Can. J. Math., Vol. XXXVI, No. 4, 1984, pp. 615-684

SPHERICAL HARMONICS, THE WEYL TRANSFORM AND THE FOURIER TRANSFORM ON THE HEISENBERG GROUP

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Introduction. In the early days of quantum mechanics, Weyl asked the following question. Let λ be a non-zero real number, $\mathscr{H}$ a separable Hilbert space. Given certain (unbounded) operators $W_1, \ldots, W_n$, $W_1^+, \ldots, W_n^+$ on $\mathscr{H}$ satisfying

$$W_j^+ = W_j^* \quad \text{and} \quad [W_j^+, -W_j] = 2\lambda I$$

(on a dense subspace $\mathscr{D}$ of $\mathscr{H}$) with all other commutators vanishing. Given also a function $f(\zeta, \bar{\zeta})$ where $\zeta \in \mathbf{C}^n$. Let $W = (W_1, \ldots, W_n)$, $W^+ = (W_1^+, \ldots, W_n^+)$. How does one associate to f an operator $f(W, W^+)$? (Actually, Weyl phrased the question in terms of $p = \text{Re } \zeta$, $q = \text{Im } \zeta$, $P = \text{Re } W$, $Q = \text{Im } W^+$, which represent momentum and position. In this paper, however, we wish to exploit the unitary group on $\mathbf{C}^n$ and so prefer complex notation.)

If f is a polynomial, say $f(\zeta, \bar{\zeta}) = \zeta_1^2\bar{\zeta}_1$, we want to associate to f a polynomial in (W, W^+). But which polynomial; $W_1^2W_1^+$, $W_1^+W_1^2$ or even $(1/2)[W_1^2W_1^+ + W_1W_1^+W_1]$? The choice is apparently arbitrary. Nevertheless, the first two possibilities listed are stand-outs. To formalize this, suppose that P is the monomial $P(\zeta) = \zeta^\rho\bar{\zeta}^\gamma$ (ρ, γ multi-indices; we are using multi-index conventions). We set

$$\tau(P) = (W^+)^\gamma W^\rho, \quad \tau'(P) = W^\rho(W^+)^\gamma.$$

We extend τ, τ' to all polynomials by linearity. As for more general functions f, Weyl's construction is this (modified for complex notation). Define $\mathscr{F}':\mathscr{S}(\mathbf{C}^n) \to \mathscr{S}(\mathbf{C}^n)$ as follows:

$$(\mathscr{F}'F)(\zeta) = \int_{\mathbf{C}^n} \exp(-z \cdot \bar{\zeta} + \bar{z} \cdot \zeta)F(z)dV.$$

(Dot denotes dot product.) This is a modification of the usual Fourier transform $\mathscr{F}$, the relation being that

$$\mathscr{F}'F(\zeta) = \mathscr{F}F(-2i\zeta).$$

Weyl shows that $i(-z \cdot W^+ + \bar{z} \cdot W)$ is essentially self-adjoint, so that $\exp(-z \cdot W^+ + \bar{z} \cdot W)$ is unitary. Thus one can define a map

Received April 11, 1983. This research was supported in part by NSF grant MCS8102399A02.

https://doi.org/10.4153/CJM-1984-039-0 Published online by Cambridge University Press

$$\mathscr{G}:\mathscr{S}(\mathbf{C}^n) \rightarrow \mathscr{B}(\mathscr{H})$$

which is analogous to $\mathscr{F}'$: namely

$$\mathscr{G}F = \int_{\mathbf{C}^n} \exp(-z \cdot W^+ + \bar{z} \cdot W)F(z)dV.$$

We call $\mathscr{G}$ the Weyl transform. We then define $\mathscr{W}:\mathscr{S}(\mathbf{C}^n) \rightarrow \mathscr{B}(\mathscr{H})$ by

$$\mathscr{W}F = \mathscr{G} \circ \mathscr{F}'^{-1}f.$$

This then is the operator associated to f, which we shall refer to as the Weyl correspondent of f. Again, there are other possibilities for $\mathscr{W}$, but this one is in many ways the simplest.

But what of the previous question? What is the Weyl correspondent of a polynomial? Let P be a polynomial; then $\Lambda = \mathscr{F}'^{-1}(P)$ is a distribution, a linear combination of δ and its derivatives. One can then make sense of $\mathscr{G}(\Lambda) = \mathscr{W}P$ in a variety of ways; for example, one could expand $\exp(-z \cdot W^+ + \bar{z} \cdot W)$ as a formal power series and perform obvious manipulations. The relation of the above notions to harmonic polynomials P is explored in Part A of this paper. As a small sample of this, we assert:

PROPOSITION 2.7. *If P is harmonic, $\mathscr{W}(P) = \tau(P) = \tau'(P)$.*

In fact, the operators $\{\mathscr{W}(P)|P$ harmonic and homogeneous$\}$ are operator analogues of spherical harmonics. There is a complete theory for them analogous to the classical theory of spherical harmonics. Part A constitutes a defense of this last statement. The main objective of Part B is the computation of an exact formula for the group F. T. (Fourier transform) of certain regular homogeneous distributions on $\mathbf{H}^n$ (the Heisenberg group), using the theory of Part A. The paper is essentially self-contained.

In Section 7, we give a few applications to the computations of formulae for certain kernels that arise on $\mathbf{H}^n$. However, the most important application to date of this work was to the study by the author and E. M. Stein of singular convolution operators on the Heisenberg group [**7**].

I would like to thank E. M. Stein for many helpful discussions.

Part A. Spherical Harmonics and the Weyl Transform

1. Summary of basic properties of the Weyl transform. We begin with a rapid summary of the properties of $\mathscr{G}$. Some short heuristic arguments for the assertions made are included. For complete proofs we refer to the first section of [**8**]. We use the prefix "I" for the results of [**8**]. Thus Lemma I.1.1 refers to Lemma 1.1 of [**8**]. We have, however, changed one piece of notation used in [**8**]: the meanings of W and W^+ will be the reverse from the usage in [**8**]. The reason for the change is to keep in accordance with the philosophy enunciated in the introduction. We regret our earlier usage in [**8**].

Many concepts to be introduced in Part A depend on the parameter $\lambda \in \mathbf{R}^* = \mathbf{R}\backslash\{0\}$. As such, when we define these concepts, they will carry λ as a subscript. Usually, however, λ will be fixed and we shall omit this subscript entirely and without further comment.

To define $\mathscr{G}_\lambda$ formally we first let $\mathscr{H}_\lambda$ be a separable complex Hilbert space with fixed orthonormal basis $\{E_{\alpha,\lambda}\}_{\alpha \in (\mathbf{Z}^+)^n}$. (Here $\mathbf{Z}^+ = \{0, 1, 2, \dots\}$.) At times we shall identify all the $\mathscr{H}_\lambda$ with each other in such a way that $E_{\alpha,\lambda}$ is identified with $E_{\alpha,\mu}$ for all λ, μ. Let

$$\mathscr{D}_\lambda = \{\text{finite linear combinations of the } E_{\alpha,\lambda}\}$$

and let

$$\mathscr{O}(\mathscr{H}_\lambda) = \{\text{linear operators } S | S : \mathscr{D}_\lambda \to \mathscr{H}_\lambda\}.$$

Sometimes if S is an operator on $\mathscr{H}$ such that $\mathscr{D} = \mathscr{D}_\lambda \subset \mathscr{D}(S)$, we shall think of S as an element of $\mathscr{O}(\mathscr{H})$; we mean more properly that $S|_{\mathscr{D}} \in \mathscr{O}(\mathscr{H})$. We let

$$\mathscr{B}(\mathscr{H}_\lambda) = \{\text{bounded operators on } \mathscr{H}_\lambda\};$$

there is an obvious injection $\mathscr{B}(\mathscr{H}) \subset \mathscr{O}(\mathscr{H})$ obtained by restricting an operator to $\mathscr{D}$. For v, $w \in \mathscr{H}$ we set

$$v \cdot w = \sum v_\alpha w_\alpha \quad \text{if } v = \sum v_\alpha E_\alpha,\ w = \sum w_\alpha E_\alpha.$$

