}}} \frac{\log q}{q} \sum_{\substack{A_0 q\le p\le A_1 q \\ p \text{ prime}}} \frac{1}{(A_1 - A_0)^L} \prod_{l=1}^{L} f_l\left(c_0\left(\frac{p + a_l}{q}\right)\right) = \prod_{l=1}^{L} \left(\int f_l\, d\mu\right).$$

We give a detailed proof of (i) and sketch the changes needed for the proof of (ii).

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2 Outline of the Proof

Several fundamental ideas appear in the paper [8] and in the thesis [13]. The key to the treatment of the sum $c_0(r/q)$ lies in its relation to the sum

$$Q\Big(\frac{r}{q}\Big) := \sum_{m=1}^{q-1} \cot\Big(\frac{\pi m r}{q}\Big)\Big\lfloor \frac{mr}{q} \Big\rfloor.$$

The second author established the following representation in his thesis [13] (see also [8, Proposition 1.8]):

$$c_0\Big(\frac{r}{q}\Big) = \frac{1}{r}\, c_0\Big(\frac{1}{q}\Big) - \frac{1}{r} Q\Big(\frac{r}{q}\Big).$$

We now have to consider simultaneously the values

$$f_1\bigg(Q\Big(\frac{r+a_1}{q}\Big)\bigg), \dots, f_L\bigg(Q\Big(\frac{r+a_L}{q}\Big)\bigg).$$

By the Weierstrass approximation theorem, this question can be reduced to the study of the joint distribution of the products

$$\prod(k_1, \dots, k_L) := Q\Big(\frac{r+a_1}{q}\Big)^{k_1} \cdots Q\Big(\frac{r+a_L}{q}\Big)^{k_L} \tag{2.1}$$

for L-tuplets $(k_1, \dots, k_L)$ of non-negative integers.

In a similar fashion as in the previous papers, we will break up the range of summation into subintervals in which

$$\Big\lfloor \frac{(r+a_l)m}{q} \Big\rfloor$$

assumes constant values.

Definition 2.1 For $j \in \mathbb{N}$, $l \in \{1, \dots, L\}$, we set

$$S_j^{(l)} := \{(r+a_l)m : qj \le (r+a_l)m < q(j+1)\}$$

and write

$$S_j^{(l)} := \{qj + s_j^{(l)}, qj + s_j^{(l)} + (r+a_l), \dots, qj + s_j^{(l)} + (r+a_l)d_j^{(l)}\}.$$

We also define $t_j^{(l)}$ by

$$qj + s_j^{(l)} + d_j^{(l)}(r+a_l) + t_j^{(l)} := q(j+1).$$

In [8,12] the map

$$j \longrightarrow s_j$$

and its inverse

$$s \longrightarrow j(s)$$

were very important. They are now replaced by L maps

$$j \longrightarrow s_j^{(l)} \ \ (1 \le l \le L)$$

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and their inverses

$$s \longrightarrow j_l(s).$$

We also have L pairs of congruences

$$s_j^{(l)} \equiv -qj\big(\bmod\ (r+a_l)\big) \quad \text{and} \quad t_j^{(l)} \equiv q(j+1)(\bmod(r+a_l)). \tag{2.2}$$

Each of the sums

$$Q\Big(\frac{r+a_l}{q}\Big)$$

is dominated by small values of $s_j^{(l)}$ and $t_j^{(l)}$, because of the poles of the function $\cot(\pi x)$ at $x = 0$ and $x = 1$. We denote this partial sum by

$$Q_0\Big(\frac{r+a_l}{q}\Big),$$

and thus we obtain the decomposition

$$Q\Big(\frac{r+a_l}{q}\Big) = Q_0\Big(\frac{r+a_l}{q}\Big) + Q_1\Big(\frac{r+a_l}{q}\Big), \quad \text{(see Definition 5.1)}.$$

