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SOME QUADRATIC AND CUBIC SUMMATION FORMULAS FOR BASIC HYPERGEOMETRIC SERIES

MIZAN RAHMAN

ABSTRACT An identity of L Carlitz for a bibasic hypergeometric series is used to find some summation formulas for series in which the bases are either q and q^2 or q and q^3, $0 < q < 1$ In general, these series are neither balanced nor very-well-poised in the usual sense

1. **Introduction.** A basic hypergeometric series in base q with $r+1$ numerator parameters and r denominator parameters is defined by

$$(1.1) \qquad {}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix} ; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_n}{(q, b_1, \ldots, b_r; q)_n} z^n,$$

where

$$(1.2) \qquad (a_1, a_2, \ldots, a_k; q)_n = (a_1; q)_n \cdots (a_k; q)_n \,,$$
$$(a_i; q)_n = \begin{cases} 1, & \text{if } n = 0, \\ (1-a_i)(1-qa_i)\cdots(1-q^{n-1}a_i), & \text{if } n = 1, 2, \ldots \end{cases}$$

and z is a complex number. We shall assume throughout the paper that $0 < q < 1$. If one of the a's in the numerator of the ${}_{r+1}\phi_r$ series is of the form q^{-n}, n a nonnegative integer, then the series is a polynomial of degree n in z, otherwise we shall assume that $|z| < 1$ which guarantees the convergence of the infinite series.

The series (1.1) is called *balanced* if $b_1 b_2 \cdots b_r = q a_1 a_2 \cdots a_{r+1}$ and $z = q$; it is called *well-poised* if $a_2 b_1 = a_3 b_2 = \cdots = a_{r+1} b_r = q a_1$, and *very-well-poised* if, in addition, $a_2 = -a_3 = q a_1^{1/2}$. Most of the known summation and transformation formulas for higher order basic hypergeometric series (that is, ${}_3\phi_2$, ${}_4\phi_3$, etc.) are for series that are either balanced or well-poised or both. The main building blocks for these formulas (see for example, Bailey [2] and Slater [11]) are the q-binomial theorem

$$(1.3) \qquad {}_1\phi_0\left[\begin{matrix} a \\ - \end{matrix} ; q, z\right] = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_\infty}{(z; q)_\infty}, \quad |z| < 1,$$

the q-Gauss formula

$$(1.4) \qquad {}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix} ; q, c/ab\right] = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}, \quad |c/ab| < 1,$$

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the q-Saalschütz formula

$$(1.5)\qquad {}_3\phi_2\left[\begin{matrix} q^{-n}, & a, & b \\ & c, & abq^{1-n}/c \end{matrix}; q, q\right] = \frac{(c/a, c/b; q)_n}{(c, c/ab; q)_n},$$

and an easily proved identity for a very-well-poised series

$$(1.6)\qquad {}_4\phi_3\left[\begin{matrix} a, & qa^{1/2}, & -qa^{1/2}, & q^{-n} \\ & a^{1/2}, & -a^{1/2}, & aq^{n+1} \end{matrix}; q, q^n\right] = \delta_{n,0},$$

where n is a nonegative integer.

A more general formula containing two independent bases p and q, that reduces to (1.6) when $p = q$, is given by

$$(1.7)\qquad \sum_{k=0}^{n} \frac{(a;p)_k(1-ap^kq^k)(q^{-n};q)_k}{(q;q)_k(1-a)(apq^n;p)_k} q^{nk} = \delta_{n,0}.$$

This is equivalent to eq. (2.1) of L. Carlitz [3] and was rediscovered by Gessel and Stanton [6,7] in their work on q-Lagrange inversion formulas. A closely related formula was also given in Al-Salam and Verma [1]. Clearly, (1.7) opens up the possibility of generating summation and transformation formulas for truly bibasic series; however, in this paper we shall only consider the cases when $p = q^2, q^{1/2}$ or q^3.

We shall first give an elementary proof of (1.7) based on hypergeometric series manipulations and then proceed to prove the following summation formulas

(1.8)

$$\sum_{k=0}^{\infty} \frac{(a;q^2)_k(1-aq^{3k})(b,c,aq/bc;q)_k}{(q;q)_k(1-a)(aq^2/b,aq^2/c,bcq;q^2)_k} q^{k(k+1)/2} = \frac{(aq^2,bq,cq,aq^2/bc;q^2)_\infty}{(q,aq^2/b,aq^2/c,bcq;q^2)_\infty},$$

(1.9)

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{(a;q^2)_n(1-aq^{3n})(b,aq/b;q^2)_n(c,d,aq/cd;q)_n}{(q;q)_n(1-a)(aq/b,b;q)_n(aq^2/c,aq^2/d,cdq;q^2)_n} q^n \\ &\quad + \frac{(b/a,qb^2/a,aq^2,b^2q^2/ac,b^2q^2/ad,b^2cdq/a^2;q^2)_\infty(c,d,aq/cd;q)_\infty}{(a/b,bq,b^3q^2/a^2,aq^2/c,aq^2/d,cdq;q^2)_\infty(bc/a,bd/a,bq/cd;q)_\infty} \\ &\quad \cdot \sum_{n=0}^{\infty} \frac{(b^3/a^2;q^2)_n(1-b^3q^{3n}/a^2)(b^2/a,bq/a;q^2)_n(bc/a,bd/a,bq/cd;q)_n}{(q;q)_n(1-b^3/a^2)(bq/a,b^2/a;q)_n(b^2q^2/ac,b^2q^2/ad,b^2cdq/a^2;q^2)_n} q^n \\ &= \frac{(aq^2,cq,dq,aq^2/cd,bq/c,bq/d,bcd/a,b/a;q^2)_\infty}{(q,aq^2/c,aq^2/d,cdq,bc/a,bd/a,bq/cd,bq;q^2)_\infty},\end{aligned}$$

(1.10)

$$\begin{aligned}&\sum_{k=0}^{\infty} \frac{(a;q)_k(1-aq^{3k})(d,q/d;q)_k(b,c,a^2q/bc;q^2)_k}{(q^2;q^2)_k(1-a)(aq^2/d,adq;q^2)_k(aq/b,aq/c,bc/a;q)_k} q^k \\ &\quad = \frac{(aq,aq/bc;q)_\infty(adq/b,adq/c,aq^2/bd,aq^2/cd;q^2)_\infty}{(aq/b,aq/c;q)_\infty(adq,adq/bc,aq^2/d,aq^2/bcd;q^2)_\infty} \\ &\qquad + \frac{aq}{bc} \frac{(b,c,a^2q^3/bc^2,a^2q^3/b^2c;q^2)_\infty(aq,d,q/d;q)_\infty}{(adq,adq/bc,aq^2/d,aq^2/bcd;q^2)_\infty(aq/b,aq/c,bc/a;q)_\infty} \\ &\qquad \cdot {}_3\phi_2\left[\begin{matrix} a^2q/bc, & adq/bc, & aq^2/bcd \\ & a^2q^3/bc^2, & a^2q^3/b^2c \end{matrix}; q^2, q^2\right],\end{aligned}$$

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$$\sum_{n=0}^{\infty} \frac{(bcdq^{-2};q^3)_n(1-bcdq^{4n-2})(b,c,d;q)_n}{(q;q)_n(1-bcdq^{-2})(cdq,bdq,bcq;q^3)_n} q^{n^2} \tag{1.11}$$
$$\cdot {}_4\phi_3\left[\begin{matrix} q^{-n}, & q^{1-n}, & q^{2-n}, & bcdq^{3n} \\ & bq^2, & cq^2, & dq^2 \end{matrix}; q^3, q^3\right] = \frac{(bcdq,bq,cq,dq;q^3)_\infty}{(q,cdq,bdq,bcq;q^3)_\infty},$$

and

$$\sum_{n=0}^{\infty} \frac{(bcdq^{-1};q^3)_n(1-bcdq^{4n-1})(b,c,d;q)_n}{(q;q)_n(1-bcdq^{-1})(cdq^2,bdq^2,bcq^2;q^3)_n} q^{n^2+n} \tag{1.12}$$
$$\cdot {}_4\phi_3\left[\begin{matrix} q^{-n-1}, q^{-n}, q^{1-n}, bcdq^{3n} \\ bq, cq, dq \end{matrix}; q^3, q^3\right] = \frac{(bcdq^2,bq^2,cq^2,dq^2;q^3)_\infty}{(q^2,cdq^2,bdq^2,bcq^2;q^3)_\infty}.$$

Apart from the assumption $0 < q < 1$ the only restrictions on these formulas are that no zero factors occur in the denominators on either side and a limit needs to be taken whenever an indeterminate form appears. Specifically, (1.11) is not valid if any of $bq^2, cq^2, dq^2, bcq, bdq, cdq$ is of the form $q^{-3k}, k = 0, 1, \ldots$; similarly (1.12) fails to hold if any of $bq, cq, dq, bcq^2, bdq^2, cdq^2$ is of this form.

Note that the infinite series in (1.8)–(1.10) have a well-poised structure although none of them is a very-well-poised series in the usual sense. The same is true for the series in (1.11) and (1.12), however, the ${}_4\phi_3$ series in both are balanced in base q^3. Note also that (1.9) and (1.10) are nonterminating extensions of [6, (1.14)] and [6, (1.4)], respectively.