If $S \in \mathscr{O}(\mathscr{H})$ we let S^+ be the operator in $\mathscr{O}(\mathscr{H})$ such that

$$Sv \cdot w = v \cdot S^+ w,$$

if there is such an operator in $\mathscr{O}(\mathscr{H})$. Let e_k denote $(0, \dots, 1, \dots, 0) \in (\mathbf{Z}^+)^n$ with the 1 in the kth position. On $\mathscr{D}$, we define weighted shift operators $\widetilde{W}_{k\lambda}$, $\widetilde{W}^+_{k\lambda}$ for $1 \leqq k \leqq n$ as follows:

$$\widetilde{W}_{k\lambda} E_\alpha = (2\alpha_k|\lambda|)^{\frac{1}{2}} E_{\alpha - e_k}, \quad \text{zero if } \alpha_k = 0$$

$$\widetilde{W}^+_{k\lambda} E_\alpha = [2(\alpha_k + 1)|\lambda|]^{\frac{1}{2}} E_{\alpha + e_k}$$

for $\lambda > 0$. The right sides are to be reversed if $\lambda < 0$. $\widetilde{W}_{k\lambda}$ and $\widetilde{W}^+_{k\lambda}$ are closable; we denote their closures by $W_{k\lambda}$ and $W^+_{k\lambda}$. We set $\mathscr{D}^k_\lambda = \mathscr{D}(W_{k\lambda})$; then

$$\mathscr{D}^k_\lambda = \mathscr{D}(W^+_{k\lambda}) = \{v = \sum v_\alpha E_{\alpha,\lambda} \in \mathscr{H}_\lambda | \sum \alpha_k |v_\alpha|^2 < \infty\}.$$

We have

$$[W^+_{j\lambda}, -W_{k\lambda}] = 2\delta_{jk}\lambda I \quad \text{on} \quad \mathscr{D}_\lambda, \quad [W_{j\lambda}, W_{k\lambda}] = 0$$

(where I denotes the identity operator). If $\mathscr{H}$ is identified with $L^2(\mathbf{R}^n)$ and the $\{E_\alpha\}$ with the Hermite functions, W_k and W_k^+ are the annihilation and creation operators. When $\lambda = \frac{1}{2}$ these are $2^{-\frac{1}{2}}(x_k \pm \partial/\partial x_k)$.

We shall not use Hermite functions but instead will sometimes use the following representation of our operators on the Bargmann space. For $\lambda > 0$, let

$$\mathscr{H}'_\lambda = \{F \text{ holomorphic on } \mathbf{C}^n |$$
$$(2\lambda/\pi)^n \int_{\mathbf{C}^n} |F(w)|^2 e^{-2\lambda|w|^2} dV = \|F\|^2 < \infty\}.$$

$\mathscr{H}'_\lambda$ is easily seen [**1**] to be a Hilbert space with orthonormal basis $E'_{\alpha,\lambda}$ where

$$E'_{\alpha\lambda}(w) = [(2\lambda)^{\frac{1}{2}} w]^\alpha / (\alpha!)^{\frac{1}{2}} \quad (\alpha \in (\mathbf{Z}^+)^n).$$

(Here $\alpha! = \prod \alpha_j!$, $z^\alpha = \prod z_j^{\alpha_j}$ if $z \in \mathbf{C}^n$.) For $\lambda < 0$, set

$$\mathscr{H}'_\lambda = \mathscr{H}'_{-\lambda}, \quad E'_{\alpha\lambda} = E'_{\alpha-\lambda}.$$

Identify $\mathscr{H}_\lambda$, $E_{\alpha,\lambda}$ with $\mathscr{H}'_\lambda$, $E'_{\alpha\lambda}$. If $\lambda > 0$, $W_{j\lambda}$, $W_{j\lambda}^+$ are then identified with the operations of multiplication by $2|\lambda|w_j$ and $\partial/\partial w_j$ respectively, while if $\lambda < 0$ the situation is reversed. In the future, when we use this representation, we will simply say "in the Bargmann representation" and omit the primes.

Through use of the Bargmann representation, one can give a subspace $\mathscr{A}_\lambda$ of $\mathscr{H}$ such that

$$\mathscr{D} \subset \mathscr{A} \subset \bigcap_{j=1}^{n} \mathscr{D}^j$$

and such that $i(-z \cdot W^+ + \bar{z} \cdot W)$ is essentially self-adjoint on $\mathscr{A}$. Thus

$$V_z = \exp(-z \cdot W^+ + \bar{z} \cdot W)$$

is a well-defined unitary operator on $\mathscr{H}$. $\mathscr{A}$ may be chosen so that the power series for $\exp(-z \cdot W^+)\varphi$ and $\exp(\bar{z} \cdot W)\varphi$ converge absolutely to elements of $\mathscr{A}$, for any $\varphi \in \mathscr{A}$. One can then prove (a complex form of) the Weyl relations, that

$$\begin{aligned} V_z &= e^{\lambda|z|^2} \exp(\bar{z} \cdot W) \exp(-z \cdot W^+) \\ &= e^{-\lambda|z|^2} \exp(-z \cdot W^+) \exp(\bar{z} \cdot W) \quad (*) \end{aligned}$$

on $\mathscr{A}$. The second equality follows from the fact that

$$[\exp(z \cdot W)F](w) = F(w + z),$$
$$[\exp(z \cdot W^+)F](w) = e^{2\lambda z \cdot w} F(w) \text{ for } F \in \mathscr{A}$$

in the Bargmann representation. These relations are trivial on the formal level. The first equality of (*) is proved through use of Stone's theorem. For all this, see Lemma I.1.1.

One also checks, using the Weyl relations, that

(1.1) $V_z V_w = \exp(2i\lambda \operatorname{Im} z \cdot \bar{w}) V_{z+w}$.

On $\mathbf{C}^n$ we define the differential operators

$$\mathscr{Z}_{j\lambda} = \partial/\partial z_j + \lambda \bar{z}_j, \quad \bar{\mathscr{Z}}_{j\lambda} = \partial/\partial \bar{z}_j - \lambda z_j.$$

Then

(1.2) $[\mathscr{Z}_j, \bar{\mathscr{Z}}_k] = -2\delta_{jk}\lambda$.

Now $\mathscr{F}':L^1(\mathbf{C}^n) \to C(\mathbf{C}^n)$ (see the introduction) satisfies

$$[\mathscr{F}'(\partial F/\partial z_j)](\zeta) = \bar{\zeta}_j(\mathscr{F}'F)(\zeta),$$
$$[\mathscr{F}'(\partial F/\partial \bar{z}_j)](\zeta) = -\zeta_j(\mathscr{F}'F)(\zeta) \quad \text{if } F \in \mathscr{S}.$$

One wishes analogously to define $\mathscr{G}_\lambda:L'(\mathbf{C}^n) \to \mathscr{B}(\mathscr{H})$ in such a way that

(1.3) $\mathscr{G}(\mathscr{Z}_j F) = (\mathscr{G}F)W_j^+,$

$$\mathscr{G}(\bar{\mathscr{Z}}_j F) = -(\mathscr{G}F)W_j \text{ on } \mathscr{D}^j \text{ if } F \in \mathscr{S}.$$

By (1.2), this is at least a conceivable objective. As in the introduction, one has only to set

$$\mathscr{G}F = \int V_z F(z) dV.$$

Computing formally with the Weyl relations, one checks (1.3) at once.