The function $\cot(\pi x)$ is antisymmetric

$$\cot\big(\pi(1-x)\big) = -\cot(\pi x).$$

Therefore, there will be considerable cancellation in the sums

$$Q_1\Big(\frac{r+a_l}{q}\Big),$$

which will thus be small. By the binomial theorem, the products

$$\prod(k_1, \dots, k_L)$$

in (2.1) will be linear combinations of products of the form

$$Q_{\epsilon_1}\Big(\frac{r+a_{l_1}}{q}\Big)^{h_1} \cdots Q_{\epsilon_M}\Big(\frac{r+a_{l_M}}{q}\Big)^{h_M} \tag{2.3}$$

with $\epsilon_g \in \{0,1\}$.

We will show that only the products with $\epsilon_g = 0$ for $1 \le g \le M$ will give a substantial contribution.

The asymptotic size of these products will be determined by localising the solutions of the congruences (2.1) simultaneously for all l. Whereas in [8] Kloosterman sums could be used for this localisation, here we need results on more general exponential sums in finite fields, due to Weil [15, 16], as well as Fouvry and Michel [6].

The contribution of the other products in (2.3) is small, since at least one $\epsilon_g = 1$.

For the discussion of these factors,

$$Q_1\Big(\frac{r+a_{l_g}}{q}\Big)^{h_g},$$

we can refer to the results of [8].

3 Exponential Sums in Finite Fields

Lemma 3.1 *Let q be a prime number. Let $a_1, \ldots, a_L$ be distinct non-negative integers, $n, m_1, \ldots, m_L$ integers not all 0,*

$$R(x) = nx + \frac{m_1}{x+a_1} + \cdots + \frac{m_L}{x+a_L}.$$

Then we have

$$\sum_{\substack{x=1 \\ x \neq -a_l,\ 1\leq l\leq L}}^{q-1} e\Big(\frac{R(x)}{q}\Big) = O(q^{1/2}).$$

The constant implied by the O-symbol may depend on L.

Proof This follows from the work of Weil [15, 16]. ∎

Lemma 3.2 *Let $\mathbb{F}_r$ be the finite field with r elements and let ψ be a non-trivial additive character over $\mathbb{F}_r$, f a rational function of the form*

$$f(x) = \frac{P(x)}{Q(x)},$$

P and Q relatively prime monic non-constant polynomials,

$$S(f; r, x) := \sum_{p\leq x} \psi(f(p)).$$

(p denotes the p-fold sum of the element 1 in $\mathbb{F}_r$). Then we have

$$S(f; r, x) \ll r^{3/16+\epsilon}\, x^{25/32}.$$

The implied constant depends only on ϵ and the degrees of P and Q.

Proof This is due to Fouvry and Michel [6]. ∎

4 Localizations of the Solutions of the Congruences

We now come to the localization of the solutions of the congruences in (2.2). This will be achieved in Lemma 4.1, which is a multidimensional version of [8, Lemma 4.3]. As in [8, Lemma 4.3], as well as our Lemma 4.1 the localization of variables is achieved by Fourier analysis. Thus the calculations in this paper bear a close resemblance to those of [8].

Lemma 4.1 *Let q be prime. Let $1/2 < A_0 < A_1 < 1$ and $r \in \mathbb{N}$. Let $q^*(r; l)$ be defined by $qq^*(r; l) \equiv 1(\mathrm{mod}(r + a_l))$. Let $\alpha_1, \ldots, \alpha_L \in (0, 1)$, $\delta > 0$, such that*

$$\alpha_l + \delta < 1 \text{ for } 1 \leq l \leq L.$$

Then we have

$$N(\alpha_1, \ldots \alpha_L, \delta) := \Big|\Big\{r : r \in \mathbb{N},\ A_0 q \leq r \leq A_1 q,\ \alpha_l \leq \frac{q^*(r; l)}{r} \leq \alpha_l + \delta\Big\}\Big|$$
$$= \delta^L (A_1 - A_0) q (1 + o(1)), \quad q \longrightarrow \infty.$$