2. **Proof of (1.7).** Let us assume that $\max(|p|, |q|, |ap|) < 1$. Clearly (1.7) is true for $n = 0$. Denoting the sum on the left side of (1.7) by f_n we then have to prove that $f_n = 0$ for $n \geq 1$. Since $1 - ap^kq^k = q^k(1-ap^k) + (1-q^k)$, we can split up f_n into two series:

$$f_n = \sum_{k=0}^{n} \frac{(q^{-n};q)_k(ap;p)_k}{(q;q)_k(apq^n;p)_k} q^{(n+1)k} + \sum_{k=1}^{n} \frac{(q^{-n};q)_k(a;p)_k}{(q;q)_{k-1}(apq^n;p)_k} \frac{q^{nk}}{1-a} \tag{2.1}$$
$$= \sum_{k=0}^{n} \frac{(q^{-n};q)_k(ap;p)_k}{(q;q)_k(apq^n;p)_k} q^{(n+1)k} - (1-q^n)\sum_{k=0}^{n-1} \frac{(q^{1-n};q)_k(ap;p)_k}{(q;q)_k(apq^n;p)_{k+1}} q^{nk}.$$

By the q-binomial theorem (1.3),

$$\sum_{j=0}^{\infty} \frac{(q^n;p)_j}{(p;p)_j}(ap^{k+1})^j = \frac{(ap^{k+1}q^n;p)_\infty}{(ap^{k+1};p)_\infty}$$

and

$$\sum_{j=0}^{\infty} \frac{(pq^n;p)_j}{(p;p)_j}(ap^{k+1})^j = \frac{(ap^{k+2}q^n;p)_\infty}{(ap^{k+1};p)_\infty}.$$

Hence

$$f_n = \frac{(ap;p)_\infty}{(apq^n;p)_\infty}\left\{\sum_{j=0}^{\infty} \frac{(q^n;p)_j}{(p;p)_j}(ap)^j \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k}(p^jq^{n+1})^k\right.$$
$$\left. - (1-q^n)\sum_{j=0}^{\infty} \frac{(pq^n;p)_j}{(p;p)_j}(ap)^j \sum_{k=0}^{n-1} \frac{(q^{1-n};q)_k}{(q;q)_k}(p^jq^n)^k\right\} \tag{2.2}$$
$$= \frac{(ap;p)_\infty}{(apq^n;p)_\infty}\left\{\sum_{j=0}^{\infty} \frac{(q^n;p)_j(qp^j;q)_\infty}{(p;p)_j(p^jq^{n+1};q)_\infty}(ap)^j - \sum_{j=0}^{\infty} \frac{(q^n;p)_{j+1}(qp^j;q)_\infty}{(p;p)_j(p^jq^n;q)_\infty}(ap)^j\right\}$$
$$= 0.$$

Since (1.7) is a finite-series identity we may now drop the assumptions $|p| < 1$ and $|ap| < 1$. George Gasper [4] pointed out to me that a slight modification of this proof yields the infinite series identity

$$\sum_{n=0}^{\infty} \frac{(a;p)_k(1-ap^kq^k)(b^{-1};q)_k}{(q;q)_k(1-a)(apb;p)_k} b^k = 0, \tag{2.3}$$

provided $\max(|p|,|q|,|b|) < 1$.

3. **Quadratic summation formulas I.** Let $p = q^2$ in (1.7), from which it follows that for a nonnegative integer n,

$$\sum_{k=0}^{2n} \frac{(a;q^2)_k(1-aq^{3k})(q^{-2n};q)_k}{(q;q)_k(1-a)(aq^{2n+2};q^2)_k} q^{2nk} = \delta_{n,0} \tag{3.1}$$

and

$$\sum_{k=0}^{2n+1} \frac{(a;q^2)_k(1-aq^{3k})(q^{-2n-1};q)_k}{(q;q)_k(1-a)(aq^{2n+3};q^2)_k} q^{(2n+1)k} = 0. \tag{3.2}$$

So, for arbitrary bounded sequences $\{\lambda_n\}$ and $\{\mu_n\}$ we have

$$\begin{aligned}\lambda_0 &= \sum_{n=0}^{\infty}\sum_{k\le 2n} \frac{(-1)^k q^{\binom{k}{2}}(a;q^2)_k(1-aq^{3k})\lambda_n}{(q;q)_{2n-k}(q;q)_k(1-a)(aq^2;q^2)_{n+k}} \\ &= \sum_{k\,\text{even}} \frac{(-1)^k q^{\binom{k}{2}}(a;q^2)_k(1-aq^{3k})}{(q;q)_k(1-a)} \sum_{n\ge 0} \frac{\lambda_{n+k/2}}{(q;q)_{2n}(aq^2;q^2)_{3k/2+n}} \\ &\quad + \sum_{k\,\text{odd}} \frac{(-1)^k q^{\binom{k}{2}}(a;q^2)_k(1-aq^{3k})}{(q;q)_k(1-a)} \sum_{n\ge 0} \frac{\lambda_{n+\frac{k+1}{2}}}{(q;q)_{2n+1}(aq^2;q^2)_{\frac{3k+1}{2}+n}}\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}0 &= \sum_{k\,\text{even}} \frac{q^{\binom{k}{2}}(a;q^2)_k(1-aq^{3k})}{(q;q)_k(1-a)} \sum_{n\ge 0} \frac{\mu_{n+k/2}}{(q;q)_{2n+1}(aq^3;q^2)_{3k/2+n}} \\ &\quad - \sum_{k\,\text{odd}} \frac{q^{\binom{k}{2}}(a;q^2)_k(1-aq^{3k})}{(q;q)_k(1-a)} \sum_{n\ge 0} \frac{\mu_{n+\frac{k-1}{2}}}{(q;q)_{2n}(aq^3;q^2)_{\frac{3k-1}{2}+n}}.\end{aligned} \tag{3.4}$$

The key to the whole set of calculations that follows is that we use both (3.3) and (3.4). The λ- and μ-sequences have to be chosen so that the corresponding series are convergent. Being aware of the q-Saalschütz formula (1.5) and its nonterminating extension

$$\begin{aligned}&{}_3\phi_2\left[\begin{matrix}a, & b, & c\\ & e, & f\end{matrix};q,q\right] + \frac{(q/e,a,b,c,fq/e;q)_\infty}{(e/q,aq/e,bq/e,cq/e,f;q)_\infty} \\ &\quad {}_3\phi_2\left[\begin{matrix}aq/e, & bq/e, & cq/e\\ & q^2/e, & fq/e\end{matrix};q,q\right] = \frac{(q/e,f/a,f/b,f/c;q)_\infty}{(aq/e,bq/e,cq/e,f;q)_\infty}\end{aligned} \tag{3.5}$$

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where $abcq = ef$, known as Sears' formula [9], we choose

$$\lambda_n = (b, c, aq/bc; q^2)_n q^{2n}, \quad \mu_n = (bq, cq, aq^2/bc; q^2)_n q^{2n}. \tag{3.6}$$

We now substitute (3.6) and add a multiple, say A, of (3.4) to (3.3). The sum of the two even series gives

$$\sum_{k\,\text{even}} \frac{q^{\binom{k}{2}+k}(a;q^2)_k(1-aq^{3k})}{(q;q)_k(1-a)} \left\{ \frac{(b,c,aq/bc;q^2)_{k/2}}{(aq^2;q^2)_{3k/2}} {}_3\phi_2\left[\begin{matrix} bq^k, & cq^k, & aq^{k+1}/bc \\ & q, & aq^{3k+2} \end{matrix}; q^2, q^2\right] + A\frac{(bq,cq,aq^2/bc;q^2)_{k/2}}{(aq^3;q^2)_{3k/2}(1-q)} {}_3\phi_2\left[\begin{matrix} bq^{k+1}, & cq^{k+1}, & aq^{k+2}/bc \\ & q^3, & aq^{3k+3} \end{matrix}; q^2, q^2\right] \right\}. \tag{3.7}$$

If we choose

$$A = -q\frac{(aq^3, b, c, aq^2/bc; q^2)_\infty}{(aq^2, bq, cq, aq^2/bc; q^2)_\infty}$$

then by (3.5), (3.7) reduces to

$$\frac{(q, aq^2/b, aq^2/c, bcq; q^2)_\infty}{(aq^2, bq, cq, aq^2/bc; q^2)_\infty} \sum_{k\,\text{even}} \frac{(a;q^2)_k(1-aq^{3k})(b,c,aq/bc;q)_k}{(q;q)_k(1-a)(aq^2/b, aq^2/c, bcq; q^2)_k} q^{k(k+1)/2}. \tag{3.8}$$

It turns out that the two odd series in (3.3) and (3.4) also add up to the same as (3.8) with a negative sign. Since $\lambda_0 = 1$, we have the summation formula (1.8).

