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ON THE HYPERSTABILITY OF A PEXIDERISED σ-QUADRATIC FUNCTIONAL EQUATION ON SEMIGROUPS

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Abstract

Motivated by the notion of Ulam stability, we investigate some inequalities connected with the functional equation

$$f(xy) + f(x\sigma(y)) = 2f(x) + h(y), \quad x, y \in G,$$

for functions f and h mapping a semigroup $(G, \cdot)$ into a commutative semigroup $(E, +)$, where the map $\sigma : G \to G$ is an endomorphism of G with $\sigma(\sigma(x)) = x$ for all $x \in G$. We derive from these results some characterisations of inner product spaces. We also obtain a description of solutions to the equation and hyperstability results for the σ-quadratic and σ-Drygas equations.

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1. Introduction

Let $(G, \cdot)$ be a semigroup and let $\sigma : G \to G$ be an endomorphism of G (that is, $\sigma(xy) = \sigma(x)\sigma(y)$ for all $x, y \in G$) and an involution (that is, $\sigma(\sigma(x)) = x$ for all $x \in G$). We consider some inequalities connected with the functional equation

$$f(xy) + f(x\sigma(y)) = 2f(x) + h(y) \tag{1.1}$$

for functions f and h mapping G into a commutative semigroup $(E, +)$. In this way, we obtain several results concerning characterisations of inner product spaces, solutions to (1.1) and hyperstability of the σ-quadratic and σ-Drygas equations.

In the particular situation where $h(x) \equiv 2f(x)$, (1.1) is just the σ-quadratic equation

$$f(xy) + f(x\sigma(y)) = 2f(x) + 2f(y) \tag{1.2}$$

introduced by Stetkær [26], which means that (1.1) is a partially pexiderised version of (1.2) (see, for example, [1]). Note that (1.2) is a natural generalisation of the well-known quadratic equation (with G being an abelian group), that is,

$$f(xy) + f(xy^{-1}) = 2f(x) + 2f(y). \tag{1.3}$$

Therefore, every solution $f : G \to E$ to (1.2) will be called a σ-quadratic function.

All solutions $f : G \to E$ to (1.2) have been described in [25], in the situation when G is commutative and E is an abelian group that is uniquely divisible by 2 (that is, for every $x \in E$ there is a unique $y \in E$ with $x = 2y$); each of them has the form

$$f(x) \equiv L(x, x) + a(x)$$

with some homomorphism $a : G \to E$ and a symmetric $L : G^2 \to E$ such that

$$a(\sigma(x)) = a(x), \quad x \in G, \tag{1.4}$$

$$L(xz, y) = L(x, y) + L(z, y), \quad x, y, z \in G, \tag{1.5}$$

$$L(\sigma(x), y) = -L(x, y), \quad x, y \in G. \tag{1.6}$$

It is easily seen that, in the case where G is an abelian group and $\sigma(x) \equiv x^{-1}$, (1.6) holds for any L satisfying (1.5) and condition (1.4) yields $a(x) \equiv 0$.

Next, let us observe that, with $h(x) \equiv 0$, equation (1.1) takes the form

$$f(xy) + f(x\sigma(y)) = 2f(x), \quad x, y \in G. \tag{1.7}$$

This is a natural generalisation of the Jensen functional equation

$$f(xy) + f(xy^{-1}) = 2f(x), \quad x, y \in G, \tag{1.8}$$

which is better known in the equivalent additively written form

$$f(\tfrac{1}{2}(x + y)) = \tfrac{1}{2}(f(x) + f(y)).$$

Under the assumption that G is commutative and E is a 2-cancellative abelian group (that is, $2x \neq 0$ for every $x \in E$ with $x \neq 0$), every solution $f : G \to E$ to (1.7) has the form (see [25])

$$f(x) = b(x) + c, \quad x \in G, \tag{1.9}$$

with some constant $c \in E$ and a homomorphism $b : G \to E$ such that

$$b(\sigma(x)) = -b(x), \quad x \in G. \tag{1.10}$$

Clearly, if G is an abelian group and $\sigma(x) \equiv x^{-1}$, then we obtain in this way the description of all solutions to (1.8).