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Proof In the sequel, we assume $1 \le l \le L$. If q is sufficiently large, we have

$$1 \le r + a_1 < \cdots < r + a_L < q,$$

and therefore $(r + a_l, q) = 1$ for $1 \le l \le L$. We let $(r + a_l)^*$ be determined by

$$(r + a_l)(r + a_l)^* \equiv 1(\bmod q).$$

The Diophantine equation $qx + (r + a_l)y = 1$ has exactly one solution $(x_{0,l}, y_{0,l})$ with

$$-\left\lfloor \frac{r + a_l}{2} \right\rfloor < x_{0,l} \le \left\lfloor \frac{r + a_l}{2} \right\rfloor, \quad -\left\lfloor \frac{q}{2} \right\rfloor < y_{0,l} \le \frac{q}{2}.$$

We have

$$q^*(r; l) \equiv x_{0,l} \ (\bmod\ (r + a_l)),$$
$$(r + a_l)^* \equiv y_{0,l} \ (\bmod\ q).$$

Therefore, for $\beta_l \in (-1/2, 1/2)$ and $\delta > 0$ with

$$\beta_l + \delta < \frac{1}{2} \quad \text{and} \quad \beta - \delta > -\frac{1}{2} \quad \text{for} \quad 1 \le l \le L,$$

we have

$$\left|\left\{r : A_0 q \le r \le A_1 q, \frac{y_{0,l}}{q} \in [\beta_l, \beta_l + \delta]\right\}\right| \tag{4.1}$$
$$= \left|\left\{r : A_0 q \le r \le A_1 q, \frac{x_{0,l}}{r} \in [-(\beta_l + \delta), -\beta_l]\right\}\right| + O(1)$$
$$= \left|\left\{r : A_0 q \le r \le A_1 q, \frac{q^*(r, l)}{r}(\bmod 1) \in [-(\beta_l + \delta), -\beta_l]\right\}\right| + O(1),$$

where

$$\frac{q^*(r, l)}{r}(\bmod 1) \in [-(\beta_l + \delta), -\beta_l]$$

stands for

$$\frac{q^*(r, l)}{r} \in \begin{cases} [1 - (\beta_l + \delta), 1 - \beta_l], & \text{if } \beta_l \ge 0, \\ [-(\beta_l + \delta), -\beta_l], & \text{if } \beta_l < 0. \end{cases}$$

Let $\Delta > 0$, such that $\beta_l + \delta + \Delta \le \frac{1}{2}$, $0 \le v_l \le \Delta$. We define the function

$$\chi_{1,l}(u, v) := \begin{cases} 1 & \text{if } u \in [\beta_l + \Delta - v,\ \beta_l + \delta - \Delta + v), \\ 0 & \text{otherwise,} \end{cases} \tag{4.2}$$

and

$$\chi_{2,l}(u, v) := \begin{cases} 1 & \text{if } u \in [\beta_l - \Delta + v,\ \beta_l + \Delta - v), \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

as well as the functions $\lambda_{1,l}, \lambda_{2,l}$ by

$$\lambda_{i,l}(u) := \Delta^{-1} \int_0^\Delta \chi_{i,l}(u, v)\, dv \ \text{ for } i = 1, 2.$$

Let the function

$$\widetilde{\chi}_l(r, \beta) := \begin{cases} 1 & \text{if } \frac{(r+a_l)^*}{q} \in [\beta_l, \beta_l + \delta] \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

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Since $\lambda_{i,l}$ for $i = 1, 2$ is obtained from $\chi_{i,l}$ by averaging over r and since $0 \le \chi_{i,l}(u, v) \le 1$ it follows that $0 \le \lambda_{i,l}(u) \le 1$ for $i = 1, 2$. From (4.2), we have

$$\lambda_{1,l}\Big(\frac{(r+a_l)^*}{q}\Big) = 0, \quad \text{if } \frac{(r+a_l)^*}{q} \notin [\beta_l, \beta_l + \delta).$$