We remark that the operators

$$\mathscr{Z}_{j\lambda}^R = \partial/\partial z_j - \lambda \bar{z}_j \quad \text{and} \quad \mathscr{Z}_{j\lambda}^R = \partial/\partial \bar{z}_j + \lambda z_j$$

also behave nicely under $\mathscr{G}$. Namely, if $F \in \mathscr{S}$, then $\mathscr{G}F:\mathscr{H} \to \mathscr{D}^j$, $\mathscr{G}(\mathscr{Z}_j^R F) = W_j^+(\mathscr{G}F)$, and

$$\mathscr{G}(\bar{\mathscr{Z}}_j^R F) = -W_j(\mathscr{G}F).$$

Again this is checked with the Weyl relations.

An operator which is frequently useful is

$$\mathscr{L}_{0\lambda} = -(1/2) \sum_{j=1}^n (\mathscr{Z}_j \bar{\mathscr{Z}}_j + \bar{\mathscr{Z}}_j \mathscr{Z}_j).$$

We also define $\tilde{A}_\lambda \in \mathscr{O}(\mathscr{H})$ by

$$\tilde{A}_\lambda = (1/2) \sum (\tilde{W}_k \tilde{W}_k^+ + \tilde{W}_k^+ \tilde{W}_k) \quad \text{on } \mathscr{D}_\lambda.$$

$\tilde{A}_\lambda$ is closable; denote its closure by A_λ. Then

$$\mathcal{D}(A) = \{v \in \mathcal{H} | \sum |\alpha|^2 |v_\alpha|^2 < \infty\}.$$

Here $|\alpha| = \sum \alpha_i$. Note

$$AE_\alpha = 2|\lambda| \nu_\alpha E_\alpha$$

where, here and elsewhere, $\nu_\alpha = |\alpha| + n/2$. If $F \in \mathcal{S}$,

$$\mathcal{G}(\mathcal{L}_0 F) = (\mathcal{G}F)A \text{ on } \mathcal{D}(A).$$

If F is a function on $\mathbf{C}^n$, let $M_k F$, $M_k F (k = 1, \dots, n)$ be the functions defined by

$$(M_k F)(z) = z_k F(z), \quad (\tilde{M}_k F)(z) = \bar{z}_k F(z).$$

Then

$$\mathcal{F}'(M_k F) = -\partial \mathcal{F}' F / \partial \bar{\zeta}_k,$$

$$\mathcal{F}'(\bar{M}_k F) = \partial \mathcal{F}' F / \partial \zeta_k \quad \text{if } F \in \mathcal{S}.$$

An analogous result holds for $\mathcal{G}$ if $\partial/\partial\zeta_k$, $\partial/\partial\bar{\zeta}_k$ are replaced by certain unbounded derivations. Thus, for $1 \leqq k \leqq n$, we define the operators $D_{k\lambda}$, $\tilde{D}_{k\lambda}: \mathcal{O}(\mathcal{H}) \to \mathcal{O}(\mathcal{H})$ as follows. The domain of the operators is

$$\mathcal{D}(D_k) = \mathcal{D}(\tilde{D}_k) = \{S \in \mathcal{O}(\mathcal{H}) | S: \mathcal{D} \to \mathcal{D}^k\}$$

and for $S \in \mathcal{D}(D_k)$ we set

$$D_k S = (2\lambda)^{-1} [S, W_k^+], \quad \tilde{D}_k S = -(2\lambda)^{-1} [S, W_k].$$

Note that

$$D_k W_j = \tilde{D}_k W_j^+ = \delta_{jk},$$

just as

$$(\partial/\partial\zeta_k)\zeta_j = (\partial/\partial\bar{\zeta}_k)\bar{\zeta}_j = \delta_{jk}.$$

Note the further analogy of $D_k S$ with $\partial f/\partial\zeta_k = \{f, \bar{\zeta}_k\}$ where $\{\ ,\ \}$ denotes Poisson bracket. We assert:

PROPOSITION 1.1. *If $F \in \mathcal{S}$ then $\mathcal{G}F \in \mathcal{D}(D_k)$ and $\mathcal{G}(M_k F) = -\tilde{D}_k(\mathcal{G}F)$, $\mathcal{G}(\tilde{M}_k F) = D_k(\mathcal{G}F)$ on $\mathcal{D}$.*

A heuristic proof would be based on the "chain rule":

$$D_k(V_z) = e^{-\lambda|z|^2} \exp(-z \cdot W^+) D_k[\exp(\bar{z} \cdot W)]$$
$$= \bar{z}_k V_z D_k(W_k) = \bar{z}_k V_z.$$

An actual proof that $D_k V_z = \bar{z}_k V_z$ is easily given in the Bargmann representation. See Proposition I.1.2.

Let

$$\mathscr{I}_1^E(\mathscr{H}_\lambda) = \{S \in \mathscr{O}(\mathscr{H}) \mid \|S\|_1^E = \sum_\alpha \|SE_\alpha\| < \infty\} \quad \text{and}$$

$$\mathscr{I}_2(\mathscr{H}_\lambda) = \{S \in \mathscr{O}(\mathscr{H}) \mid \|S\|_2 = (\sum \|SE_\alpha\|^2)^{\frac{1}{2}} < \infty\}.$$

If $S \in \mathscr{I}_1^E$, S is easily seen to be of trace class (more properly, can be extended to $\mathscr{H}$ as a trace class operator). $\|S\|_2$ is the Hilbert-Schmidt norm of S. Define $\mathscr{T}_\lambda : \mathscr{I}_1^E \to L^\infty(\mathbf{C}^n)$ by

$$(\mathscr{T}_\lambda S)(z) = \operatorname{tr}(V_{-z}S).$$

Here then is a version of the inversion and Plancherel theorems for $\mathscr{G}$.

THEOREM 1.2 (a) *If* $F \in \mathscr{S}(\mathbf{C}^n)$, *then*

$$\mathscr{G}F \in \mathscr{I}_1^E \cap \mathscr{I}_2.$$

(b) *If* $F \in L^1(\mathbf{C}^n)$ *and* $\mathscr{G}F \in \mathscr{I}_1^E$ *then*

$$F = \pi^{-n}(2|\lambda|)^n \mathscr{T}(\mathscr{G}F)$$

for almost every $z \in \mathbf{C}^n$.

(c) *If* $F \in \mathscr{S}$, *then*

$$\|F\|_2^2 = \pi^{-n}\|\mathscr{G}F\|_2^2(2|\lambda|)^n.$$

$\mathscr{G}$ *may then be extended to a constant multiple of a unitary map from* $L^2(\mathbf{C}^n)$ *onto* $\mathscr{I}_2$.

A proof of (b) may be carried out along the following lines. We shall assume $F \in C_c^\infty$ and indicate how to prove (b) for all z. We may assume $z = 0$; otherwise we replace F by T_zF where

$$(T_zF)(w) = \exp(2i\lambda \operatorname{Im} w \cdot \bar{z})F(z + w)$$

and use (1.1). One has then only to prove: (i) for some F, with $F(0) \neq 0$, one has

$$F(0) = \pi^{-n}(2|\lambda|)^n \mathscr{T}(\mathscr{G}F)(0).$$

(ii) For all F, if $F(0) = 0$, then

$$\operatorname{tr}(\mathscr{G}F) = 0.$$

For (i), we refer to Lemma I.1.3 (b), where the function $F(z) = e^{-|\lambda||z|^2}$ is used. This function satisfies

$$\mathscr{Z}_jF = \bar{\mathscr{Z}}_j^R F = 0 \quad \text{for all } j.$$

Thus

$$W_j(\mathcal{G}F) = (\mathcal{G}F)W_j^+ = 0.$$

Accordingly, $(\mathcal{G}F)E_\alpha = 0$ unless $\alpha = 0$, and $(\mathcal{G}F)E_0 = cE_0$ for some c. In Lemma I.1.3 (b), and also in Section 4 below, we show directly that

$$c = \pi^n(2|\lambda|)^{-n},$$

and this gives (i).