Finally, in the special case where $h(y) = f(y) + f(\sigma(y))$ for all $y \in G$, (1.1) takes the form

$$f(xy) + f(x\sigma(y)) = 2f(x) + f(y) + f(\sigma(y)). \tag{1.11}$$

We call it the σ-Drygas functional equation, because a particular form of it (in the case where G is an abelian group and $\sigma(x) \equiv x^{-1}$) is the classical Drygas functional equation

$$f(xy) + f(xy^{-1}) = 2f(x) + f(y) + f(y^{-1}) \tag{1.12}$$

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introduced in [13] (see also [14, 15, 28]). Generalisations of (1.11) in abelian groups have been studied by Stetkær in [26, 27].

If E has a neutral element 0, then we say that a function $f : G \to E$ is σ-even provided $f(\sigma(x)) = f(x)$ for every $x \in G$; a function $f : G \to E$ is σ-odd provided $f(\sigma(x)) + f(x) = 0$ for every $x \in G$.

It is easily seen that σ-even solutions of (1.11) are σ-quadratic functions while σ-odd solutions satisfy (1.7) and therefore have the form (1.9). This means that under the assumption that E is a group uniquely divisible by 2, for every solution $f : G \to E$ of (1.11), there exist a constant $c \in E$, homomorphisms $a, b : G \to E$ and a function $L : G^2 \to E$ such that (1.4), (1.5), (1.6) and (1.10) are valid and

$$f(x) = L(x, x) + a(x) + b(x) + c, \quad x \in G. \tag{1.13}$$

As E is assumed here to be uniquely divisible by 2, a simple verification shows that such an f satisfies (1.11) if and only if $c = 0$; therefore, the general solution to (1.11) (with E uniquely divisible by 2) has the form

$$f(x) = L(x, x) + a(x) + b(x), \quad x \in G. \tag{1.14}$$

Plainly, if $c \neq 0$, then functions of the form (1.13) are the only solutions of the functional equation

$$f(xy) + f(x\sigma(y)) + 2c = 2f(x) + f(y) + f(\sigma(y)). \tag{1.15}$$

If G has a neutral element denoted by e, then this equation can be written in the form

$$f(xy) + f(x\sigma(y)) + 2f(e) = 2f(x) + f(y) + f(\sigma(y)). \tag{1.16}$$

So, formula (1.13) (with any c) depicts all solutions $f : G \to E$ of (1.16), that is, to (1.1) with

$$h(x) \equiv f(x) + f(\sigma(x)) - 2f(e).$$

If E is 2-cancellative, then we can also say that a solution $f : G \to E$ to (1.16) satisfies (1.11) if and only if $f(e) = 0$.

Clearly, if G is an abelian group and $\sigma(x) \equiv x^{-1}$, then (1.13) becomes

$$f(x) = L(x, x) + b(x) + c, \quad x \in G, \tag{1.17}$$

because $a(x) \equiv 0$, as we have observed above. So, (1.17) depicts all solutions of the equation

$$f(xy) + f(xy^{-1}) + 2f(e) = 2f(x) + f(y) + f(y^{-1}).$$

Functions $f : G \to E$ of the form (1.17) (when G is an abelian group and $\sigma(x) \equiv x^{-1}$) are called polynomial functions of order at most 2 (see [22, 29]) and they are the only solutions of the Fréchet equation

$$\Delta_t^3 f(z) = 0, \quad t, z \in G, \tag{1.18}$$

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where Δ denotes the Fréchet difference operator given by

$$\Delta_y f(x) = \Delta_y^1 f(x) := f(xy) - f(x), \quad x, y \in G,$$

and recurrently

$$\Delta_{x_{n+1},x_n,\dots,x_1} := \Delta_{x_{n+1}} \circ \Delta_{x_n,\dots,x_1}, \quad x_1, \dots, x_{n+1} \in G,\ n \in \mathbb{N},$$
$$\Delta_t^{n+1} := \Delta_t \circ \Delta_t^n, \quad t, z \in G,\ n \in \mathbb{N}.$$

It is known that (when G is commutative and under suitable assumptions on the divisibility of E), for a given $n \in \mathbb{N}$, the two functional equations

$$\Delta_{x_n,\dots,x_1} f(z) = 0, \quad x_1, \dots, x_n, z \in G,$$
$$\Delta_t^n f(z) = 0, \quad t, z \in G,$$

have the same solutions $f : G \to E$ (see [12]), which are called polynomial functions in [22] (polynomials in [29]) of order at most $n - 1$. Functions $f : G \to E$ of the form (1.17) are the only solutions of (1.18).