To prove (1.9) we proceed as follows. First observe that by (1.5),

$${}_3\phi_2\left[\begin{matrix} q^{-k}, & q^{1-k}, & aq^{2k} \\ & dq, & aq^2/d \end{matrix}; q^2, q^2\right] = \frac{(d, aq/d; q^2)_k}{(aq/d, d; q)_k} q^{-\binom{k}{2}}. \tag{3.9}$$

Hence

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{(a;q^2)_k(1-aq^{3k})(d,aq/d;q^2)_k(b,c,aq/bc;q)_k}{(q;q)_k(1-a)(aq/d,d;q)_k(aq^2/b,aq^2/c,bcq;q^2)_k} q^k \\ &= \sum_k \sum_j \frac{(a;q^2)_{k+j}(1-aq^{3k})(b,c,aq/bc;q)_k q^{\binom{k-2j}{2}+k}}{(q;q)_{k-2j}(1-a)(aq^2/b,aq^2/c,bcq;q^2)_k(q^2;q^2)_j(aq^2/d,dq;q^2)_j}. \end{aligned} \tag{3.10}$$

Setting $k-2j = n$ and interchanging the order of summation which can be easily justified because of the assumption $0 < q < 1$, the double sum on the right side of (3.10) becomes

$$\begin{aligned} &\sum_{j=0}^{\infty} \frac{(a;q^2)_{3j}(1-aq^{6j})(b,c,aq/bc;q)_{2j}q^{2j}}{(q^2;q^2)_j(1-a)(aq^2/b,aq^2/c,bcq;q^2)_{2j}(dq,aq^2/d;q^2)_j} \\ &\quad \cdot \sum_{n=0}^{\infty} \frac{(aq^{6j};q^2)_n(1-aq^{6j+3n})(bq^{2j},cq^{2j},aq^{2j+1}/bc;q)_n}{(q;q)_n(1-aq^{6j})(aq^{4j+2}/b,aq^{4j+2}/c,bcq^{4j+1};q^2)_n} q^{n(n+1)/2} \\ &= \sum_{j=0}^{\infty} \frac{(a;q^2)_{3j}(1-aq^{6j})(b,c,aq/bc;q)_{2j}}{(q^2;q^2)_j(1-a)(aq^2/b,aq^2/c,bcq;q^2)_{2j}} \\ &\quad \cdot \frac{q^{2j}}{(dq,aq^2/d;q^2)_j} \frac{(aq^{6j+2},bq^{2j+1},cq^{2j+1},aq^{2j+2}/bc;q^2)_\infty}{(q,aq^{4j+2}/b,aq^{4j+2}/c,bcq^{4j+1};q^2)_\infty}, \end{aligned} \tag{3.11}$$

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by (1.8). Simplifying the sum on the right side of (3.11) we find that

$$(3.12)\quad \sum_{k=0}^{\infty} \frac{(a;q^2)_k(1-aq^{3k})(d,aq/d;q^2)_k(b,c,aq/bc;q)_k}{(q;q)_k(1-a)(aq/d,d;q)_k(aq^2/b,aq^2/c,bcq;q^2)_k} q^k = \frac{(aq^2,bq,cq,aq^2/bc;q^2)_\infty}{(q,aq^2/b,aq^2/c,bcq;q^2)_\infty} {}_3\phi_2\left[\begin{matrix} b, & c, & aq/bc \\ & dq, & aq^2/d \end{matrix} ;q^2,q^2\right].$$

Note that the ${}_3\phi_2$ series is balanced and so is summable by (1.5) if it terminates. Suppose $aq/bc = q^{-2n}, n = 0,1,2,\ldots$. Then, after some simplifications we obtain

$$(3.13)\quad \sum_{k=0}^{2n} \frac{(bcq^{-2n-1};q^2)_k(1-bcq^{3k-2n-1})(d,bcq^{-2n}/d;q^2)_k(b,c,q^{-2n};q)_k}{(q;q)_k(1-bcq^{-2n-1})(bcq^{-2n}/d,d;q)_k(cq^{1-2n},bq^{1-2n},bcq;q^2)_k} q^k = \frac{(q,q/bc,dq/b,dq/c;q^2)_n}{(q/b,q/c,dq,dq/bc;q^2)_n}.$$

Since $(q^{-2n};q^2)_\infty = 0$, it also follows from (3.12) that

$$(3.14)\quad \sum_{k=0}^{\infty} \frac{(bcq^{-2n-2};q^2)_k(1-bcq^{3k-2n-2})(d,bcq^{-2n-1}/d;q^2)_k(b,c,q^{-2n-1};q)_k}{(q;q)_k(1-bcq^{-2n-2})(bcq^{-2n-1}/d,d;q)_k(cq^{-2n},bq^{-2n},bcq;q^2)_k} q^k = 0, \quad n = 0,1,2,\ldots.$$

Formulas (3.13) and (3.14) are the same as (6.14) of [6].

To prove (1.9), which is the nonterminating extension of (3.13) and (3.14), note that

$$(3.15)\quad {}_3\phi_2\left[\begin{matrix} b, & c, & aq/bc \\ & dq, & aq^2/d \end{matrix} ;q^2,q^2\right] = \frac{(d/a,dq/b,dq/c,bcd/a;q^2)_\infty}{(bd/a,cd/a,dq/bc,dq;q^2)_\infty} - \frac{(d/a,b,c,aq/bc,qd^2/a;q^2)_\infty}{(a/d,bd/a,cd/a,dq/bc,dq;q^2)_\infty} {}_3\phi_2\left[\begin{matrix} bd/a, & cd/a, & dq/bc \\ & dq^2/a, & qd^2/a \end{matrix} ;q^2,q^2\right].$$

However, (3.12) enables us to express the balanced ${}_3\phi_2$ series on the right side of (3.15) in terms of a bibasic series similar to the one on the left side of (3.12). Combining these results and interchanging b and d we obtain (1.9).

If we let $a \to 0, b \to 0$ in (1.9) such that a/b = constant and then rename the parameters, we obtain a summation formula for

$$(3.16)\quad {}_4\phi_3\left[\begin{matrix} a, & b, & c, & -c \\ & \sqrt{abq}, & -\sqrt{abq}, & c^2 \end{matrix} ;q,q\right] + \frac{(a,b;q)_\infty(q/c^2,abq^3/c^4;q^2)_\infty}{(aq/c^2,bq/c^2;q)_\infty(c^2/q,abq;q^2)_\infty} \cdot {}_4\phi_3\left[\begin{matrix} aq/c^2, & bq/c^2, & q/c, & -q/c \\ & \sqrt{abq^3}/c^2, & -\sqrt{abq^3}/c^2, & q^2/c^2 \end{matrix} ;q,q\right]$$

which, by virtue of Bailey's transformation formula [2, 8.5 (3)] can be expressed as

$$\tag{3.17} \begin{aligned} &{}_8W_7(-c\sqrt{ab/q}; a, b, c, -c, -\sqrt{abq}/c; q, c\sqrt{q/ab}) \\ &\quad = \frac{(-q, c^2/a, -c\sqrt{abq}, c\sqrt{aq/b}; q)_\infty (a^2q^2, bq, c^2, c^2q^2/b^2; q^2)_\infty}{(c^2, -aq, c\sqrt{q/ab}, -c\sqrt{bq/a}; q)_\infty (c^2/a, abq, aq^2, aqc^2/b^2; q^2)_\infty}. \end{aligned}$$

where $|qc^2/ab| < 1$, and

$$\tag{3.18} \begin{aligned} &{}_{r+3}W_{r+2}(a; b_1, b_2, \ldots, b_r; q, z) \\ &\quad \equiv {}_{r+3}\phi_{r+2}\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b_1, \ldots, b_r \\ & \sqrt{a}, & -\sqrt{a}, & aq/b_1, \ldots, aq/b_r \end{matrix} ; q, z\right]. \end{aligned}$$

Formula (3.17) was found previously by Gasper and Rahman [5].

Before closing this section we would like to point out that (3.12) is equivalent to a formula due to Gosper which was communicated to various people including Dennis Stanton who brought it to my attention. In order to prove this equivalence we first use (3.15) in (3.12) and rewrite the formula in Gosper's notation

$$\tag{3.19} \begin{aligned} &\sum_{k=0}^{\infty} \frac{(a^2b^2c^2q^{-1}; q^2)_k (1 - a^2b^2c^2q^{3k-1})(abcd, abcd^{-1}; q^2)_k (a^2, b^2, c^2; q)_k}{(q; q)_k (1 - a^2b^2c^2q^{-1})(abcd^{-1}, abcd; q)_k (b^2c^2q, c^2a^2q, a^2b^2q; q^2)_k} q^k \\ &\quad = \frac{(a^2b^2c^2q, a^2q, b^2q, c^2q, dq/abc, bcdq/a, cdaq/b, dabq/c; q^2)_\infty}{(q, b^2c^2q, c^2a^2q, a^2b^2q, abcdq, adq/bc, bdq/ac, cdq/ab; q^2)_\infty} \\ &\qquad - \frac{(a^2b^2c^2q, a^2q, b^2q, c^2q, a^2, b^2, c^2, qd/abc, d^2q^2; q^2)_\infty}{(q, b^2c^2q, c^2a^2q, a^2b^2q, adq/bc, bdq/ac, cdq/ab, abc/qd; q^2)_\infty} \\ &\qquad \cdot {}_3\phi_2\left[\begin{matrix} adq/bc, & bdq/ac, & cdq/ab \\ & d^2q^2, & dq^3/abc \end{matrix} ; q^2, q^2\right]. \end{aligned}$$

Since

$$\tag{3.20} \begin{aligned} &{}_3\phi_2\left[\begin{matrix} abq/bc, & bdq/ac, & cdq/ab \\ & d^2q^2, & dq^3/abc \end{matrix} ; q^2, q^2\right] \\ &\quad = \lim_{e\to\infty} {}_8W_7(d^3q^3/abce; adq/bc, bdq/ac, cdq/ab, dq^3/abce, d^2q^2/e; q^2, q^2), \end{aligned}$$

it follows, by an iterate of the following limit case of Jackson's transformation formula [11, (3.4.2.4)]

$$\tag{3.21} \begin{aligned} {}_8W_7(a; b, c, d, e, f; q, a^2q^2/bcdef) &= \frac{(aq, aq/ef, a^2q^2/bcde, a^2q^2/bcdf; q)_\infty}{(aq/e, aq/f, a^2q^2/bcdef, a^2q^2/bcd; q)_\infty} \\ &\quad \cdot {}_8W_7(a^2q/bcd; aq/cd, aq/bd, aq/bc, e, f; aq/ef), \end{aligned}$$

that

(3.22)

$$ {}_3\phi_2\left[\begin{matrix} adq/bc, & bdq/ca, & cdq/ab \\ & d^2q^2, & dq^3/abc \end{matrix};q^2,q^2\right] = q^{-1}\frac{(adq/bc,bdq/ac,cdq/ab;q^2)_\infty}{(d^2,d^2q^4,dq^3/abc;q^2)_\infty} \cdot\sum_{n=0}^{\infty}\frac{(1-d^2q^{4n+2})(bcdq/a,cdaq/b,dabq/c;q^2)_n}{(1-d^2q^2)(adq^3/bc,bdq^3/ac,cdq^3/ab;q^2)_n}\Big(-\frac{d}{abc}\Big)^n q^{(n+1)^2}. $$