Similarly, from (4.3), we have

$$\lambda_{2,l}\Big(\frac{(r+a_l)^*}{q}\Big) = 1, \quad \text{if } \frac{(r+a_l)^*}{q} \notin [\beta_l, \beta_l + \delta).$$

Thus, we obtain

$$\lambda_{1,l}\Big(\frac{(r+a_l)^*}{q}\Big) \le \widetilde{\chi}_l(r, \beta) \le \lambda_{2,l}\Big(\frac{(r+a_l)^*}{q}\Big). \tag{4.5}$$

We have the Fourier expansion

$$\lambda_{i,l}(u) = \sum_{n=-\infty}^{\infty} a_l(n) e(nu).$$

The Fourier coefficients $a_l(n)$ are computed as follows.

For $i = 1$,

$$a_l(0) = \Delta^{-1} \int_0^{\Delta} \Big(\int_{\beta_l+\Delta-v}^{\beta_l+\delta-\Delta+v} 1\, du \Big)\, dv = \delta - \Delta,$$

as well as

$$\begin{aligned} a_l(n) &= \Delta^{-1} \int_0^{\Delta} \Big(\int_{\beta_l+\Delta-v}^{\beta_l+\delta-\Delta+v} e(-nu)\, du \Big)\, dv \\ &= \Delta^{-1} \int_0^{\Delta} -\frac{1}{2\pi i n} \big(e(-n(\beta_l + \delta - \Delta + v)) - e(-n(\beta_l + \Delta - v)) \big)\, dv \\ &= -\frac{1}{4\pi^2 n^2}\, \Delta^{-1} (e(-n(\beta_l + \delta)) - e(-n(\beta_l + \delta - \Delta)) \\ &\qquad - e(-n\beta_l) + e(-n(\beta_l + \Delta))). \end{aligned}$$

From the above and an analogous computation, for $i = 2$, we obtain

$$a_l(0) = \delta + R_{1,l}, \ \text{where } |R_{1,l}| \le \Delta$$

and

$$a_l(n) = \begin{cases} O(\Delta) & \text{if } |n| \le \Delta^{-1}, \\ O(\Delta^{-1} n^2) & \text{if } |n| > \Delta^{-1}. \end{cases}$$

Let $\Delta_1 > 0$, such that $A_0 - \Delta_1 > 1/2$, $A_1 + \Delta_1 < 1$ and $0 \le v \le \Delta_1$.

We define the functions

$$\chi_3(u, v) := \begin{cases} 1 & \text{if } u \in [A_0 + v - \Delta_1, A_1 - v + \Delta_1], \\ 0 & \text{otherwise}, \end{cases} \tag{4.6}$$

and

$$\chi_4(u, v) := \begin{cases} 1 & \text{if } u \in [A_0 + \Delta_1 - v, A_1 + \Delta_1 + v], \\ 0 & \text{otherwise}. \end{cases} \tag{4.7}$$

as well as the functions λ_3, λ_4 by

$$\lambda_i(u) := \Delta_1^{-1} \int_0^{\Delta_1} \chi_i(u, v)\, dv \quad \text{for } i = 3, 4.$$

Let the function

$$\chi^*(r, \beta) := \begin{cases} 1 & \text{if } A_0 \le \frac{r}{q} \le A_1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.8}$$

Since λ_i for $i = 3, 4$ is obtained from χ_i by averaging over r, and since

$$0 \le \chi_i(u, v) \le 1 \quad \text{for } i = 3, 4$$

we obtain $0 \le \lambda_i(u) \le 1$ for $i = 3, 4$.