By analogy we can say that every solution $f : G \to E$ of equation (1.16) is a σ-polynomial of order at most 2.

Throughout this paper, $\mathbb{R}^+$ stands for the set of nonnegative real numbers, $\mathbb{N}$ stands for the set of positive integers and Y^X denotes the family of all functions mapping a nonempty set X into a nonempty set Y. Unless explicitly stated otherwise, we assume that $(E, +)$ is a commutative semigroup, d is a metric in E that is invariant (that is, $d(x + z, y + z) = d(x, y)$ for $x, y, z \in E$), $(G, \cdot)$ is a semigroup and $\sigma : G \to G$ is an endomorphism and involution.

2. The main result

The following theorem is the main result of this paper. It has been motivated by the issue of Ulam stability, which concerns approximate solutions of functional equations. Very roughly, this form of stability means that a function satisfying an equation approximately (in some sense) must be near an exact solution to the equation. It has been studied in connection with a question of Ulam from 1940 about approximate homomorphisms of groups (for more details, see [9, 16–18, 20, 21]). The next theorem deals with approximate solutions of the functional equation (1.1).

Theorem 2.1. *Let $\varepsilon : G \times G \to \mathbb{R}^+$ be a function such that there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in G satisfying one of the following two conditions.*

$$\liminf_{n\to\infty} \varepsilon(x, yu_n) = \liminf_{n\to\infty} \varepsilon(x, y\sigma(u_n)) = 0, \quad x, y \in G, \tag{2.1}$$

$$\liminf_{n\to\infty} \varepsilon(u_n x, y) = 0, \quad x, y \in G. \tag{2.2}$$

Suppose that $f, h : G \to E$ satisfy the inequality

$$d(f(xy) + f(x\sigma(y)), 2f(x) + h(y)) \le \varepsilon(x, y), \quad x, y \in G. \tag{2.3}$$

Then the following two statements are valid.

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(a) *If (2.1) holds and G has a neutral element e, then f is a solution of the functional equation (1.16).*
(b) *If (2.2) holds, then h is a solution of (1.2).*

We provide a proof of the theorem in the last part of the paper. First, we will present some comments and (in the next two sections) show several consequences of Theorem 2.1.

REMARK 2.2. Let $\varepsilon : G \times G \to \mathbb{R}^+$ be a function such that there exist $\alpha \in G$ and $p \in [0, 1)$ satisfying

$$\varepsilon(x, y\alpha) \le p\varepsilon(x, y), \quad \varepsilon(x, y\sigma(\alpha)) \le p\varepsilon(x, y), \quad x, y \in G.$$

Then, by induction,

$$\varepsilon(x, y\alpha^n) \le p^n\varepsilon(x, y), \quad \varepsilon(x, y\sigma(\alpha^n)) \le p^n\varepsilon(x, y)$$

for every $x, y \in G$ and $n \in \mathbb{N}$. So, ε satisfies (2.1) with $u_n = \alpha^n$ for $n \in \mathbb{N}$.

Analogously, if there are $\alpha \in G$ and $r \in [0, 1)$ with

$$\varepsilon(\alpha x, y) \le r\varepsilon(x, y), \quad x, y \in G,$$

then (2.2) holds also with the sequence $u_n = \alpha^n$ for $n \in \mathbb{N}$.

Another example of suitable functions $\varepsilon : G \times G \to \mathbb{R}^+$ satisfying conditions (2.1) and (2.2) is provided in Corollary 3.1 and Remark 3.2.

REMARK 2.3. We give two very simple natural examples of endomorphisms $\sigma : G \to G$ which are involutions, in the field of complex numbers $\mathbb{C}$.

First, if G is any multiplicative or additive subsemigroup of $\mathbb{C}$ with $\bar{z} \in G$ for each $z \in G$, then we can take $\sigma(z) \equiv \bar{z}$. Second, if $G = \{z \in \mathbb{C} : \Re z \ge 0, \Im z \ge 0\}$ or $G = \mathbb{C}$ with the operation being the usual addition of complex numbers, then we can take $\sigma(a + ib) \equiv b + ia$; if $G = \mathbb{C}$, then we also can take $\sigma(a + ib) \equiv -(b + ia)$.