Using (3.22) in (3.19) we obtain Gosper's formula

$$ \begin{aligned} &\sum_{n=0}^{\infty}\frac{(a^2b^2c^2q^{-1};q^2)_n(1-a^2b^2c^2q^{3n-1})(abcd,abcd^{-1};q^2)_n(a^2,b^2,c^2;q)_n}{(q;q)_n(1-a^2b^2c^2q^{-1})(abcd^{-1},abcd;q)_n(b^2c^2q,c^2a^2q,a^2b^2q;q^2)_n}q^n \\ &\quad=\frac{(a^2b^2c^2q,a^2q,b^2q,c^2q,dq/abc,bcdq/a,cdaq/b,dabq/c;q^2)_\infty}{(q,b^2c^2q,c^2a^2q,a^2b^2q,abcdq,adq/bc,bdq/ac,cdq/ab;q^2)_\infty} \\ &\quad-\frac{(a^2,b^2,c^2;q)_\infty(a^2b^2c^2q;q^2)_\infty}{(q;q)_\infty(b^2c^2q,c^2a^2q,a^2b^2q,abcdq,abcd^{-1}q;q^2)_\infty} \\ &\quad\cdot\sum_{n=0}^{\infty}(1-d^2q^{4n+2})\frac{(bcdq/a,cdaq/b,dabq/c;q^2)_n}{(adq/bc,bdq/ac,cdq/ab;q^2)_{n+1}}\Big(-\frac{d}{abc}\Big)^n q^{(n+1)^2}. \end{aligned} \tag{3.23} $$

4. **Quadratic summation formulas II.** Let us now consider the $p^2=q$ case of (1.7). For an arbitrary sequence $\{A_n\}_0^\infty$ we then have

$$ \begin{aligned} A_0 &= \sum_{n=0}^{\infty}\frac{A_n}{(q^2;q^2)_n(aq;q)_{2n}}\sum_{k=0}^{n}\frac{(a;q)_k(1-aq^{3k})(q^{-2n};q^2)_k}{(q^2;q^2)_k(1-a)(aq^{2n+1};q)_k}q^{2nk} \\ &= \sum_{k=0}^{\infty}\frac{(a;q)_k(1-aq^{3k})(-1)^kq^{k^2-k}}{(q^2;q^2)_k(1-a)(aq;q)_{3k}}\sum_{n=0}^{\infty}\frac{A_{k+n}}{(q^2;q^2)_n(aq^{3k+1};q)_{2n}}, \end{aligned} \tag{4.1} $$

where it is assumed that the infinite series over n is absolutely convergent. If we now set $A_j=(b,a^2q^{2n+1}/b,q^{-2n};q^2)_j$, $j=0,1,\ldots$, where n is a fixed nonnegative integer, then this series becomes a terminating balanced ${}_3\phi_2$ series which can be summed by the q-Saalschütz formula (1.5). This leads us to the summation formula

$$ \sum_{k=0}^{n}\frac{(a;q)_k(1-aq^{3k})(b,a^2q^{2n+1}/b,q^{-2n};q^2)_k}{(q^2;q^2)_k(1-a)(aq/b,bq^{-2n}/a,aq^{2n+1};q)_k}q^{-\binom{k}{2}}a^{-k}=\frac{(aq;q)_{2n}}{(aq/b;q)_{2n}}b^{-n}. \tag{4.2} $$

Since

$$ {}_3\phi_2\left[\begin{matrix} q^{-2k}, & aq^k, & aq^{k+1} \\ & dq, & a^2q^2/d \end{matrix};q^2,q^2\right]=\frac{(d/a,aq/d;q)_k}{(dq,a^2q^2/d;q^2)_k}a^kq^{\binom{k}{2}+k} \tag{4.3} $$

by (1.5), we find that

(4.4)
$$\sum_{k=0}^{n}\frac{(a;q)_k(1-aq^{3k})(d/a,aq/d;q)_k(b,a^2q^{2n+1}/b,q^{-2n};q^2)_k}{(q^2;q^2)_k(1-a)(a^2q^2/d,dq;q^2)_k(aq/b,bq^{-2n}/a,aq^{2n+1};q)_k}q^k$$
$$=\sum_{k=0}^{n}\sum_{j=0}^{k}\frac{(a;q)_{2j+k}(1-aq^{3k})(b,a^2q^{2n+1}/b,q^{-2n};q^2)_k}{(q^2;q^2)_j(q^2;q^2)_{k-j}(aq/b,bq^{-2n}/a,aq^{2n+1};q)_k}\cdot\frac{(-1)^jq^{j(j+1)-\binom{k}{2}-2kj}a^{-k}}{(1-a)(dq,a^2q^2/d;q^2)_j}$$
$$=\sum_{j=0}^{n}\frac{(a;q)_{3j}(1-aq^{3j})(b,a^2q^{2n+1}/b,q^{-2n};q^2)_j(-1)^jq^{-3\binom{j}{2}}}{(q^2;q^2)_j(1-a)(aq/b,bq^{-2n}/a,aq^{2n+1};q)_j(dq,a^2q^2/d;q^2)_j}$$
$$\cdot\sum_{k=0}^{n-j}\frac{(aq^{3j};q)_k(1-aq^{3j+3k})(bq^{2j},a^2q^{2n+2j+1}/b,q^{2j-2n};q^2)_k}{(q^2;q^2)_k(1-aq^{3j})(aq^{j+1}/b,bq^{j-2n}/a,aq^{2n+j+1};q)_k}q^{-\binom{k}{2}}(aq^{3j})^{-k}.$$

By (4.2) the series over k equals $(aq^{3j+1};q)_{2n-2j}(bq^{2j})^{j-n}/(aq^{j+1}/b;q)_{2n-2j}$. The right side of (4.4) then simplifies to

$$\frac{(aq;q)_{2n}}{(aq/b;q)_{2n}}b^{-n}{}_3\phi_2\left[\begin{matrix}q^{-2n}, & b, & a^2q^{2n+1}/b \\ & dq, & a^2q^2/d\end{matrix};q^2,q^2\right].$$

But this ${}_3\phi_2$ series is balanced and so, by (1.5) we get the summation formula

(4.5)
$$\sum_{k=0}^{n}\frac{(a;q)_k(1-aq^{3k})(d/a,aq/d;q)_k(b,a^2q^{2n+1}/b,q^{-2n};q^2)_k}{(q^2;q^2)_k(1-a)(a^2q^2/d,dq;q^2)_k(aq/b,bq^{-2n}/a,aq^{2n+1};q)_k}q^k$$
$$=\frac{(aq;q)_{2n}(dq/b,a^2q^2/bd;q^2)_n}{(aq/b;q)_{2n}(dq,a^2q^2/d;q^2)_n},$$

which is the same as [6, (1.4)].

The $n\longrightarrow\infty$ limit of this formula gives

(4.6)
$$\sum_{k=0}^{\infty}\frac{(a;q)_k(1-aq^{3k})(d,q/d;q)_k(b;q^2)_k}{(q^2;q^2)_k(1-a)(aq^2/d,adq;q^2)_k(aq/b;q)_k}\left(\frac{aq}{b}\right)^kq^{\binom{k}{2}}$$
$$=\frac{(aq,aq^2,adq/b,aq^2/bd;q^2)_\infty}{(aq/b,aq^2/b,aq^2/d,adq;q^2)_\infty}.$$

Using (4.6) we shall now derive (1.10). First observe that, by (3.5)

(4.7)
$${}_3\phi_2\left[\begin{matrix}bq^{2k}, & bq^{-k}/a, & bq^{1-k}/a \\ & b^2cq/a^2, & bq^2/c\end{matrix};q^2,q^2\right]$$
$$=\frac{(a^2q/b^2c,q^2/c;q^2)_\infty(aq/c;q)_\infty}{(a^2q/bc,bq^2/c;q^2)_\infty(aq/bc;q)_\infty}\frac{(c,a^2q/bc;q^2)_k}{(aq/c,bc/a;q)_k}\left(\frac{b}{a}\right)^kq^{-\binom{k}{2}}$$
$$-\frac{(b,qa^2/b^2c,a^2q^3/bc^2;q^2)_\infty(b/a;q)_\infty}{(a^2q/bc,b^2c/a^2q,bq^2/c;q^2)_\infty(aq/bc;q)_\infty}\frac{(aq/b,a^2q/bc;q^2)_k}{(b,bc/a;q^2)_k}\left(\frac{b^2c}{a^2q}\right)^k$$
$$\cdot{}_3\phi_2\left[\begin{matrix}a^2q^{2k+1}/bc, & aq^{1-k}/bc, & aq^{2-k}/bc \\ & a^2q^3/bc^2, & a^2q^3/b^2c\end{matrix};q^2,q^2\right].$$