From (4.6), we have

$$\lambda_3\Big(\frac{r}{q}\Big) = 0, \quad \text{if } \frac{r}{q} \notin (A_0, A_1).$$

From (4.7), we have

$$\lambda_4\Big(\frac{r}{q}\Big) = 1, \quad \text{if } \frac{r}{q} \in (A_0, A_1).$$

Therefore, we obtain

$$\lambda_3\Big(\frac{r}{q}\Big) \le \chi^*(r, \beta) \le \lambda_4\Big(\frac{r}{q}\Big). \tag{4.9}$$

By an analogous computation as for λ_1, λ_2, we obtain the Fourier expansions

$$\lambda_i(u) = \sum_{n=-\infty}^{\infty} c(n) e(nu), \quad \text{for } i = 3, 4,$$

with $c(0) = A_1 - A_0 + R_2$, where $|R_2| \le \Delta_1$ and

$$c(n) := \begin{cases} O(1), & \text{if } |n| \le \Delta_1^{-1}, \\ O(\Delta_1^{-1} n^{-2}), & \text{if } |n| > \Delta_1^{-1}. \end{cases}$$

From (4.1), (4.4), (4.5), (4.8), and (4.9), setting $\beta = -\alpha$, we get the following

$$\begin{aligned} &\sum_{1 \le r \le q-1} \Bigg(\prod_{l=1}^{L} \lambda_{1,l}\Big(\frac{(r + a_l)^*}{q}\Big) \Bigg) \lambda_3\Big(\frac{r}{q}\Big) \\ &\qquad \le N(\alpha_1, \ldots, \alpha_L, \delta) \\ &\qquad \le \sum_{1 \le r \le q-1} \Bigg(\prod_{l=1}^{L} \lambda_{2,l}\Big(\frac{(r + a_l)^*}{q}\Big) \Bigg) \lambda_4\Big(\frac{r}{q}\Big). \end{aligned}$$

We obtain

$$\begin{aligned} &\sum_{1 \le r \le q-1} \Bigg(\prod_{l=1}^{L} \lambda_{1,l}\Big(\frac{(r + a_l)^*}{q}\Big) \Bigg) \lambda_3\Big(\frac{r}{q}\Big) = \\ &\qquad \sum_{m_1, \ldots, m_L, n = -\infty}^{\infty} a(m_1) a(m_2) \cdots a(m_L) c(n) E(n, m_1, \ldots, m_L, q), \end{aligned} \tag{4.10}$$

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with

$$E(n, m_1, \dots, m_L, q) := \sum_{1 \le r \le q-1} e\Big(\frac{nr + m_1(r+a_1)^* + \cdots + m_L(r+a_L)^*}{q}\Big).$$

For $(n, m_1, \dots, m_L) \neq (0, 0, \dots, 0)$, we estimate $E(n, m_1, \dots, m_L, q)$ by Lemma 3.1 and obtain

$$E(n, m_1, \dots, m_L, q) = O(q^{1/2}).$$

From (4.10), we get

$$\sum_{1 \le r \le q-1} \Big(\prod_{l=1}^{L} \lambda_{1,l}\Big(\frac{(r+a_l)^*}{q}\Big)\Big) \lambda_3\Big(\frac{r}{q}\Big) = (\delta + R_1)^L (A_1 - A_0 + R_2) q + o(q), \tag{4.11}$$

for $|R_1| \le \Delta$ and $|R_2| \le \Delta$. The same computation also gives

$$\sum_{1 \le r \le q-1} \Big(\prod_{l=1}^{L} \lambda_{2,l}\Big(\frac{(r+a_l)^*}{q}\Big)\Big) \lambda_4\Big(\frac{r}{q}\Big) = (\delta + R_1)^L (A_1 - A_0 + R_2) q + o(q). \tag{4.12}$$

Since Δ and Δ_1 can be chosen to be arbitrarily small, it follows that (4.11) and (4.12) imply Lemma 4.1. ∎

5 Decomposition of the Sums Q

We start from the decompositions

$$Q\Big(\frac{r+a_l}{q}\Big) = \sum_{j=1}^{r-1} j \sum_{h=0}^{d_j} \cot\Big(\pi \frac{s_j^{(l)} + hr}{q}\Big).$$

We further decompose $Q(\frac{r+a_l}{q})$ as in the following definition.