There are many other more involved examples of endomorphisms which are also involutions (see, for example, [7]).

3. Some consequences

The next corollary generalises the well-known parallelogram law characterisation of inner product spaces due to Jordan and von Neumann [19], stating that a normed space X is an inner product space if and only if

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad x, y \in X. \tag{3.1}$$

For more information and various related results, we refer to [2, 3].

COROLLARY 3.1. *Let X be a normed space, $\sigma_0 : X \to X$ be an additive involution and $\psi : X^2 \to \mathbb{R}^+$ be such that there is a sequence $(u_n)_{n\in\mathbb{N}}$ in X with $\lim_{n\to\infty} \|u_n\| = \infty$ and*

$$\liminf_{n\to\infty} \psi(u_n + x, z) = 0, \quad x, z \in X. \tag{3.2}$$

Further, suppose that one of the following two conditions is valid.

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(i) *There exist* $s > 0$ *and* $h : X \to \mathbb{R}$ *with*

$$\big|\, \|x+y\|^2 + \|x+\sigma_0(y)\|^2 - 2\|x\|^2 - h(y) \,\big| \le \psi(x,y) \quad \textit{for } x, y \in X, \|x\| > s, \|x+y\| > s.$$

(ii) *There exist* $s > 0$ *and* $f : X \to \mathbb{R}$ *with*

$$\big| f(x+y) + f(x+\sigma_0(y)) - 2f(x) - \|y\|^2 \big| \le \psi(x,y) \quad \textit{for } x, y \in X, \|x\| > s, \|x+y\| > s.$$

Then X *is an inner product space.*

Proof. Let $B_s := \{x \in X : \|x\| \le s\}$ and $\varepsilon : X \times X \to \mathbb{R}^+$ be given by

$$\varepsilon(x,y) := \psi(x,y), \qquad x \in X \setminus B_s, y \in X, x+y \notin B_s,$$

and, for $x, y \in X$ such that $x \in B_s$ or $x+y \in B_s$,

$$\varepsilon(x,y) := \begin{cases} \big|\, \|x+y\|^2 + \|x+\sigma_0(y)\|^2 - 2\|x\|^2 - h(y) \,\big| & \text{if (i) holds,} \\ \big|\, f(x+y) + f(x+\sigma_0(y)) - 2f(x) - \|y\|^2 \,\big| & \text{if (ii) holds.} \end{cases}$$

Then it is easily seen that (2.2) holds for $(G, \cdot) = (X, +)$, $(E, +) = (\mathbb{R}, +)$ and $\sigma := \sigma_0$.

Hence, if (i) holds, then (2.3) is valid for $f(x) \equiv \|x\|^2$ and Theorem 2.1 implies that the function f is a solution of (1.11). Thus, condition (1.14) holds with some homomorphisms $a, b : G \to E$ and a function $L : G^2 \to E$ such that (1.4), (1.5), (1.6) and (1.10) are satisfied. Since f is even, we must have $a(x) + b(x) \equiv 0$. Consequently, $f(x) = L(x,x)$ for $x \in X$, which yields (3.1).

If (ii) holds, then (2.3) is valid with $h(x) = \|x\|^2$ for $x \in X$, which means that h is a solution to (1.2) (again in view of Theorem 2.1) and consequently $f(x) \equiv L(x,x) + a(x)$ with some homomorphism $a : G \to E$ and a symmetric $L : G^2 \to E$ such that (1.4)–(1.6) are valid. As before, $a(x) \equiv 0$ (because f is even), which yields (3.1).

So, in either case we have obtained the statement. □

Remark 3.2. Let X be a normed space and let $k \in \mathbb{N}$, $\beta_i, \gamma_i : X \to \mathbb{R}^+$, $s_i, v_i \in (0, \infty)$, $p_i, r_i, t_i \in (-\infty, 0)$, $A_i, D_i \ge 0$, $C_i > 0$ for $i = 1, \dots, k$ and

$$\psi(w,z) := \sum_{i=1}^{k} (A_i \|z+w\|^{p_i} + D_i(\gamma_i(z) + C_i\|w\|^{s_i})^{t_i} + \beta_i(z)\|w\|^{r_i})^{v_i}, \quad \text{for } z \in X, w \in X \setminus \{0\}, z \ne -w,$$

$$\psi(0,z) := 0, \qquad \psi(z,-z) := 0 \quad \text{for } z \in X.$$

Then it is easily seen that (3.2) holds for every sequence $(u_n)_{n\in\mathbb{N}}$ in X with

$$\lim_{n\to\infty} \|u_n\| = \infty.$$

Theorem 2.1 also yields the next corollary, which describes the solutions to the functional equation (1.1).