For (ii), note

$$F = \sum_{k=1}^{n} (M_kF_k + \widetilde{M}_k\widetilde{F}_k)$$

for certain $F_1, \ldots, F_n, \widetilde{F}_1, \ldots, \widetilde{F}_n \in C_c^\infty$. So

$$\mathcal{G}F = -\sum \widetilde{D}_k(\mathcal{G}F_k) + \sum D_k(\mathcal{G}\widetilde{F}_k).$$

Now if S is suitable, $\mathrm{tr}(D_kS) = 0$; this follows heuristically from

$$\mathrm{tr}(SW_k^+) = \mathrm{tr}(W_k^+S).$$

(ii) then follows. A proof of the ordinary Fourier inversion formula may be given along the same outline, noting that for suitable G,

$$\int_{\mathbf{C}^n} (\partial/\partial\zeta_k)G = 0.$$

The formula of (c) can be proved as a consequence of (b). See Theorem I.1.6 for the complete proof.

Frequently it is desirable to reduce problems about $\mathcal{G}_\lambda$ to the case $\lambda = 1/2$. This is accomplished through the simple relations which follow. For each λ, $\mu \in \mathbf{R}^*$ define the unitary $\mathcal{U}_{\lambda,\mu}:\mathcal{H}_\lambda \to \mathcal{H}_\mu$ which identifies $\mathcal{H}_\lambda$ and $\mathcal{H}_\mu$, so that

$$\mathcal{U}_{\lambda,\mu}(E_{\alpha,\lambda}) = E_{\alpha,\mu}.$$

We abbreviate $\mathcal{U}_\lambda = \mathcal{U}_{\lambda,-\lambda}$. Then

$$(V_z^\lambda)^+ = \mathcal{U}_{-\lambda}V_z^{-\lambda}\mathcal{U}_\lambda = V_{-\bar{z}}^\lambda \text{ on } \mathcal{D}_\lambda,$$

so that if F is in L^1 or L^2,

(1.4) $$(\mathcal{G}_\lambda F)^+ = \mathcal{U}_{-\lambda}(\mathcal{G}_{-\lambda}F)\mathcal{U}_\lambda = \mathcal{G}_\lambda G$$

on $\mathcal{D}_\lambda$, where $G(z) = F(-\bar{z})$. Say $\lambda > 0$. Then

$$V^{\frac{1}{2}}_{\sqrt{2\lambda}z} = V_z^\lambda$$

so that

(1.5) $$\mathcal{G}_\lambda F = \mathcal{U}_{\frac{1}{2},\lambda}(\mathcal{G}_{\frac{1}{2}}H)\mathcal{U}_{\lambda,\frac{1}{2}}$$

https://doi.org/10.4153/CJM-1984-039-0 Published online by Cambridge University Press

if

$$H(z) = F((2\lambda)^{-\frac{1}{2}}z)(2\lambda)^{-n}.$$

If X, Y and Z are spaces and $g:X \to Y, f:Y \to Z$, we sometimes write $g^{\cdot} f$ to mean $f \circ g$. Thus, for example, if $f:\mathbf{C}^n \to \mathbf{C}^n$, $U \in U(n)$ (the unitary group), then $U^{\cdot} f = f \circ U$. There is thus a natural action of $U(n)^{\cdot}$ on many spaces of functions on $\mathbf{C}^n$, for example $L^p(\mathbf{C}^n)$.

Next we examine the way in which $U(n)^{\cdot}$ acts on $\mathcal{O}(\mathcal{H})$. For this, we return to the Bargmann representation. First note that, in this representation, if $F \in \mathcal{H}$,

$$(1.6) \quad \begin{aligned} (V_z^{\lambda}F)(w) &= F(w + \bar{z}) \exp[-2\lambda(w \cdot \bar{z} + |z|^2/2)] \text{ if } \lambda > 0 \\ &= F(w - \bar{z}) \exp[2\lambda(-w \cdot \bar{z} + |z|^2/2)] \text{ if } \lambda < 0 \end{aligned}$$

One checks this first for $F \in \mathcal{A}$ then extends to $F \in \mathcal{H}$, using the fact that the power series of F converges to F both pointwise and in $\mathcal{H}$.

The natural action of $U(n)$ on $\mathbf{C}^n$ induces a unitary representation of $U(n)^{\cdot}$ on $\mathcal{H}$, as one sees in the Bargmann representation. Note $U(n)^{\cdot}:\mathcal{D} \to \mathcal{D}$.

Now if $U \in U(n)$, define $\bar{U} \in U(n)$ by $\bar{U} = jUj$ where $j:\mathbf{C}^n \to \mathbf{C}^n$ is given by $jz = \bar{z}$. We define

$$\pi_{\lambda}(U^{\cdot}):\mathcal{O}(\mathcal{H}) \to \mathcal{O}(\mathcal{H})$$

as follows: if $S \in \mathcal{O}(\mathcal{H})$,

$$(1.7) \quad \begin{aligned} \pi_{\lambda}(U^{\cdot})S &= \bar{U}^{\cdot} S(\bar{U}^{\cdot})^* \quad \text{if } \lambda > 0; \\ \pi_{\lambda}(U^{\cdot})S &= U^{\cdot} S(U^{\cdot})^* \quad \text{if } \lambda < 0 \end{aligned}$$

where the domain of these operators is $\mathcal{D}$. Similarly, if S is any operator on $\mathcal{H}$ such that $\mathcal{D} \subset \mathcal{D}(S)$, we define $\pi_{\lambda}(U^{\cdot})S$ by (1.7) where the domain is now the natural domain (i.e., $\bar{U}^{\cdot} \mathcal{D}(S)$ if $\lambda > 0$, $U^{\cdot} \mathcal{D}(S)$ if $\lambda < 0$).

PROPOSITION 1.3. *If* $F \in L^1(\mathbf{C}^n)$ *or* $L^2(\mathbf{C}^n)$,

$$\mathcal{G}(U^{\cdot} F) = \pi(U^{\cdot})\mathcal{G}F \quad \textit{for each } U \in U(n).$$

Proof. Note that it is an easy consequence of (1.6) that if $U \in U(n)$,

$$V_{Uz} = \pi((U^{\cdot})^*)V_z.$$

So if $F \in L^1(\mathbf{C}^n)$,

$$\begin{aligned} \mathcal{G}(U^{\cdot} F) &= \int V_z(U^{\cdot} F)(z)dV = \int V_{U^*z}F(z)dV \\ &= \int \pi(U^{\cdot})V_zF(z)dV = \pi(U^{\cdot})\mathcal{G}F \end{aligned}$$

(integrals over $\mathbf{C}^n$). If instead $F \in L^2(\mathbf{C}^n)$, approximate F by a sequence of functions in $L^1 \cap L^2$ and note that

$$\|\pi(U^{\cdot})S\|_2 = \|S\|_2$$

for any Hilbert-Schmidt operator S on $\mathscr{H}$.