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Hence

$$
\begin{aligned}
&\sum_{k=0}^{\infty}\frac{(a;q)_k(1-aq^{3k})(d,q/d;q)_k(b,c,a^2q/bc;q^2)_k}{(q^2;q^2)_k(1-a)(aq^2/d,adq;q^2)(aq/b,aq/c,bc/a;q)_k}q^k\\
&=\frac{(a^2q/bc,bq^2/c;q^2)_\infty(aq/bc;q)_\infty}{(a^2q/b^2c,q^2/c;q^2)_\infty(aq/c;q)_\infty}\sum_{j=0}^{\infty}\frac{(b;q^2)_j(b/a;q)_{2j}q^{2j}}{(q^2,bq^2/c,b^2cq/a^2;q^2)_j}\\
&\quad\cdot\sum_{k=0}^{\infty}\frac{(a;q)_k(1-aq^{3k})(d,q/d;q)_k(bq^{2j};q^2)_k}{(q^2;q^2)_k(1-a)(aq^2/d,adq;q^2)_k(aq^{1-2j}/b;q)_k}\left(\frac{a}{b}q^{1-2j}\right)^k q^{\binom{k}{2}}\\
&\quad+\frac{(b,a^2q^3/bc^2;q^2)_\infty(b/a;q)_\infty}{(q^2/c,b^2c/a^2q;q^2)_\infty(aq/c;q)_\infty}\sum_{j=0}^{\infty}\frac{(a^2q/bc;q^2)_j(aq/bc;q)_{2j}q^{2j}}{(q^2,a^2q^3/bc^2,a^2q^3/b^2c;q^2)_j}\\
(4.8)\qquad&\quad\cdot\sum_{k=0}^{\infty}\frac{(a;q)_k(1-aq^{3k})(d,q/d;q)_k(a^2q^{2j+1}/bc;q^2)_k}{(q^2;q^2)_k(1-a)(aq^2/d,adq;q^2)_k(bcq^{-2j}/a;q)_k}\left(\frac{bcq^{-2j}}{a}\right)q^{\binom{k}{2}}\\
&=\frac{(aq,aq/bc;q)_\infty(a^2q/bc,bq^2/c,aq^2/bd,adq/b;q^2)_\infty}{(aq/b,aq/c;q)_\infty(a^2q/b^2c,q^2/c,adq,aq^2/d;q^2)_\infty}\\
&\quad\cdot{}_3\phi_2\left[\begin{matrix}b, & bd/a, & bq/ad\\ & bq^2/c, & b^2cq/a^2\end{matrix};q^2,q^2\right]\\
&\quad+\frac{(aq,b/a;q)_\infty(b,a^2q^3/bc^2,bcq/ad,bcd/a;q^2)_\infty}{(aq/c,bc/a;q)_\infty(b^2c/a^2q,q^2/c,adq,aq^2/d;q^2)_\infty}\\
&\quad\cdot{}_3\phi_2\left[\begin{matrix}a^2q/bc, & adq/bc, & aq^2/bcd\\ & a^2q^3/bc^2, & a^2q^3/b^2c\end{matrix};q^2,q^2\right],
\end{aligned}
$$

by (4.6). Applying (3.5) on the first ${}_3\phi_2$ series on the right side of (4.8) we get

(4.9)

$$
\begin{aligned}
&\sum_{k=0}^{\infty}\frac{(a;q)_k(1-aq^{3k})(d,q/d;q)_k(b,c,a^2q/bc;q^2)_k}{(q^2;q^2)_k(1-a)(aq^2/d,adq;q^2)_k(aq/b,aq/c,bc/a;q)_k}q^k\\
&=\frac{(aq,aq/bc;q)_\infty(adq/b,adq/c,aq^2/bd,aq^2/cd;q^2)_\infty}{(aq/b,aq/c;q)_\infty(adq,adq/bc,aq^2/d,aq^2/bcd;q^2)_\infty}\\
&\quad+\frac{(aq;q)_\infty(b,a^2q^3/bc^2;q^2)_\infty}{(aq/c;q)_\infty(b^2c/a^2q,q^2/c,adq,aq^2/d;q^2)_\infty}\\
&\left\{\frac{(bcq/ad,bcd/a;q^2)(b/a;q)_\infty}{(bc/a;q)_\infty}-\frac{(bd/a,bq/ad,aq^2/bd,adq/b;q^2)_\infty(aq/bc;q)_\infty}{(adq/bc,aq^2/bcd;q^2)_\infty(aq/b;q)_\infty}\right\}\\
&\quad{}_3\phi_2\left[\begin{matrix}a^2q/bc, & adq/bc, & aq^2/bcd\\ & a^2q^3/bc^2, & a^2q^3/b^2c\end{matrix},q^2,q^2\right].
\end{aligned}
$$

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However, by [11, (7.4.4)],

(4.10)

$$\begin{aligned}(aq/bc, aq^2/bc, bcq/a, bc/a, bd/a, aq^2/bd, adq/b, bq/ad; q^2)_\infty \\ - (b/a, aq^2/b, bq/a, aq/b, adq/bc, bcq/ad, bcd/a, aq^2/bcd; q^2)_\infty \\ = -\frac{aq}{bc}(d, q/d, q^2/d, dq, c, q^2/c, a^2q^3/b^2c, b^2c/a^2q; q^2)_\infty \\ = -\frac{aq}{bc}(d, q/d; q)_\infty(c, q^2/c, a^2q^3/b^2c, b^2c/a^2q; q^2)_\infty\end{aligned}$$

and so the coefficient of the ${}_3\phi_2$ series on the right side of (4.9) simplifies to

$$\frac{aq}{bc}\frac{(b, c, a^2q^3/bc^2, a^2q^3/b^2c, ; q^2)_\infty(aq, d, q/d; q)_\infty}{(adq, adq/bc, aq^2/d, aq^2/bcd; q^2)_\infty(aq/b, aq/c, bc/a; q)_\infty}$$

which completes the proof of (1.10). It can be shown that (1.10) is equivalent to Gosper's formula (1.7_q) in his second letter to Gessel and Stanton.

It is clear that if either b and c is of the form $q^{-2n}, n = 0, 1, \ldots$, then (1.10) reduces to (4.5), but it is not clear how (1.10) becomes (4.5) if $a^2q/bc = q^{-2n}$. The problem is that the right side of (1.10) is not symmetric in $b, c, a^2q/bc$, as it should be. So our final task in this section is to transform the right side of (1.10) to a form where the symmetry becomes obvious. To that end let us first make the changes: $a \longrightarrow abcq$, $b \longrightarrow acq$, $c \longrightarrow abq$ so that $a^2q/bc \longrightarrow bcq$ and (1.10) conforms to Gosper's notation. So (1.10) is rewritten as

$$\begin{aligned}(4.11)\quad &\sum_{k=0}^{\infty}\frac{(abcq;q)_k(1-abcq^{3k+1})(d,q/d;q)_k(abq,acq,bcq;q^2)_k}{(q^2;q^2)_k(1-abcq)(abcq^3/d,abcdq^2;q^2)_k(cq,bq,aq;q)_k}q^k \\ &= \frac{(abcq^2, a^{-1}; q)_\infty(cdq, bdq, cq^2/d, bq^2/d; q^2)_\infty}{(bq, cq; q)_\infty(abcq^3/d, abcdq^2, d/a, q/ad; q^2)_\infty} \\ &+ a^{-1}\frac{(abq, acq, bq^2/a, cq^2/a; q^2)_\infty(abcq^2, d, q/d; q)_\infty}{(abcq^3/d, abcdq^2, d/a, q/ad; q^2)_\infty(aq, bq, cq; q)_\infty} \\ &\cdot {}_3\phi_2\left[\begin{matrix}bcq, & d/a, & q/ad \\ & bq^2/a, & cq^2/a\end{matrix}; q^2, q^2\right].\end{aligned}$$

By Bailey's summation formula [11, (IV. 15)] for nonterminating very-well-poised balanced ${}_8\phi_7$ series,

(4.12)

$$\begin{aligned}&{}_8W_7(abcq; bcq, caq, abq, d, q/d; q^2, q^2) \\ &= \frac{(abcq^2, a^{-1}; q)_\infty(bdq, cdq, bq^2/d, cq^2/d; q^2)_\infty}{(bq, cq; q)_\infty(abcq^3/d, abcdq^2, d/a, q/ad; q^2)_\infty} \\ &+ a^{-1}\frac{(abcq^3, abq, acq, d, q/d, bq^2/d, cq^2/a, bcq^3/d, bcdq^2; q^2)_\infty}{(bcq^3/a, aq^2, bq, bq^2, cq, cq^2, d/a, q/ad, abcq^3/d, abcdq^2; q^2)_\infty} \\ &\cdot {}_8W_7(bcq/a; bcq, bq, cq, d/a, q/ad; q^2, q^2).\end{aligned}$$

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However, by [2, 8.5(3)],

(4.13)

$$
\begin{aligned}
{}_8W_7(bcq/a; bcq, bq, cq, d/a, q/ad; q^2, q^2) &= \frac{(bcq^3/a, abcq^2, dq, q^2/d; q^2)_\infty}{(aq, q^2/a, bcq^3/d, bcdq^2; q^2)_\infty} \\
&\quad \cdot {}_3\phi_2\left[\begin{matrix} bcq, & d/a, & q/ad \\ & bq^2/a, cq^2/a & \end{matrix}; q^2, q^2\right] \\
&\quad + \frac{(bcq^3/a, bcq, d/a, q/ad, bq^3, cq^3, q/a; q^2)_\infty}{(q^2/a, 1/aq, bq^2/a, cq^2/a, bcq^3/d, bcdq^2, q^2; q^2)_\infty} \\
&\quad \cdot {}_4\phi_3\left[\begin{matrix} dq, & q^2/d, & q^2, & abcq^2 \\ & aq^3, bq^3, cq^3 & & \end{matrix}; q^2, q^2\right].
\end{aligned}
$$