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Corollary 3.3. *Assume that G is commutative and has a neutral element e, and E is a group uniquely divisible by 2. Functions $f, h : G \to E$ satisfy equation (1.1) if and only if there exist a constant $c \in E$, homomorphisms $a, b : G \to E$ and a symmetric function $L : G^2 \to E$ such that (1.4), (1.5), (1.6) and (1.10) are valid and*

$$f(x) = L(x, x) + a(x) + b(x) + c, \qquad x \in G, \tag{3.3}$$

$$h(x) = 2L(x, x) + 2a(x), \qquad x \in G. \tag{3.4}$$

Proof. Let $f, h : G \to E$ satisfy equation (1.1). According to Theorem 2.1 with $\varepsilon(x, y) \equiv 0$ and d being the discrete metric in E, f is a solution of (1.16). Further, in view of the description of solutions to (1.16) provided in the introduction, there exist $c \in E$, homomorphisms $a, b : S \to E$ and a symmetric function $L : G^2 \to E$ such that (1.4), (1.5), (1.6), (1.10) and (3.3) are valid.

Now, simple calculations show that (1.1) holds for $x, y \in G$, with such an f, if and only if h is of the form (3.4); this also means that the converse implication is true. □

Remark 3.4. It is easily seen that Corollary 3.3 yields at once descriptions of solutions to the equations discussed in the introduction. Namely, equation (1.1) becomes (1.2) when $h = 2f$, which (in view of (3.3) and (3.4)) is the case if $b(x) \equiv 0$ and $c = 0$ and then we get $f(x) = L(x, x) + a(x)$ for $x \in G$ with symmetric L and homomorphic a satisfying (1.4)–(1.6). Next, (1.2) with $\sigma(x) \equiv x^{-1}$ is equation (1.3); note that (1.4) implies that $a(x) \equiv 0$, which means that $f(x) \equiv L(x, x)$ with any symmetric L satisfying (1.5) ((1.5) implies (1.6) for such σ).

If we take $h(x) \equiv 0$ in (1.1), then we obtain (1.7). Clearly, $h(x) \equiv 0$ in (3.4) means that $L(x, x) + a(x) \equiv 0$ and, consequently, by (3.3), $f(x) = b(x) + c$ for every $x \in G$ (with (1.10) fulfilled). Equation (1.8) is (1.7) with $\sigma(x) \equiv x^{-1}$ and then (1.10) holds for each homomorphism $b : G \to E$.

Equation (1.1) with $h(x) \equiv f(x) + f(\sigma(x))$ has the form (1.11). In this case (1.4), (1.6), (1.10), (3.3) and (3.4) yield $c = 0$, which means that every solution $f : G \to E$ to (1.11) is depicted by (1.14). For (1.12) (that is, $\sigma(x) \equiv x^{-1}$), (1.4) gives $a(x) \equiv 0$.

Finally, if $h(x) \equiv f(x) + f(\sigma(x)) - 2f(e)$ in (1.1), then we obtain (1.16) (or (1.15)), which has solutions described by (3.3); again with $a(x) \equiv 0$ when $\sigma(x) \equiv x^{-1}$.

4. Hyperstability results

One of the notions connected with the issue of Ulam stability is that of hyperstability. The following definition describes the main ideas for equations in several variables (for more details, see [11]).

Definition 4.1. Let X be a nonempty set, (Y, d) be a metric space, $\varepsilon : X^n \to \mathbb{R}^+$ (with $n \in \mathbb{N}$) be an arbitrary function and let $\mathcal{F}_1, \mathcal{F}_2$ be two operators mapping a nonempty set $\mathcal{D} \subset Y^X$ into Y^{X^n}. We say that the equation

$$\mathcal{F}_1\varphi(x_1, \ldots, x_n) = \mathcal{F}_2\varphi(x_1, \ldots, x_n), \qquad x_1, \ldots, x_n \in X, \tag{4.1}$$

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is ε-hyperstable provided every $\varphi_0 \in \mathcal{D}$, which satisfies

$$d(\mathcal{F}_1\varphi_0(x_1,\dots,x_n),\mathcal{F}_2\varphi_0(x_1,\dots,x_n)) \le \varepsilon(x_1,\dots,x_n), \quad x_1,\dots,x_n \in X,$$

satisfies equation (4.1).