2. The Weyl correspondence and polynomials. For $T \in \mathscr{S}'(\mathbf{C}^n)$, $F \in \mathscr{S}(\mathbf{C}^n)$ we let $(T|R)$ denote the sesquilinear pairing:

$$(T|R) = \overline{T(F)}.$$

Let

$$c_{n\lambda} = \pi^{-n}(2|\lambda|)^n.$$

If $R \in \mathscr{O}(\mathscr{H})$ we say that $\mathscr{G}(T) = R$ if

$$(T|F) = c_{n\lambda} \sum_{\alpha} (RE_{\alpha}|\,(\mathscr{G}F)E_{\alpha}) \quad \text{for all } F \in \mathscr{S}(\mathbf{C}^n),$$

with absolute convergence. The definition is in agreement with the polarization of Plancherel. As an example, note $\mathscr{G}(\delta) = I$. It must be checked that the definition makes sense, that is, that if $R_1, R_2 \in \mathscr{O}(\mathscr{H})$ and

$$\sum_{\alpha} (R_1E_{\alpha}|\,(\mathscr{G}F)E_{\alpha}) = \sum_{\alpha} (R_2E_{\alpha})\,|\,(\mathscr{G}F)E_{\alpha}) \text{ for all } F \in \mathscr{S}(\mathbf{C}^n),$$

with absolute convergence, then $R_1 = R_2$ (on $\mathscr{D}$). To see this, observe that by the discussion of (i) of Theorem 1.2, there exists $F_{00} \in \mathscr{S}(\mathbf{C}^n)$ such that

$$[\mathscr{G}(F_{00})]E_{\gamma} = \delta_{0\gamma}E_0;$$

$F_{00}(z)$ is a constant multiple of $e^{-|\lambda|\,|z|^2}$. Consequently, there exists $F_{\alpha\beta} \in \mathscr{S}(\mathbf{C}^n)$ such that

$$[\mathscr{G}(F_{\alpha\beta})]E_{\gamma} = \delta_{\alpha\gamma}E_{\beta}.$$

Indeed, because of the shifting properties of the W_j, $F_{\alpha\beta}$ is just a constant multiple of $(\mathscr{Z}^R)^{\beta}_{\bar{\mathscr{Z}}}\alpha F_{00}$. Accordingly, if R_1, R_2 are as above, then for any α, β,

$$(R_1E_{\alpha}|E_{\beta}) = \sum_{\gamma} (R_1E_{\gamma}|\,(\mathscr{G}F_{\alpha\beta})E_{\gamma}) = (R_2E_{\alpha}|E_{\beta});$$

so $R_1 = R_2$.

Observe that if $\mathscr{G}T = R$, then

$$\mathscr{G}\mathscr{Z}_jT = RW_j^+, \quad \mathscr{G}\bar{\mathscr{Z}}_jT = -RW_j.$$

Indeed,

$$(\tilde{\mathscr{Z}}_j T|F) = (T|\mathscr{Z}_j F) = -c_{n\lambda} \sum_{\alpha} (RE_\alpha|(\mathscr{G}F)W_j^+ E_\alpha)$$

$$= -c_{n\lambda} \sum_{\alpha} (RW_j E_{\alpha+e_j}|(\mathscr{G}F)E_{\alpha+e_j})$$

$$= -c_{n\lambda} \sum_{\beta} (RW_j E_\beta|(\mathscr{G}F)E_\beta),$$

so that $\mathscr{G}(\tilde{\mathscr{Z}}_j T) = -RW_j$; similarly for $\mathscr{Z}_j T$. Further, if $R:\mathscr{D} \to \mathscr{D}$,

$$\mathscr{G}\mathscr{Z}_j^R T = W_j^+ R, \quad \mathscr{G}\tilde{\mathscr{Z}}_j^R T = -W_j R,$$

as a similar argument shows. We then have:

PROPOSITION 2.1. (a) *Suppose P is a polynomial in ζ, $\bar{\zeta}(\zeta \in \mathbf{C}^n)$, and* deg $P = k$. *Let $\Lambda = \mathscr{F}'^{-1}p$, a linear combination of δ and its derivatives. Then there exists $R \in \mathscr{O}(\mathscr{H})$ such that $\mathscr{G}\Lambda = R$. R is a (non-commuting) polynomial in $W_1, \dots, W_n, W_1^+, \dots, W_n^+$; we write $R = Q(W, W^+)$. We may choose Q so that* deg $Q = k$ *and such that $Q(\zeta, \bar{\zeta}) = P(\zeta, \bar{\zeta})$ in the sense that if, in Q, we replace W by ζ and W^+ by $\bar{\zeta}$, we obtain P. We write $\mathscr{W}P = R$, or $\mathscr{W}P = Q$.*

(b) *If $P(z) = |\zeta|^2$, then $\mathscr{W}P = A$. If $P(\zeta) = \zeta_j^p \bar{\zeta}_k^p$, $j \neq k$, then $\mathscr{W}P = W_j^p (W_k^+)^q$; this is valid even if $p = 0$ or $q = 0$.*

(c) *Every non-commuting polynomial Q' in the W, W^+ equals $\mathscr{W}P'$ for some polynomial P'; we write $P' = \mathscr{W}^{-1}Q'$.*

Proof. (a) We argue by induction on k, the case $k = 0$ being immediate. Assume it is known when k is replaced by $k - 1$; it suffices to prove (a) when P is a monomial of degree k. Then for some j, and some monomial P_1 of degree $k - 1$,

$$P(\zeta, \bar{\zeta}) = \zeta_j P_1(\zeta, \bar{\zeta}) \quad \text{or} \quad P(\zeta, \bar{\zeta}) = \bar{\zeta}_j P_1(\zeta, \bar{\zeta}).$$

We assume the former; the proof in the latter case is similar. Suppose $\mathscr{W}P_1 = Q$. Now

$$\Lambda = \mathscr{F}'^{-1}P = -(\partial/\partial\bar{z}_j)\mathscr{F}'^{-1}P_1 = -(1/2)(\mathscr{Z}_j + \tilde{\mathscr{Z}}_j^R)\mathscr{F}'^{-1}P_1,$$

so that

$$\mathscr{G}\Lambda = (1/2)(W_j Q_1 + Q_1 W_j),$$

and we are done.

(b) The proof of (a) shows that if $P(\zeta) = |\zeta|^2$, then $\mathscr{W}P = A$. If $P(\zeta) = \zeta_j^p \bar{\zeta}_k^q$, $j \neq k$, then

$$\mathscr{F}'^{-1}P = (-\partial/\partial\bar{z}_j)^P (\partial/\partial z_k)^q \delta = (-\tilde{\mathscr{Z}}_j)^p \mathscr{Z}_k^q \delta,$$

since $z_j \delta = \bar{z}_k \delta = 0$. Thus

$$\mathscr{W}P = W_j^p(W_k^+)^q.$$

(c) Any such Q' evidently equals $\mathscr{G}(\Lambda)$, where $\Lambda = q(\mathscr{Z}, \tilde{\mathscr{Z}})\delta$, q being a non-commuting polynomial in the $\mathscr{Z}_1, \ldots, \mathscr{Z}_n, \tilde{\mathscr{Z}}_1, \ldots, \tilde{\mathscr{Z}}_n$. Since Ω is supported at 0, it equals $\mathscr{F}'^{-1}P'$ for some P', as desired.