Combining (4.11), (4.12) and (4.13) we obtain the desired formula

(4.14)

$$
\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(abcq; q)_k(1 - abcq^{3k+1})(d, q/d; q)_k(abq, bcq, caq; q^2)_k}{(q^2; q^2)_k(1 - abcq)(abcq^3/d, abcdq^2; q^2)_k(cq, aq, bq; q)_k} q^k \\
&= {}_8\phi_7\left[\begin{matrix} abcq, & q^2\sqrt{abcq}, & -q^2\sqrt{abcq}, & d, & q/d, & abq, & bcq, & caq \\ & \sqrt{abcq}, & -\sqrt{abcq}, & abcq^3/d, & abcdq^2, & cq^2, & aq^2, & bq^2 \end{matrix}; q^2, q^2\right] \\
&\quad + \frac{(abcq^3, abq, bcq, caq, d, q/d; q^2)_\infty}{(q^2, aq^2, bq^2, cq^2, abcq^3/d, abcdq^2; q^2)_\infty} \frac{q}{(1 - aq)(1 - bq)(1 - cq)} \\
&\quad \cdot {}_4\phi_3\left[\begin{matrix} q^2, & abcq^2, & dq, & q^2/d \\ & aq^3, & bq^3, & cq^3 \end{matrix}; q^2, q^2\right].
\end{aligned}
$$

It is clear that if any one of $d, q/d, abq, bcq$ and caq is of the form q^{-2n}, then the second term on the right vanishes and the ${}_8\phi_7$ series can be summed by Jackson's formula [11, (IV. 8)]. On the other hand, if d or q/d is q^{-n} where n is an odd integer then the appropriate formula to use is (4.11) since the coefficient of the ${}_3\phi_2$ series on the right side vanishes.

5. **Proof of the cubic summation formulas.** We now set $p = q^3$ in (1.7) and obtain the following formulas

$$\sum_{k=0}^{3n} \frac{(a; q^3)_k(1 - aq^{4k})(q^{-3n}; q)_k}{(q; q)_k(1 - a)(aq^{3n+3}; q^3)_k} q^{3nk} = \delta_{n,0}, \tag{5.1}$$

$$\sum_{k=0}^{3n+1} \frac{(a; q^3)_k(1 - aq^{4k})(q^{-3n-1}; q)_k}{(q; q)_k(1 - a)(aq^{3n+4}; q^3)_k} q^{(3n+1)k} = 0, \tag{5.2}$$

$$\sum_{k=0}^{3n+2} \frac{(a; q^3)_k(1 - aq^{4k})(q^{-3n-2}; q)_k}{(q; q)_k(1 - a)(aq^{3n+5}; q^3)_k} q^{(3n+2)k} = 0. \tag{5.3}$$

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Similar to the quadratic case we then easily deduce the following summation formulas

$$\begin{aligned}\lambda_0 = &\sum_{k\equiv 0\ (\mathrm{mod}\,3)}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}(a;q^3)_k(1-aq^{4k})}{(q;q)_k(1-a)} \sum_{n=0}^{\infty} \frac{\lambda_{n+k/3}}{(q;q)_{3n}(aq^3;q^3)_{n+4k/3}} \\ (5.4)\qquad &+ \sum_{k\equiv 1\ (\mathrm{mod}\,3)}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}(a;q^3)_k(1-aq^{4k})}{(q;q)_k(1-a)} \sum_{n=0}^{\infty} \frac{\lambda_{n+\frac{k+2}{3}}}{(q;q)_{3n+2}(aq^3;q^3)_{n+\frac{4k+2}{3}}} \\ &+ \sum_{k\equiv 2\ (\mathrm{mod}\,3)}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}(a;q^3)_k(1-aq^{4k})}{(q;q)_k(1-a)} \sum_{n=0}^{\infty} \frac{\lambda_{n+\frac{k+1}{3}}}{(q;q)_{3n+1}(aq^3;q^3)_{n+\frac{4k+1}{3}}}\end{aligned}$$

$$(5.5)\qquad 0 = \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}(a;q^3)_k(1-aq^{4k})}{(q;q)_k(1-a)} g_k,$$

$$(5.6)\qquad 0 = \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}(a;q^3)_k(1-aq^{4k})}{(q;q)_k(1-a)} h_k,$$

where

$$(5.7)\qquad g_k = \begin{cases} \sum_{n=0}^{\infty} \frac{\mu_{n+\frac{k}{3}}}{(q,q)_{3n+1}(aq^4,q^3)_{n+\frac{4k}{3}}}, & \text{if } k \equiv 0 \ (\mathrm{mod}\,3), \\ \sum_{n=0}^{\infty} \frac{\mu_{n+\frac{k-1}{3}}}{(q,q)_{3n}(aq^4,q^3)_{n+\frac{4k-1}{3}}}, & \text{if } k \equiv 1 \ (\mathrm{mod}\,3), \\ \sum_{n=0}^{\infty} \frac{\mu_{n+\frac{k+1}{3}}}{(q,q)_{3n+2}(aq^4,q^3)_{n+\frac{4k+1}{3}}}, & \text{if } k \equiv 2 \ (\mathrm{mod}\,3), \end{cases}$$

and

$$(5.8)\qquad h_k = \begin{cases} \sum_{n=0}^{\infty} \frac{\nu_{n+\frac{k}{3}}}{(q,q)_{3n+2}(aq^5,q^3)_{n+\frac{4k}{3}}}, & \text{if } k \equiv 0 \ (\mathrm{mod}\,3), \\ \sum_{n=0}^{\infty} \frac{\nu_{n+\frac{k-1}{3}}}{(q,q)_{3n+1}(aq^5,q^3)_{n+\frac{4k-1}{3}}}, & \text{if } k \equiv 1 \ (\mathrm{mod}\,3), \\ \sum_{n=0}^{\infty} \frac{\nu_{n+\frac{k-2}{3}}}{(q,q)_{3n}(aq^5,q^3)_{n+\frac{4k-2}{3}}}, & \text{if } k \equiv 2 \ (\mathrm{mod}\,3), \end{cases}$$

where $\{\lambda_n\}$, $\{\mu_n\}$ and $\{\nu_n\}$ are arbitrary complex sequences such that the corresponding series on the right sides of (5.4)–(5.8) are convergent. Since $(a;q)_{3n} = (a,aq,aq^2;q^3)_n$, the simplest choice that can be made for these sequences is

$$(5.9)\qquad \begin{aligned}\lambda_n &= (b,c,d,aq^3/bcd;q^3)_n q^{3n}, \\ \mu_n &= (bq,cq,dq,aq^4/bcd;q^3)_n q^{3n}, \\ \nu_n &= (bq^2,cq^2,dq^2,aq^5/bcd;q^3)_n q^{3n}\end{aligned}$$

so that each of the series in (5.4), (5.7) and (5.8) is a balanced and nonterminating ${}_4\phi_3$ series in base q^3. But nonterminating ${}_4\phi_3$ series are neither summable nor transformable to another single series and so the only logical thing to do is to add suitable multiples of

(5.5) and (5.6) to (5.4) so that multiples of two ${}_4\phi_3$ series combine to give a very-well-poised ${}_8\phi_7$ series by [2, 8.5 (3)]. Thus, adding

$$-q(b,c,d,aq^3/bcd,aq^4;q^3)/\{(bq,cq,dq,aq^4/bcd,aq^3;q^3)_\infty\}$$

times (5.5) to (5.4) we obtain

$$\sum_{k=0}^{\infty}\frac{(-1)^k q^{k(k+1)/2}(a;q^3)_k(1-aq^{4k})}{(q;q)_k(1-a)}r_k=\frac{(aq^3;q^3)_\infty}{(q,b,c,d,aq^3/bcd;q^3)_\infty},\tag{5.10}$$

where

$$r_k=\frac{(bcq^{2k+1},bdq^{2k+1},cdq^{2k+1},aq^{4k+3};q^3)_\infty}{(bq^{k+1},cq^{k+1},dq^{k+1},bq^k,cq^k,dq^k,bcdq^{3k+1},aq^{k+3}/bcd;q^3)_\infty}\tag{5.11}$$

$$\cdot {}_8W_7(bcdq^{3k-2};bcdq^{3k},bq^k,cq^k,dq^k,bcdq^{-k-2}/a;q^3,aq^{k+4}/bcd),\quad \text{if } k\equiv 0 \pmod 3,$$

$$r_k=-\frac{1}{1-q}\frac{(bcq^{2k+2},bdq^{2k+2},cdq^{2k+2},aq^{4k+5};q^3)_\infty}{(bq^{k+2},cq^{k+2},dq^{k+2},bq^k,cq^k,dq^k,bcdq^{3k+4},aq^{k+5}/bcd;q^3)_\infty}$$

$$\cdot {}_8W_7(bcdq^{3k+1};bcdq^{3k},bq^{k+2},cq^{k+2},dq^{k+2},bcdq^{-k-1}/a;q^3,aq^{k+3}/bcd),$$
$$\text{if } k\equiv 1 \pmod 3,$$

$$r_k=\frac{q}{1-q}\frac{(bcq^{2k+3},bdq^{2k+3},cdq^{2k+3},aq^{4k+4};q^3)_\infty}{(bq^{k+2},cq^{k+2},dq^{k+2},bq^{k+1},cq^{k+1},dq^{k+1},bcdq^{3k+4},aq^{k+4}/bcd;q^3)_\infty}$$

$$\cdot {}_8W_7(bcdq^{3k+1};bcdq^{3k},bq^{k+1},cq^{k+1},dq^{k+1},bcdq^{-k}/a;q^3,aq^{k+5}/bcd),\quad \text{if } k\equiv 2 \pmod 3,$$

assuming, of course, that $|aq^3/bcd|<1$.