The following two corollaries are special cases of Theorem 2.1 and describe hyperstability results for equations (1.2) and (1.11). They complement several recent results in [4–6, 8, 23, 24, 30], where stability of particular forms of the functional equation

$$\sum_{i=1}^{m} A_i f\Big(\sum_{j=1}^{n} a_i x_i\Big) + A = 0 \tag{4.2}$$

has been studied for functions f mapping a linear space X over $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$ into a linear space Y over $\mathbb{F} \in \{\mathbb{R},\mathbb{C}\}$ with fixed $m, n \in \mathbb{N}$, $A \in Y$ and $A_1,\dots,A_m \in \mathbb{F}$, $a_1,\dots,a_n \in \mathbb{K}$. Clearly, if $\sigma(x) \equiv x^{-1}$, equation (1.2) can be considered to be a special case of (4.2) (when $G = X$ and $E = Y$), but this is not true for an arbitrary homomorphic involution σ. The results on the stability of (1.2) in [8] are in somewhat similar settings to this paper (but with the function ε being constant); however, no hyperstability outcomes have been provided there.

Corollary 4.2. *Let $\varepsilon : G \times G \to \mathbb{R}^+$ be a function such that (2.1) holds with a sequence $(u_n)_{n\in\mathbb{N}}$ in G. If G has a neutral element e and $f : G \to E$ satisfies the inequality*

$$d(f(xy) + f(x\sigma(y)) + 2f(e), 2f(x) + f(y) + f(\sigma(y))) \le \varepsilon(x,y), \quad x, y \in G,$$

then f is a solution of (1.16).

Proof. Taking $h(x) \equiv f(x) + f(\sigma(x))$ in Theorem 2.1 gives the desired result. □

Corollary 4.3. *Let E be 2-cancellative and $\varepsilon : G \times G \to \mathbb{R}^+$ be a function such that (2.2) holds with a sequence $(u_n)_{n\in\mathbb{N}}$ in G. If $f : G \to E$ satisfies the inequality*

$$d(f(xy) + f(x\sigma(y)), 2f(x) + 2f(y)) \le \varepsilon(x,y), \quad x, y \in G, \tag{4.3}$$

then f is a solution of equation (1.2).

Proof. Taking $h(x) \equiv 2f(x)$ in Theorem 2.1, we deduce that h is a solution to (1.2), whence so is f by 2-cancellativity of E. □

Following observations in [10], on the stability of some inhomogeneous functional equations, we present two hyperstability results for the inhomogeneous versions of (1.2) and (1.16).

Corollary 4.4. *Let $(E,+)$ be a commutative group, $F : G \times G \to E$ and $\varepsilon : G^2 \to \mathbb{R}^+$ satisfy (2.2) with a sequence $(u_n)_{n\in\mathbb{N}}$ in G. Let $f : G \to E$ be a function such that*

$$d(f(xy) + f(x\sigma(y)), 2f(x) + 2f(y) + F(x,y)) \le \varepsilon(x,y), \quad x, y \in G.$$

Assume that the functional equation

$$f(xy) + f(x\sigma(y)) = 2f(x) + 2f(y) + F(x,y) \tag{4.4}$$

admits a solution $f_0 : G \to E$. Then f is a solution of (4.4).

Proof. Let $f_1(x) := f(x) - f_0(x)$ for $x \in G$. Then

$$\begin{aligned}
&d(f_1(xy) + f_1(x\sigma(y)), 2f_1(x) + 2f_1(y)) \\
&\quad = d(f(xy) + f(x\sigma(y)) - (f_0(xy) + f_0(x\sigma(y))), \\
&\qquad\qquad 2f(x) + 2f(y) + F(x,y) - (2f_0(x) + 2f_0(y) + F(x,y))) \\
&\quad \le d(f(xy) + f(x\sigma(y)), 2f(x) + 2f(y) + F(x,y)) \\
&\qquad\quad + d(-(f_0(xy) + f_0(x\sigma(y))), -(2f_0(x) + 2f_0(y) + F(x,y))) \\
&\quad = d(f(xy) + f(x\sigma(y)), 2f(x) + 2f(y) + F(x,y)) \le \varepsilon(x,y), \quad x, y \in G.
\end{aligned}$$