We also define $\mathscr{W}:\mathscr{S}(\mathbf{C}^n) \to \mathscr{I}_1^E$ by

$$\mathscr{W}G = \mathscr{G}\mathscr{F}'^{-1}F.$$

It is a consequence of the definitions that $\mathscr{W}P$ is determined by the rule

$$(P|G) = (2|\lambda|)^n \sum_\alpha ((\mathscr{W}P)E_\alpha|(\mathscr{W}G)E_\alpha) \quad \text{for all } G \in \mathscr{S}(\mathbf{C}^n).$$

Let $\mathscr{P} = \{\text{polynomials in } \zeta \text{ and } \bar{\zeta}\}$ $(\zeta \in \mathbf{C}^n)$. Each $P \in \mathscr{P}$ may be written in the form

$$(2.1) \quad P(\zeta) = \sum a_{\rho\gamma}\zeta^\rho\bar{\zeta}^\gamma.$$

Here the sum is taken over all multi-indices $\rho \in (\mathbf{Z}^+)^n$, $\gamma \in (\mathbf{Z}^+)^n$; $a_{\rho\gamma} = 0$ for all but finitely many (ρ, γ); and we use multi-index conventions. For $p, q \in \mathbf{Z}^+$, $m \in \mathbf{Z}^n$, let

$$\mathscr{P}_{pq} = \{P \in \mathscr{P} \text{ as in (2.1) with } a_{\rho\gamma} = 0$$
$$\text{unless } |\rho| = p, |\gamma| = q\}.$$

Note

$$\Delta:\mathscr{P}_{pq} \to \mathscr{P}_{p-1,q-1}.$$

Let

$$\mathscr{H}_{pq} = \{P \in \mathscr{P}_{pq}|\Delta P = 0\}.$$

Elements of $\mathscr{H}_{pq}$ are called (solid) (bigraded) spherical harmonics. For the time being, we note only this important fact about $\mathscr{H}_{pq}$, which is well known.

PROPOSITION 2.2. *The natural action of $U(n)^{\cdot}$ on $\mathscr{H}_{pq}$ is irreducible. If the actions on $\mathscr{H}_{pq}$ and $\mathscr{H}_{p_1q_1}$ are equivalent, then $(p, q) = (p_1, q_1)$.*

Proof. Fix (p, q); write $z \in \mathbf{C}^n$ as (z, z') where $z' = (z_2, \ldots, z_n)$. Let

$$V = \{\text{linear maps } S:\mathscr{H}_{pq} \to \mathscr{P}|SU^{\cdot} = U^{\cdot}S \text{ for all } U \in U(n)\}.$$

It suffices to prove that $V = \{cI|c \in \mathbf{C}\}$. Suppose $S \in V$. Since $\mathscr{H}_{pq}$ is finite dimensional, there exists a unique $Z_S \in \mathscr{H}_{pq}$ with

$$(Z_S, P) = [S(P)](e_1) \quad \text{for all } P \in \mathscr{H}_{pq}$$

$(e_1 = (1, 0)$; $(\ , \ ) = L^2(S^{2n-1})$ inner product). Now Z_S is invariant under $U_1(n)^{\cdot}$ where

$$U_1(n) = \{U \in U(n)\,|Ue_1 = e_1\}.$$

Thus Z_S is a polynomial in z_1 and $|z'|^2$. (For, if Q is the polynomial over $\mathbf{C} \times \mathbf{R}$ with $Q(z, x) = Z_S(z, x, 0, \dots, 0)$ then $Q(z, x) = Q(z, -x)$ for all s; hence Q is a polynomial in z and x^2. Now use the transitivity of $U_1(n)$.) So

$$Z_S = \sum_k a_k z_1^{p-k} \bar{z}_1^{q-k} |z'|^{2k}$$

for certain constants a_k. But one easily computes that the a_k are determined completely up to a constant multiple by the condition $\Delta Z_S = 0$. So $Z_S = cZ_I$ for some $c \in \mathbf{C}$; so

$$(SP)(e_1) = cP(e_1) \quad \text{for all } P \in \mathscr{H}_{pq}.$$

But $SU^{\cdot} = U^{\cdot}S$ for all U, so $SP = cP$ for all P and $S = cI$. This completes the proof. (Z_I is called the zonal harmonic of bidegree (p, q) and pole e_1.)

Our next result is an analogue of Hecke's identity for $\mathscr{G}$. Recall that this identity asserts that if $P \in \mathscr{H}_{pq}$, and if

$$f(z) = e^{-|z|^2}P(z),$$

then

$$\mathscr{F}'f(\zeta) = (-1)^q \pi^n f(\zeta)$$

(see e.g. [**18**]; recall $\mathscr{F}'F(\zeta) = \mathscr{F}F(-2i\zeta)$).

Recall also that the fractional linear map ψ with

$$\psi(a) = (a - 1)/(a + 1)$$

takes the right half plane onto the unit disc, and

$$\psi^{-1}(s) = (1 + s)/(1 - s).$$

We say $F \in \mathscr{S}(\mathbf{C}^n)$ is *polyradial* if it is a function of $|z_1|^2, \dots, |z_n|^2$. We also call an operator $S \in \mathscr{O}(\mathscr{H})$ polyradial if there exist numbers $c_\alpha(\alpha \in (\mathbf{Z}^+)^n)$ such that $SE_\alpha = c_\alpha E_\alpha$ for all α. Theorem I.1.3 asserts, among other things, that if $F \in \mathscr{S}(\mathbf{C}^n)$ is polyradial and $S = \mathscr{G}_\lambda F$ then S is also polyradial. We discuss this point in much greater detail in Proposition 4.1 below.

THEOREM 2.3. *Suppose* $a \in \mathbf{C}$, $\operatorname{Re} a > 0$, $P \in \mathscr{H}_{pq}$. *Fix* λ, *and let*

(2.2)
$$p' = p \text{ if } \lambda > 0,\ p' = q \text{ if } \lambda < 0,\ q' = q \text{ if } \lambda > 0,$$
$$q' = p \text{ if } \lambda < 0,\ \kappa = p + q.$$

Let

$$F(z) = \exp(-a|\lambda|\,|z|^2)P(z).$$

Then $\mathcal{G}F = (-1)^q \mathcal{W}(P)S$, *where* S *is as follows. For all* α,

$$SE_\alpha = c_{|\alpha|}E_\alpha$$

where if $N \geqq p'$,

$$c_N = \pi^n[(1 - s)/2|\lambda|]^{n+\kappa}s^{N-p'}.$$

Here $s = \psi(a)$, *and our convention is that* $0^0 = 1$. *If* $N < p'$, c_N *may be defined arbitrarily, for if* $|\alpha| < p'$,

$$(\mathcal{G}F)E_\alpha = (\mathcal{W}P)E_\alpha = 0.$$

Proof. We first claim that we may reduce to the case $\lambda = 1/2$. This follows from (1.4) and (1.5), together with the following facts. Suppose $P \in \mathcal{P}$; define $JP \in \mathcal{P}$ by $JP(z) = P(\bar{z})$. Then

$$(2.2)\quad \mathcal{W}_\lambda(JP) = \mathcal{U}_{-\lambda}(\mathcal{W}_{-\lambda}P)\mathcal{U}_\lambda$$

while if $P \in \mathcal{P}_{pq}$, $\lambda > 0$, then

$$(2.3)\quad \mathcal{W}_\lambda P = (2\lambda)^{k/2}\mathcal{U}_{1/2,\lambda}(\mathcal{W}_{1/2}P)\mathcal{U}_{\lambda,1/2}.$$

These facts are easy consequences of (1.4) and (1.5) and the definition of $\mathcal{W}P$. A simple computation now shows that we may assume $\lambda = 1/2$, and we do.