Definition 5.1 We have

$$Q\Big(\frac{r+a_l}{q}\Big) := Q_0\Big(\frac{r+a_l}{q}\Big) + Q_1\Big(\frac{r+a_l}{q}\Big)$$

with

$$Q_0\Big(\frac{r+a_l}{q}\Big) := \sum_{j=1}^{q-1}{}^{*}\, j \sum_{h=0}^{d_j^{(l)}} \cot\Big(\pi \frac{s_j^{(l)} + hr}{q}\Big),$$

where $\sum_{j=1}^{q-1}{}^{*}$ means that the sum is extended over all values of j, for which

$$\Big\{\frac{\theta j q}{r + a_j}\Big\} \le q^{-1} 2^{m_1}$$

for either $\theta = 1$ or $\theta = -1$,

$$Q_1\Big(\frac{r+a_l}{q}\Big) = Q\Big(\frac{r+a_l}{q}\Big) - Q_0\Big(\frac{r+a_l}{q}\Big),$$

m_1 is a fixed positive integer.

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(For the conclusion of the proof we let $m_1 \to \infty$).

The size of $Q(\frac{r+a_l}{q})$ and also of $c_0(\frac{r+a_l}{q})$ is essentially determined by Q_0, since Q_1 is small, as we shall see in Section 7.

6 Comparison of Q_0 and g

Definition 6.1 Let $A_0 q \le r \le A_1 q$. We set

$$\alpha^{(l)} := \alpha^{(l)}(r, q) = \frac{q_l^*}{r + a_l},$$

where

$$q_l^* q \equiv 1 \,(\mathrm{mod}(r + a_l)),$$

$$g(\alpha; m_1) := \sum_{s=1}^{2^{m_1}} \frac{1 - 2\{s\alpha\}}{s},$$

$$Q(r, q, m_1, l) := \frac{rq}{\pi} g(\alpha^{(l)}; m_1).$$

The next lemma shows, that $Q_0(\frac{r+a_l}{q})$ is well approximated by $Q(r, q, m_1, l)$.

Lemma 6.2 *We have*

$$Q_0\Big(\frac{r + a_l}{q}\Big) = Q(r, q, m_1, l) + O(q 2^{m_1}).$$

Proof This follows from the result in the thesis of the second author [13] and in the paper [8, step 1 of the proof of Theorem 4.15], if r is replaced by $r + a_l$. ∎

7 The Estimate of $Q_1(p/q)$

Lemma 7.1 *We have*

$$Q_1\Big(\frac{r + a_l}{q}\Big) = O(q^2 2^{-m_1}).$$

Proof From [8, (4.113)], we have

$$Q_1\Big(\frac{r}{q}\Big) = O(q^2 2^{-m_1}).$$

Lemma 7.1 follows, if we replace r by $r + a_l$. ∎

8 The Joint Moments of the Sums $Q(r, q, m_1, l)$ and $c_0(\frac{r+a_l}{q})$

Lemma 8.1 *We have*

$$\lim_{\substack{q\to\infty \\ q \text{ prime}}} (A_1 - A_0)^{-L} q^{-L} \sum_{A_0 q \le r \le A_1 q} \prod_{l=1}^{L} Q(r, q, m_1, l)^{k_l} =$$
$$\prod_{l=1}^{L} \Big(\int_0^1 g(\alpha, m_1)^{k_l} \, d\alpha \Big).$$