Consequently, statement (b) of Theorem 2.1 is valid with f replaced by f_1. Therefore,

$$\begin{aligned}
&f(xy) + f(x\sigma(y)) - 2f(x) - 2f(y) - F(x,y) \\
&\quad = f_1(xy) + f_1(x\sigma(y)) - 2f_1(x) - 2f_1(y) + f_0(xy) \\
&\qquad + f_0(x\sigma(y)) - 2f_0(x) - 2f_0(y) - F(x,y) = 0, \quad x, y \in G. \qquad \square
\end{aligned}$$

Analogously we can prove the following result.

Corollary 4.5. *Let $(E, +)$ be a commutative group, $F : G \times G \to E$, $\varepsilon : G^2 \to \mathbb{R}^+$ satisfy (2.1) with a sequence $(u_n)_{n\in\mathbb{N}}$ in G and G have a neutral element e. Let $f : G \to E$ be a function such that*

$$d(f(xy) + f(x\sigma(y)) + 2f(e), 2f(x) + f(y) + f(\sigma(y)) + F(x,y)) \le \varepsilon(x,y), \quad x, y \in G.$$

Assume that the functional equation

$$f(xy) + f(x\sigma(y)) + 2f(e) = 2f(x) + f(y) + f(\sigma(y)) + F(x,y) \tag{4.5}$$

admits a solution $f_0 : G \to E$. Then f is a solution of (4.5).

5. Proof of Theorem 2.1

First, note that

$$\begin{aligned}
d(x_1 + x_2, y_1 + y_2) &\le d(x_1 + x_2, x_2 + y_1) + d(x_2 + y_1, y_1 + y_2) \\
&= d(x_1, y_1) + d(x_2, y_2), \quad x_1, x_2, y_1, y_2 \in E.
\end{aligned}$$

Hence, by induction, for $n \in \mathbb{N}$ and $x_i, y_i \in E, i = 1, \ldots, n$,

$$d(x_1 + \cdots + x_n, y_1 + \cdots + y_n) \le d(x_1, y_1) + \cdots + d(x_n, y_n). \tag{5.1}$$

Let $f, h : G \to E$ be functions satisfying (2.3). Write

$$D(x,y) = d(f(xy) + f(x\sigma(y)), 2f(x) + h(y)), \quad x, y \in G,$$
$$D_1(x,y) = d(2f(xy) + 2f(x\sigma(y)) + 4f(e), 4f(x) + 2f(y) + 2f(\sigma(y))), \quad x, y \in G,$$
$$D_2(x,y) = d(h(xy) + h(x\sigma(y)), 2h(x) + 2h(y)), \quad x, y \in G,$$

and

$$\begin{aligned}
w_1(x,y,z) = 2h(z) &+ f(xyz) + f(xy\sigma(z)) + f(x\sigma(y)z) + f(x\sigma(y)\sigma(z)) \\
&+ h(yz) + h(y\sigma(z)) + f(yz) \\
&+ f(\sigma(y)\sigma(z)) + f(y\sigma(z)) + f(\sigma(y)z), \quad x, y, z \in G,
\end{aligned}$$

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$$w_2(x,y,z) = f(zxy) + f(z\sigma(x)\sigma(y)) + f(zx\sigma(y)) + f(z\sigma(x)y) \\ + 4f(z) + 2f(zx) + 2f(z\sigma(x)), \quad x,y,z \in G.$$

Then

$$D_1(x,y) = d(2f(xy) + 2f(x\sigma(y)) + 4f(e) + w_1(x,y,z), \\ 4f(x) + 2f(y) + 2f(\sigma(y)) + w_1(x,y,z)) \tag{5.2}$$

and

$$D_2(x,y) = d(h(xy) + h(x\sigma(y)) + w_2(x,y,z), 2h(x) + 2h(y) + w_2(x,y,z)) \tag{5.3}$$

for every $x, y, z \in G$, because d is invariant. Consequently, (5.1) and (5.2) yield