Suppose next that $P \equiv 1$. We already observed the case $a = 1$ in the discussion of Theorem 1.2. If $a \neq 1$, we use roughly the same method. Say $\mathcal{G}F = S$. S is polyradial; say $SE_\alpha = c_\alpha E_\alpha$. If $1 \leqq j \leqq n$, note

$$\tilde{\mathcal{Z}}_j^R F = (1/2)(1 - a)z_jF,$$

so if $\mathcal{G}F = S$,

$$-W_jS = -1/2(1 - a)\tilde{D}_jS = 1/2(a - 1)(W_jS - SW_j),$$

so $W_jS = sSW_j$. Applying this to E_α, we find at once that $c_{\alpha+e_j} = sc_\alpha$, where $e_j = (0, \dots, 1, \dots 0)$. Thus $c_\alpha = Cs^{|\alpha|}$ for some constant C. We can put $c_{|\alpha|} = c_\alpha$.

$$c_N = Cs^N \quad \text{for some constant } C.$$

To compute C, use the inversion formula:

$$1 = F(0) = \pi^{-n}\operatorname{tr}(S) = \pi^{-n}C\sum_{n=0}^{\infty}\binom{N+n-1}{n-1}s^n$$

$$= C(1 - s)^{-n}\pi^{-n}$$

so the result follows if $P \equiv 1$. (We have noted that

$$\#\{E_\alpha \mid |\alpha| = N\} = \binom{N+n-1}{n-1}.)$$

Suppose next that $P(\zeta) = \zeta_j^p \bar{\zeta}_k^q, j \neq k$. (If $n = 1$, then $\mathscr{H}_{pq} = 0$ unless $p = 0$ or $q = 0$, so such a P exists even if $n = 1$, if $\mathscr{H}_{pq} \neq \{0\}$.) We change our notation. Let

$$G(z) = \exp(-a|z|^2/2),$$

$F = \mathscr{G}P$ and $R = \mathscr{G}G$ (which we have just computed). By Proposition 2.1 (b),

$$\mathscr{W}P = W_j^p (W_k^+)^q$$

and indeed, then, $(\mathscr{W}P)E_\alpha = 0$ if $|\alpha| < p$. Now

$$(\tilde{\mathscr{Z}}_j^R)^p (\mathscr{Z}_k^R)^q G = (-1)^q 2^{-\kappa}(1 - a)^p (1 + a)^q F$$
$$= (-1)^\kappa s^p (1 - s)^{-\kappa} F.$$

Suppose that $a \neq 1$, so that $s \neq 0$. Then

$$\mathscr{G}F = (-1)^\kappa s^{-p}(1 - s)^\kappa (-W_j)^p (W_k^+)^q R$$
$$= (-1)^q s^{-p}(1 - s)^\kappa \mathscr{W}(P)R.$$

The desired result follows in this case from our computation of R; in particular,

$$(\mathscr{G}F)E_\alpha = 0 \quad \text{if } |\alpha| < p.$$

If $a = 1$, we obtain the desired result by taking a limit, in $L^1(\mathbf{C}^n)$ and $\mathscr{B}(\mathscr{H})$, of the result in cases we have demonstrated.

Finally, suppose $P \in \mathscr{H}_{pq}$ is general, $P \neq 0$. Select $P_1 \in \mathscr{H}_{pq}$ with $P_1(\zeta) = \zeta_j^p \bar{\zeta}_k^q$ for some $j, k, j \neq k$; again put

$$G(z) = \exp(-a|z|^2/2).$$

Because of the irreducibility of $\mathscr{H}_{pq}$, we may assume that, for some $U \in U(n)$, $P = U^\cdot P_1$, so $F = U^\cdot (GP_1)$. Thus

$$\mathscr{G}F = \pi(U^\cdot)\mathscr{G}(GP_1) = (-1)^q \pi(U^\cdot)[\mathscr{W}(P_1)S]$$
$$= (-1)^q \bar{U}^\cdot \mathscr{W}(P_1) S (\bar{U}^\cdot)^*,$$

S as in the statement of the theorem. For $N \in \mathbf{Z}$, let

$$\mathscr{V}_N = \text{span}\{E_\alpha | \, |\alpha| = N\}.$$

Checking in the Bargmann representation, we see that if $V \in U(n)$, then

$$V^\cdot : \mathscr{V}_N \to \mathscr{V}_N.$$

Accordingly, since S is radial, $SV^\cdot = V^\cdot S$. We find

$$\mathscr{G}F = (-1)^q \bar{U}^\cdot \mathscr{W}(P_1)(\bar{U}^\cdot)^* S.$$

In order to show $\mathscr{G}F = (-1)^q \mathscr{W}(P)S$, it suffices to show this on $\mathscr{D}$. Since S is radial, it suffices to show the following simple lemma:

LEMMA 2.4. *Suppose* $P_1, P \in \mathscr{P}$, $U \in U(n)$ *and* $U^{\cdot} P_1 = P$. *Then if* $\lambda > 0$,

$$\bar{U}^{\cdot} \mathscr{W}(P_1)(\bar{U}^{\cdot})^* = \mathscr{W}(P) \quad on\ \mathscr{D}.$$

If $\lambda < 0$,

$$U^{\cdot} \mathscr{W}(P_1)(U^{\cdot})^* = \mathscr{W}(P) \quad on\ \mathscr{D}.$$

Indeed, if this is known, then in our present situation $\mathscr{G}F = (-1)^q \mathscr{W}(P)S$ follows, as does the fact that $\mathscr{W}(P)E_\alpha = 0$ if $|\alpha| < p$, so that the proof will be complete.

Proof of Lemma 2.4. Assume $\lambda > 0$. We need only show

$$(\bar{U}^{\cdot} \mathscr{W}(P_1)(\bar{U}^{\cdot})^* E_\alpha | E_\beta) = (\mathscr{W}(P)E_\alpha | E_\beta) \text{ for all } \alpha, \beta.$$

By the discussion in the second paragraph of this section, for any α, β there exists $H_{\alpha\beta} \in \mathscr{S}(\mathbf{C}^n)$ such that

$$\mathscr{W}(H_{\alpha\beta})E_\gamma = \delta_{\alpha\gamma} E_\beta.$$

Fix α, β and write $H = H_{\alpha\beta}$. Then

$$(\mathscr{W}(P)E_\alpha | E_\beta) = \sum_\gamma (\mathscr{W}(P)E_\gamma | \mathscr{W}(H)E_\gamma) = (P|H).$$

On the other hand,

$$\begin{aligned}(\bar{U}^{\cdot} \mathscr{W}(P_1)(\bar{U}^{\cdot})^* E_\alpha | E_\beta) &= \sum_\gamma (\bar{U}^{\cdot} \mathscr{W}(P_1)(\bar{U}^{\cdot})^* E_\gamma | \mathscr{W}(H)E_\gamma) \\ &= \operatorname{tr}(\bar{U}^{\cdot} \mathscr{W}(P_1)^+ (\bar{U}^{\cdot}) \mathscr{W}(H)).\end{aligned}$$

Indeed, $\bar{U}^{\cdot} \mathscr{W}(P_1)^+ (\bar{U}^{\cdot})^* \mathscr{W}(H)$ is finite rank, hence trace class, since $\mathscr{W}(H)$ has rank 1, $\bar{U}^{\cdot c}$, $(\bar{U}^{\cdot})^*: \mathscr{V}_N \to \mathscr{V}_N$ for all N, and $\mathscr{W}(P_1)^+$ simply shifts. For the same reason, $\mathscr{W}(P_1)^+ (\bar{U}^{\cdot}) \mathscr{W}(H)$ is trace class. Since $\bar{U}^{\cdot}$ is bounded, the trace equals

$$\begin{aligned}\operatorname{tr}(\mathscr{W}(P_1)^+ (\bar{U}^{\cdot}) \mathscr{W}(H) \bar{U}^{\cdot}) &= \operatorname{tr}(\mathscr{W}(P_1)^+ \mathscr{W}((U^{\cdot})^* H)) \\ &= (P_1 | (U^{\cdot})^* H) = (P|H).\end{aligned}$$

(We used Proposition 1.3.) This completes the proof if $\lambda > 0$; similarly if $\lambda < 0$.

One could prove the usual Hecke identity for $\mathbf{C}^n$ in much the same manner, and the proof can even be adapted to $\mathbf{R}^m$, m any positive integer.