Similarly, adding

$$-q^2(b,c,d,aq^3/bcd,aq^5;q^3)_\infty/\{(bq^2,cq^2,dq^2,aq^5/bcd,aq^3;q^3)_\infty\}$$

times (5.6) to (5.4) we get

$$\sum_{k=0}^{\infty}\frac{(-1)^k q^{k(k+1)/2}(a;q^3)_k(1-aq^{4k})}{(q;q)_k(1-a)}s_k=\frac{(aq^3;q^3)_\infty}{(q^2,b,c,d,aq^3/bcd;q^3)_\infty},\tag{5.12}$$

where

$$s_k=\frac{(bcq^{2k+2},bdq^{2k+2},cdq^{2k+2},aq^{4k+3};q^3)_\infty}{(bq^{k+2},cq^{k+2},dq^{k+2},bq^k,cq^k,dq^k,bcdq^{3k+2},aq^{k+3}/bcd;q^3)_\infty}\tag{5.13}$$

$$\cdot {}_8W_7(bcdq^{3k-1};bcdq^{3k},bq^k,cq^k,dq^k,bcdq^{-k-1}/a;q^3,aq^{k+5}/bcd),\quad \text{if } k\equiv 0 \pmod 3,$$

$$s_k=-\frac{q}{1-q^2}\frac{(bcq^{2k+3},bdq^{2k+3},cdq^{2k+3},aq^{4k+5};q^3)_\infty}{(bq^{k+2},cq^{k+2},dq^{k+2},bq^{k+1},cq^{k+1},dq^{k+1},bcdq^{3k+5},aq^{k+5}/bcd;q^3)_\infty}$$

$$\cdot {}_8W_7(bcdq^{3k+2};bcdq^{3k},bq^{k+2},cq^{k+2},dq^{k+2},bcdq^{-k}/a;q^3,aq^{k+4}/bcd),\quad \text{if } k\equiv 1 \pmod 3,$$

$$s_k = -\frac{(bcq^{2k+1}, bdq^{2k+1}, cdq^{2k+1}, aq^{4k+4}; q^3)_\infty}{(bq^k, cq^k, dq^k, bq^{k+1}, cq^{k+1}, dq^{k+1}, bcdq^{3k+2}, aq^{k+4}/bcd; q^3)_\infty}$$
$$\cdot {}_8W_7(bcdq^{3k-1}; bcdq^{3k}, bq^{k+1}, cq^{k+1}, dq^{k+1}, bcdq^{-k-2}/a; q^3, aq^{k+3}/bcd),$$
$$\text{if } k \equiv 2 \pmod 3.$$

Unlike the quadratic case there does not seem to be any choice of the parameters a, b, c, d that would render any of the ${}_8W_7$ series in (5.11) and (5.13) summable to ratios of q-shifted factorials, so the next best thing is to choose them so that the formulas (5.10) and (5.12) can be brought to forms that suggest they are cubic extensions of the quadratic summation formula (1.8).

First, let us take $aq^2 = bcd$ in (5.10) and (5.11). All the ${}_8W_7$ series in (5.11) then terminate and hence, by Watson's formula [2, 8.5(2)], can be transformed to balanced terminating ${}_4\phi_3$ series each of which can, in turn, be freely transformed to other ${}_4\phi_3$ series of the same type by virtue of Sears' transformation formula [10]

(5.14)
$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, & a, & b, & c \\ & d, & e, & f \end{matrix}; q, q\right] = \frac{(aq^{1-n}/e, aq^{1-n}/f; q)_n}{(e,f;q)_n}\left(\frac{bc}{d}\right)^n {}_4\phi_3\left[\begin{matrix} q^{-n}, & a, & d/b, & d/c \\ & d, & aq^{1-n}/e, & aq^{1-n}/f \end{matrix}; q, q\right], $$

where $def = abcq^{1-n}$. Thus

(5.15)
$$ {}_8W_7(bcdq^{3k-2}; bcdq^{3k}, bq^k, cq^k, dq^k, q^{-k}; q^3, q^{k+2}) $$
$$ = \frac{(bcdq^{3k+1}, q^{1-k}/b; q^3)_{\frac{k}{3}}}{(cdq^{2k+1}, q; q^3)_{\frac{k}{3}}} \cdot {}_4\phi_3\left[\begin{matrix} q^{-k}, & bcdq^{3k}, & bq^k, & bq^{k+1} \\ bq^2, & bcq^{2k+1}, & bdq^{2k+1} & \end{matrix}; q^3, q^3\right] $$
$$ = \frac{(bcdq^{3k+1}, cq^2, dq^2, q^{1-k}/b; q^3)_{\frac{k}{3}}}{(cdq^{2k+1}, q, bcq^{2k+1}, bdq^{2k+1}; q^3)_{\frac{k}{3}}}(bq^{2k-1})^{\frac{k}{3}} {}_4\phi_3\left[\begin{matrix} q^{-k}, & q^{1-k}, & q^{2-k}, & bcdq^{3k} \\ & bq^2, & cq^2, & dq^2 \end{matrix}; q^3, q^3\right] $$
$$ = \frac{(bq^2, cq^2, dq^2, bcq^{3k+1}, bdq^{3k+1}, cdq^{3k+1}, q^{k+1}, bcdq^{3k+1}; q^3)_\infty}{(bq^{k+2}, cq^{k+2}, dq^{k+2}, bcq^{2k+1}, bdq^{2k+1}, cdq^{2k+1}, q, bcdq^{4k+1}; q^3)_\infty} \cdot (-1)^{\frac{k}{3}} q^{\binom{k}{2}} {}_4\phi_3\left[\begin{matrix} q^{-k}, & q^{1-k}, & q^{2-k}, & bcdq^{3k} \\ & bq^2, & cq^2, & dq^2 \end{matrix}; q^3, q^3\right] $$

when $k \equiv 0 \pmod 3$. Similarly

$$ {}_8W_7(bcdq^{3k+1}; bcdq^{3k}, bq^{k+2}, cq^{k+2}, dq^{k+2}, q^{1-k}; q^3, q^{k+1}) $$
(5.16)
$$ = \frac{(bq^2, cq^2, dq^2, bcq^{3k+1}, bdq^{3k+1}, cdq^{3k+1}, q^{k+3}, bcdq^{3k+4}; q^3)_\infty}{(bq^{k+1}, cq^{k+1}, dq^{k+1}, bcq^{2k+2}, bdq^{2k+2}, cdq^{2k+2}, q^4, bcdq^{4k+3}; q^3)_\infty} \cdot (-1)^{\frac{k-1}{3}} q^{\binom{k}{2}} {}_4\phi_3\left[\begin{matrix} q^{-k}, & q^{1-k}, & q^{2-k}, & bcdq^{3k} \\ & bq^2, & cq^2, & dq^2 \end{matrix}; q^3, q^3\right] $$

when $k \equiv 1 \pmod 3$, and

$$(5.17)\quad \begin{aligned} &{}_8W_7(bcdq^{3k+1}; bcdq^{3k}, bq^{k+1}, cq^{k+1}, dq^{k+1}, q^{2-k}; q^3, q^{k+3}) \\ &\quad = \frac{(bq^2, cq^2, dq^2, bcq^{3k+1}, bdq^{3k+1}, cdq^{3k+1}, q^{k+2}, bcdq^{3k+4}; q^3)_\infty}{(bq^k, cq^k, dq^k, bcq^{2k+3}, bdq^{2k+3}, cdq^{2k+3}, q^4, bcdq^{4k+2}; q^3)_\infty} \\ &\quad\quad \cdot (-1)^{\frac{k-2}{3}} q^{\binom{k}{2}-1} {}_4\phi_3\left[\begin{matrix} q^{-k}, & q^{1-k}, & q^{2-k}, & bcdq^{3k} \\ & bq^2, & cq^2, & dq^2 \end{matrix}; q^3, q^3\right] \end{aligned}$$

when $k \equiv 2 \pmod 3$. Substitution of (5.15)–(5.17) in (5.11) gives

$$(5.18)\quad \begin{aligned} r_k &= \frac{(bq^2, cq^2, dq^2, bcq, bdq, cdq; q^3)_\infty}{(q;q^3)_\infty (b,c,d;q)_\infty} \frac{(b,c,d;q)_k}{(cdq, bdq, bcq; q^3)_k} q^{\binom{k}{2}} (-1)^k \\ &\quad \cdot {}_4\phi_3\left[\begin{matrix} q^{-k}, & q^{1-k}, & q^{2-k}, & bcdq^{3k} \\ & bq^2, & cq^2, & dq^2 \end{matrix}; q^3, q^3\right]. \end{aligned}$$

Formula (1.11) then follows by combining (5.10) and (5.18). By a similar calculation we find that

$$(5.19)\quad \begin{aligned} s_k &= \frac{(bq, cq, dq, bcq^2, bdq^2, cdq^2; q^3)_\infty}{(q^2;q^3)_\infty (b,c,d;q)_\infty} \frac{(b,c,d;q)_k}{(cdq^2, bdq^2, bcq^2; q^3)_k} q^{\binom{k+1}{2}} (-1)^k \\ &\quad \cdot {}_4\phi_3\left[\begin{matrix} q^{-k-1}, & q^{-k}, & q^{1-k}, & bcdq^{3k} \\ & bq, & cq, & dq \end{matrix}; q^3, q^3\right]. \end{aligned}$$

Using (5.19) in (5.12) we obtain formula (1.12).