$$\begin{aligned} D_1(x,y) &\le d(2f(xy) + h(z), f(xyz) + f(xy\sigma(z))) \\ &\quad + d(2f(x\sigma(y)) + h(z), f(x\sigma(y)z) + f(x\sigma(y)\sigma(z))) \\ &\quad + d(f(xyz) + f(x\sigma(yz)), 2f(x) + h(yz)) \\ &\quad + d(f(xy\sigma(z)) + f(x\sigma(y\sigma(z))), 2f(x) + h(y\sigma(z))) \\ &\quad + d(h(yz) + 2f(e), f(yz) + f(\sigma(yz))) \\ &\quad + d(h(y\sigma(z)) + 2f(e), f(y\sigma(z)) + f(\sigma(y)z)) \\ &\quad + d(f(yz) + f(y\sigma(z)), 2f(y) + h(z)) \\ &\quad + d(f(\sigma(y)z) + f(\sigma(y)\sigma(z)), 2f(\sigma(y)) + h(z)) \\ &= D(xy,z) + D(x\sigma(y),z) + D(x,yz) + D(x,y\sigma(z)) \\ &\quad + D(e,yz) + D(e,y\sigma(z)) + D(y,z) + D(\sigma(y),z) \end{aligned} \tag{5.4}$$

for all $x, y, z \in G$. Analogously, by (5.1) and (5.3),

$$\begin{aligned} D_2(x,y) &\le d(2f(z) + h(xy), f(zxy) + f(z\sigma(xy))) \\ &\quad + d(2f(z) + h(x\sigma(y)), f(zx\sigma(y)) + f(z\sigma(x)y)) \\ &\quad + d(f(zxy) + f(zx\sigma(y)), 2f(zx) + h(y)) \\ &\quad + d(f(z\sigma(x)y) + f(z\sigma(x)\sigma(y)), 2f(z\sigma(x)) + h(y)) \\ &\quad + d(2f(zx) + 2f(z\sigma(x)), 4f(z) + 2h(x)) \\ &\le D(z,xy) + D(z,x\sigma(y)) + D(zx,y) + D(z\sigma(x),y) + 2D(z,x) \end{aligned} \tag{5.5}$$

for all $x, y, z \in G$.

Suppose that there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in G satisfying condition (2.1). Replacing y by yu_n in (2.3),

$$D(x, yu_n) \le \varepsilon(x, yu_n)$$

for all $x, y \in G$ and $n \in \mathbb{N}$. Also, replacing y by $y\sigma(u_n)$ in (2.3),

$$D(x, y\sigma(u_n)) \le \varepsilon(x, y\sigma(u_n)), \quad x, y \in G.$$

Thus,

$$\liminf_{n\to\infty} D(x, yu_n) = \liminf_{n\to\infty} D(x, y\sigma(u_n)) = 0, \quad x, y \in G. \tag{5.6}$$

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Let $x, y, t \in G$ be fixed. Replacing z by tu_n in (5.4),

$$D_1(x,y) \le D(xy, tu_n) + D(x\sigma(y), tu_n) + D(x, ytu_n) + D(x, y\sigma(tu_n)) + D(e, ytu_n) + D(e, y\sigma(tu_n)) + D(y, tu_n) + D(\sigma(y), tu_n), \quad n \in \mathbb{N}. \tag{5.7}$$

Now, taking the limit as $n \to \infty$ and applying (5.6), we deduce from (5.7) that

$$D_1(x,y) = 0, \quad x, y \in G.$$

That is, f is a solution of equation (1.11).

Now, suppose that there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in G satisfying condition (2.2). Replacing x by $u_n x$ in (2.3),

$$D(u_n x, y) \le \varepsilon(u_n x, y), \quad x, y \in G, n \in \mathbb{N}.$$

So, by (2.2),

$$\liminf_{n\to\infty} D(u_n x, y) = 0, \quad x, y \in G. \tag{5.8}$$

Let $x, y, s \in G$ be fixed. Replacing z by $u_n s$ in (5.5), for $n \in \mathbb{N}$,

$$D_2(x,y) = D(u_n s, xy) + D(u_n s, x\sigma(y)) + D(u_n sx, y) + D(u_n s\sigma(x), y) + 2D(u_n s, x). \tag{5.9}$$

Letting $n \to \infty$ and using (5.8), we derive from (5.9) that

$$D_2(x,y) = 0, \quad x, y \in G.$$

That is, h is a solution of equation (1.2). This completes the proof.

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