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Proof We choose a partition $\mathcal{P}$ of the interval $[0,1]$:

$$0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{n-1} < \alpha_n = 1 \tag{8.1}$$

and consider upper and lower Riemann sums of the L-dimensional Riemann integral

$$I := \int_0^1 \cdots \int_0^1 g(x^{(1)}, m_1)^{k_1} \cdots g(x^{(L)}, m_1)^{k_L}\, dx^{(1)} \cdots dx^{(L)},$$

the upper sum

$$\begin{aligned}&\mathcal{U}\big(g(x^{(1)}, m_1)^{k_1}, \dots, g(x^{(L)}, m_1)^{k_L}; \mathcal{P}\big)\\ &:= \sum_{i_1=0}^{n-1} \cdots \sum_{i_L=0}^{n-1} \sup_{(\alpha^{(i_1)}, \dots, \alpha^{(i_L)} \in X_{l=1}^{L}[\alpha_{i_l}, \alpha_{i_l+1}]} \big(g(\alpha^{(i_1)}, m_1)^{k_1} \cdots g(\alpha^{(i_L)}, m_1)^{k_L}\big)\\ &\quad \times \prod_{l=1}^{L} (\alpha_{i_l+1} - \alpha_{i_l}),\end{aligned}$$

and the lower sum

$$\begin{aligned}&\mathcal{L}\big(g(x^{(1)}, m_1)^{k_1}, \dots, g(x^{(L)}, m_1)^{k_L}; \mathcal{P}\big)\\ &:= \sum_{i_1=0}^{n-1} \cdots \sum_{i_L=0}^{n-1} \inf_{(\alpha^{(i_1)}, \dots, \alpha^{(i_L)} \in X_{l=1}^{L}[\alpha_{i_l}, \alpha_{i_l+1}]} \big(g(\alpha^{(i_1)}, m_1)^{k_1} \cdots g(\alpha^{(i_L)}, m_1)^{k_L}\big)\\ &\quad \times \prod_{l=1}^{L} (a_{i_l+1} - \alpha_{i_l}).\end{aligned}$$

The function $g(x, m_1)$ is piecewise linear. Therefore, the integral I exists. It is well known from the Theory of the Riemann-integral, that for given $\epsilon > 0$, there is a partition $\mathcal{P}_\epsilon$ of the form (8.1), such that

$$\mathcal{L} \le \prod_{l=1}^{L} \Big(\int_0^1 g(x, m_1)^{k_l}\, dx \Big) \le \mathcal{U} \le \mathcal{L} + \epsilon.$$

We now let

$$N_{i_1, \dots, i_L} := \Big| \Big\{ r : A_0 q \le r \le A_1 q,\ \frac{q^*(r + a_l)}{r} \in [\alpha_{i_l}, \alpha_{i_l+1}) \text{ for } 1 \le l \le L \Big\} \Big|.$$

By Lemma 4.1, we have

$$N_{i_1, \dots, i_L} = \delta^L (A_1 - A_0) q (1 + o(1)).$$

From the asymptotics

$$\cot\Big(\frac{\pi(qj + s_j^{(l)})}{q} \Big) = \frac{q}{\pi} \frac{1}{s_j^{(l)}} (1 + o(1)),$$

$$\cot\Big(\frac{\pi(qj + d_j^{(l)}(r + a_l) + t_j^{(l)})}{q} \Big) = -\frac{q}{\pi} \frac{1}{t_j^{(l)}} (1 + o(1)),$$

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we obtain (using the notation (2.2)):

$$\sum_{1\le r\le q-1}\prod_{l=1}^{L} Q(r,q,m_1,l)^{k_l}$$

$$= \sum_{i_1=0}^{n-1}\cdots\sum_{i_L=0}^{n-1}\prod_{l=1}^{L}\Bigg(\sum_{\substack{j:\{\frac{\theta j q}{r+a_l}\}\in[\alpha_{i_l},\alpha_{i_l+1}]\\ \theta\in\{1,-1\}}}\cot\Big(\frac{\pi u_j^{(l)}}{q}\Big)^{k_l}\Bigg)$$