6. **Limiting forms of the summation formulas.** In this section we will display the limiting form of the summation formulas obtained in Sections 3–5. Use of the q-gamma function

$$(6.1)\quad \Gamma_q(x) = \frac{(q;q)_\infty (1-q)^{1-x}}{(q^x;q)_\infty}, \quad 0 < q < 1, \quad \Gamma(x) = \lim_{q \to 1} \Gamma_q(x)$$

gives us

$$(6.2)\quad \begin{aligned} &\sum_{n=0}^{\infty} \frac{(a)_n (a+\frac{3n}{2})(2b)_n (2c)_n (2a+1-2b-2c)_n}{n!(a)(1+a-b)_n (1+a-c)_n (\frac{1}{2}+b+c)_n} 4^{-n} \\ &\quad = \frac{\Gamma(\frac{1}{2})\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(b+c+\frac{1}{2})}{\Gamma(1+a)\Gamma(b+\frac{1}{2})\Gamma(c+\frac{1}{2})\Gamma(1+a-b-c)} \end{aligned}$$

as the limit of (1.8) and

$$(6.3)\quad \begin{aligned} &\sum_{n=0}^{\infty} \frac{(2a)_k (2a+3k)(2b)_k (1-2b)_k (c)_k (2a-c+n+\frac{1}{2})_k (-n)_k}{k!(2a)(1+a-b)_k (\frac{1}{2}+a+b)_k (1+2a-2c)_k (2c-2a-2n)_k (1+2a+2n)_k} \\ &\quad = \frac{(1+2a)_{2n} (\frac{1}{2}+a+b-c)_n (1+a-b-c)_n}{(1+2a-2c)_{2n} (1+a-b)_n (\frac{1}{2}+a+b)_n} \end{aligned}$$

as the limit of (1.10). This formula is the same as (1.7) in Gessel and Stanton [8]. The nonterminating extension of (6.3) is obtained by taking the $q \longrightarrow 1$ limit of (4.7):

(6.4)
$$\sum_{k=0}^{\infty}\frac{(2a)_k(2a+3k)(2d)_k(1-2d)_k(b)_k(c)_k(2a-b-c+\frac{1}{2})_k}{k!(2a)(1+a-d)_k(\frac{1}{2}+a+d)_k(1+2a-2b)_k(1+2a-2c)_k(2b+2c-2a)_k}$$
$$=\frac{\Gamma(1+2a-2b)\Gamma(1+2a-2c)\Gamma(\frac{1}{2}+a+d)\Gamma(1+a-d)\Gamma(\frac{1}{2}+a+d-b-c)\Gamma(1+a-b-c-d)}{\Gamma(1+2a)\Gamma(1+2a-2b-2c)\Gamma(\frac{1}{2}+a+d-b)\Gamma(\frac{1}{2}+a+d-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)}$$
$$+\frac{\Gamma(1+2a-2b)\Gamma(1+2a-2c)\Gamma(2b+2c-2a)\Gamma(\frac{1}{2}+a+d)\Gamma(1+a-d)}{\Gamma(1+2a)\Gamma(b)\Gamma(c)\Gamma(2d)\Gamma(1-2d)}$$
$$\cdot\frac{\Gamma(\frac{1}{2}+a+d-b-c)\Gamma(1+a-b-c-d)}{\Gamma(\frac{3}{2}+2a-2b-c)\Gamma(\frac{3}{2}+2a-b-2c)}$$
$$\cdot\,{}_3F_2\left[\begin{matrix}\frac{1}{2}+2a-b-c, & \frac{1}{2}+a+d-b-c, & 1+a-b-c-d\\ & \frac{3}{2}+2a-2b-c, & \frac{3}{2}+2a-b-2c\end{matrix};1\right].$$

The limiting form of (3.12) is

(6.5)
$$\sum_{k=0}^{\infty}\frac{(a)_k(2a+3k)(d)_k(\frac{1}{2}+a-d)_k(2b)_k(2c)_k(1+2a-2b-2c)_k}{k!(2a)(1+2a-2d)_k(2d)_k(1+a-b)_k(1+a-c)_k(\frac{1}{2}+b+c)_k}$$
$$=\frac{\Gamma(\frac{1}{2})\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(\frac{1}{2}+b+c)}{\Gamma(1+a)\Gamma(\frac{1}{2}+b)\Gamma(\frac{1}{2}+c)\Gamma(1+a-b-c)}\,{}_3F_2\left[\begin{matrix}b, & c, & \frac{1}{2}+a-b-c\\ & \frac{1}{2}+d, & 1+a-d\end{matrix};1\right]$$

which is the nonterminating extension of formula (1.8) in Gessel and Stanton [8].

Similarly,

(6.6)
$$\sum_{n=0}^{\infty}\frac{(a)_n(a+\frac{3n}{2})(b)_n(a-b+\frac{1}{2})_n(2c)_n(2d)_n(2a-2c-2d+1)_n}{n!(a)(2b)_n(2a-2b+1)_n(a-c+1)_n(a-d+1)_n(c+d+\frac{1}{2})_n}$$
$$+\frac{\Gamma(a-b)\Gamma(b+\frac{1}{2})\Gamma(3b-2a+1)\Gamma(a-c+1)\Gamma(a-d+1)\Gamma(c+d+\frac{1}{2})\Gamma(2b-2a+2c)}{\Gamma(b-a)\Gamma(2b-a+\frac{1}{2})\Gamma(a+1)\Gamma(2b-a-c+1)\Gamma(2b-a-d+1)\Gamma(2b-2a+c+d+\frac{1}{2})\Gamma(2c)}$$
$$\cdot\frac{\Gamma(2b-2a+2d)\Gamma(2b-2c-2d+1)}{\Gamma(2d)\Gamma(2a-2c-2d+1)}\sum_{n=0}^{\infty}\frac{(3b-2a)_n(3b-2a+\frac{3n}{2})(2b-a)_n(b-a+\frac{1}{2})_n}{n!(3b-2a)(4b-2a)_n(2b-2a+1)_n}$$
$$\cdot\frac{(2b-2a+2c)_n(2b-2a+2d)_n(2b-2c-2d+1)_n}{(2b-a-c+1)_n(2b-a-d+1)_n(2b-2a+c+d+\frac{1}{2})_n}$$
$$=\frac{\Gamma(\frac{1}{2})\Gamma(a-c+1)\Gamma(a-d+1)\Gamma(b-a+c)\Gamma(b-a+d)\Gamma(c+d+\frac{1}{2})\Gamma(b-c-d+\frac{1}{2})\Gamma(b+\frac{1}{2})}{\Gamma(a+1)\Gamma(b-a)\Gamma(c+\frac{1}{2})\Gamma(d+\frac{1}{2})\Gamma(a-c-d+1)\Gamma(b-c+\frac{1}{2})\Gamma(b-d+\frac{1}{2})\Gamma(b+c+d-a)},$$

(6.7)
$$\sum_{n=0}^{\infty}\frac{(a+b+c-\frac{2}{3})_n(a+b+c-\frac{2}{3}+\frac{4n}{3})(3a)_n(3b)_n(3c)_n}{n!(a+b+c-\frac{2}{3})(b+c+\frac{1}{3})_n(c+a+\frac{1}{3})_n(a+b+\frac{1}{3})_n}3^{-2n}$$
$$\cdot\,{}_4F_3\left[\begin{matrix}-\frac{n}{3}, & \frac{1-n}{3}, & \frac{2-n}{3}, & a+b+c+n\\ & a+\frac{2}{3}, & b+\frac{2}{3}, & c+\frac{2}{3}\end{matrix};1\right]$$
$$=\frac{\Gamma(\frac{1}{3})\Gamma(b+c+\frac{1}{3})\Gamma(c+a+\frac{1}{3})\Gamma(a+b+\frac{1}{3})}{\Gamma(a+\frac{1}{3})\Gamma(b+\frac{1}{3})\Gamma(c+\frac{1}{3})\Gamma(a+b+c+\frac{1}{3})},$$

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and

$$\sum_{n=0}^{\infty} \frac{(a+b+c-\frac{1}{3})_n(a+b+c-\frac{1}{3}+\frac{4n}{3})(3a)_n(3b)_n(3c)_n}{n!(a+b+c-\frac{1}{3})(b+c+\frac{2}{3})_n(c+a+\frac{2}{3})_n(a+b+\frac{2}{3})_n} 3^{-2n} \tag{6.8}$$
$$\cdot {}_4F_3\left[\begin{matrix} -\frac{n+1}{3}, & \frac{-n}{3}, & \frac{1-n}{3}, & a+b+c+n \\ & a+\frac{1}{3}, & b+\frac{1}{3}, & c+\frac{1}{3} \end{matrix}; 1\right]$$
$$= \frac{\Gamma(\frac{2}{3})\Gamma(b+c+\frac{2}{3})\Gamma(c+a+\frac{2}{3})\Gamma(a+b+\frac{2}{3})}{\Gamma(a+\frac{2}{3})\Gamma(b+\frac{2}{3})\Gamma(c+\frac{2}{3})\Gamma(a+b+c+\frac{2}{3})},$$

are the limits of (1.9), (1.11) and (1.12), respectively.

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Department of Mathematics and Statistics
Carleton University
Ottawa, Ontario
K1S 5B6

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