(where we set $u_j^{(l)} = s_j^{(l)}$ if $\theta = 1$, and $u_j^{(l)} = -t_j^{(l)}$ if $\theta = -1$)

$$= \Bigg(\prod_{l=1}^{L}\Big(\int_0^1 g(\alpha,m_1)^{k_l}\,d\alpha\Big)\Bigg)q^L(1+o(1)),$$

which proves Lemma 8.1. ∎

Lemma 8.2 *We have*

$$\lim_{\substack{q\to\infty\\ q\ prime}}(A_1-A_0)^{-L}q^{-L}\sum_{A_0q\le r\le A_1q}\prod_{l=1}^{L}Q\Big(\frac{r+a_l}{q}\Big)^{k_l} = \prod_{l=1}^{L}\Big(\int_0^1 g(\alpha)^{k_l}\,d\alpha\Big).$$

Proof The proof is a modification of the procedure of Step 7 of the proof of [8, Theorem 4.15].

We use the decomposition

$$Q\Big(\frac{r+a_l}{q}\Big) = Q_0\Big(\frac{r+a_l}{q}\Big) + Q_1\Big(\frac{r+a_l}{q}\Big).$$

From Lemma 6.2, we obtain

$$Q_0\Big(\frac{r+a_l}{q}\Big) = Q(r,q,m_1,l) + O(q2^{m_1}).$$

From Lemma 7.1, we have

$$Q\Big(\frac{r+a_l}{q}\Big) = Q(r,q,m_1,l) + Q_1\Big(\frac{r+a_l}{q}\Big) + O(q2^{m_1}).$$

We then expand the factors of the product

$$\prod_{l=1}^{L}Q\Big(\frac{r+a_l}{q}\Big)^{k_l}$$

by applications of the Multinomial Theorem, which leads to a sum of products of the type

$$\prod_{l=1}^{L}Q(r,q,m_1,l)^{h_{1,l}}Q_1\Big(\frac{r+a_l}{q}\Big)^{h_{2,l}}.$$

Only products with $h_{1,l} = k_l$, $h_{2,l} = 0$ contribute substantially, the contributions being given in Lemma 8.1.

The other terms can be estimated by the application of Hölder's inequality as in Step 7 of the proof of [8, Theorem 4.15]. ∎

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Lemma 8.3 *We have*

$$\lim_{\substack{q\to\infty \\ q\ prime}} \frac{1}{\phi(q)} \sum_{A_0 q \le r \le A_1 q} (A_1 - A_0)^{-(k_1+\cdots+k_L)} \prod_{l=1}^{L} c_0\left(\frac{r+a_l}{q}\right)^{k_l}$$
$$= \prod_{l=1}^{L} \left(\int_0^1 g(x)^{k_l}\, dx \right).$$

Proof In [8, Theorem 5.2 (c)] a more general one-dimensional version of Lemma 8.3 has been proved. The proof of Lemma 8.3 has been obtained by modifying this proof to obtain results for monomials

$$\prod_{l=1}^{L} c_0\left(\frac{r+a_l}{q}\right)^{k_l}$$

in several variables. ■

9 Conclusion of the Proof and Concluding Remarks

By the Weierstrass approximation theorem, each of the functions f_l in Theorem 1.3 can be approximated arbitrarily closely by a polynomial

$$p_l(x) = \sum_{j=0}^{C_l} e_{j,l}\, x^j,$$

where $C_l \in \mathbb{N}_0$. The product

$$\prod_{l=1}^{L} f_l\left(c_0\left(\frac{r+a_l}{q}\right)\right)$$

then becomes the sum of products as considered in Lemma 8.3.

The Theorem follows from the fact that g has a continuous distribution function ([8, Theorem 5.2]).

The proof of Theorem 1.3(ii) can be obtained from the proof of (i) by applying Lemma 3.2 instead of Lemma 3.1 and making the obvious changes otherwise.

Open Problem Can the result of Theorem 1.3(i) be modified, such that the primality of q is not requested?

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