Theorem 2.3 is our first, and most important, indication that the operators $\mathscr{W}(P)$ (P harmonic) play the role in $\mathscr{O}(\mathscr{H})$ that harmonic polynomials play in analysis of functions on $\mathbf{C}^n$. Our main goal now is to draw this analogy closer. The operator $\Omega_\lambda = \sum D_k\widetilde{D}_k$ plays the role of the Laplacian, in the sense indicated in this proposition:

PROPOSITION 2.5. *Suppose $P \in \mathscr{P}$. Then P is harmonic if and only if $\Omega(\mathscr{W}(P)) = 0$.*

Proof. It suffices to show that if $Q \in \mathscr{P}$,

(2.4) $\widetilde{D}_k\mathscr{W}(Q) = \mathscr{W}(\bar{\partial}_k Q), \quad D_k\mathscr{W}(Q) = \mathscr{W}(\partial_k Q).$

(Here $\partial_k = \partial/\partial\zeta_k$, $\bar{\partial}_k = \partial/\partial\bar{\zeta}_k$.) These can be easily seen in more than one way, the simplest being to note

$$\begin{aligned}\mathscr{W}(\bar{\partial}_k Q) &= \mathscr{G}(-z_k\mathscr{F}'^{-1}Q)\\ &= (2\lambda)^{-1}\mathscr{G}([\widetilde{\mathscr{Z}}_k - \widetilde{\mathscr{Z}}_k^R]\mathscr{F}'^{-1}Q)\\ &= (2\lambda)^{-1}[W_k\mathscr{W}(Q) - \mathscr{W}(Q)W_k] = \widetilde{D}_k\mathscr{W}(Q);\end{aligned}$$

similarly for the second identity.

We note, by the way, that a formal computation shows

$$\Omega = \sum D_k\widetilde{D}_k = \sum \widetilde{D}_kD_k$$

also.

The next result is more interesting. Let $S_n = S^{2n-1} \subset \mathbf{C}^n$, so that rS_n is the sphere of radius r centered at 0. It is known, and proved below, that the $\mathscr{H}_{pq}$ spaces, restricted to rS_n, are orthogonal and have $L^2(rS_n)$ as their direct sum. We already have every indication that the spaces

$$\mathscr{V}_N = \text{span}\,\langle E_\alpha |\, |\alpha| = N\rangle$$

form the analogues of "spheres" in $\mathscr{H}$. Indeed, firstly we already know that $U^{\cdot}:\mathscr{V}_N \to \mathscr{V}_N$ if $U \in U(n)$. Secondly, if $P(\zeta) = |\zeta|^2$, $\mathscr{W}P = A$ is "constant" on each $\mathscr{V}_N$; that is, A is a constant multiple of the identity when restricted to each $\mathscr{V}_N$. We call such operators "radial," and S in Theorem 2.3 was radial. Since

$$AE_\alpha = 2\nu_\alpha|\lambda|E_\alpha,$$

we think of $\mathscr{V}_N$ in analogy to rS_n where

$$r^2 \sim (2N + n)\,|\lambda|.$$

Let

$$\mathscr{O}(\mathscr{V}_N) = \{\text{linear operators from } \mathscr{V}_N \text{ to } \mathscr{H}\};$$

this we intend to think of in analogy to $L^2(rS_n)$. The inner product we wish to use on $\mathscr{O}(\mathscr{V}_N)$ is

$$(R, S)_N = (2|\lambda|)^{n-1} \sum_{|\alpha|=N} (RE_\alpha|SE_\alpha).$$

Our next result will show, in particular, that for any N, the restrictions of the $\mathscr{W}(\mathscr{H}_{pq})$ to $\mathscr{V}_N$ are mutually orthogonal.

If $P, Q \in \mathscr{P}$, we define their Fisher inner product $\langle P, Q\rangle$ to be $\overline{[P(\partial)}Q](0)$. Here, if P is as in (2.1),

$$P(\bar{\partial}) = \sum a_{\rho\gamma}\bar{\partial}^\rho\partial^\gamma.$$

If $P \in \mathscr{P}_{pq}$ and $Q \in \mathscr{P}_{p_1q_1}$, observe that

$$\langle P, Q\rangle = 0 \quad \text{unless } (p, q) = (p_1, q_1).$$

If $(p, q) = (p_1, q_1)$ and if $Q(\zeta) = \sum b_{\rho\gamma}\zeta^\rho\bar{\zeta}^\gamma$, then

$$\langle P, Q\rangle = \sum \bar{a}_{\rho\gamma}b_{\rho\gamma}\rho!\gamma!.$$

Thus, $\langle P, Q\rangle$ is indeed an inner product. If $F, G \in L^2(S_n)$, we let (F, G) be their inner product, $\int \bar{F}GdS$. If $N, k \in \mathbf{Z}^+$, we write

$$N^{(k)} = N(N - 1)\dots(N - k + 1).$$

THEOREM 2.6. (a) *Suppose* $P \in \mathscr{H}_{pq}$, $Q \in \mathscr{P}_{p_1q_1}$ *and that* $p_1 \leqq p$ *or* $q_1 \leqq q$. *Then*

$$(Q, P) = 2\pi^n(k + n - 1)!^{-1}\langle Q, P\rangle,$$

while

$$(\mathscr{W}(Q), \mathscr{W}(P))_N = a(N, p, q, \lambda)(Q, P)$$

where

$$\text{(2.5)} \qquad a(N, p, q, \lambda) = (2\pi^n)^{-1}(N + q' + n - 1)^{(\kappa+n-1)}(2|\lambda|)^{\kappa+n-1}.$$

(*Notation as in* (2.2).)

(b) *In particular, the* $\mathscr{H}_{pq}$ *spaces are mutually orthogonal, in either the* $L^2(S_n)$ *or* $\mathscr{O}(\mathscr{V}_N)$ *inner product.*

Proof. We need only prove (a), since (b) is an immediate consequence. For the first identity, note that

$$(Q, P) = 2\Gamma([\kappa + \kappa_1]/2 + n - 1)^{-1} \int_{\mathbf{C}^n} \overline{Q(\zeta)}P(\zeta)e^{-|\zeta|^2}dV$$

as an integration in polar coordinates shows. However, if $G(\zeta) = P(\zeta)e^{-|\zeta|^2}$,

$$\begin{aligned}\int_{\mathbf{C}^n} \overline{Q(\zeta)}P(\zeta)e^{-|\zeta|^2}dV &= (Q|G)\\ &= \pi^{2n}(\mathscr{F}'^{-1}Q|\mathscr{F}'^{-1}G)\\ &= \pi^{2n}(-1)^{q_1}[\overline{Q(\bar{\partial})}(\mathscr{F}'^{-1}G)](0)\end{aligned}$$

https://doi.org/10.4153/CJM-1984-039-0 Published online by Cambridge University Press

$$= \pi^n(-1)^{q_1+q}[\overline{Q(\bar{\partial})}(P(z)e^{-|z|^2})](0)$$
$$= \pi^n(-1)^{q_1+q}\langle Q, P\rangle$$

by Hecke's identity, and the fact that $p_1 \leqq p$ or $q_1 \leqq q$. Since $\langle Q, P\rangle = 0$ unless $(p, q) = (p_1, q_1)$, the first identity follows. More generally, for any $a > 0$, and any λ,

$$\langle Q, P\rangle = [\overline{Q(\bar{\partial})}(P(z)e^{-a|\lambda|\,|z|^2})](0)$$
$$= (-1)^q(\mathscr{F}'^{-1}Q|F_a)$$

where

$$F_a(z) = e^{-a|\lambda|\,|z|^2}P(z).$$

Thus

$$\langle Q, P\rangle = (-1)^q\pi^{-n}(2|\lambda|\,)^n \sum (\,(\mathscr{W}Q)E_\alpha|\,(\mathscr{G}F_a)E_\alpha),$$

with absolute convergence. Indeed, $(\mathscr{W}Q)^+ = \mathscr{W}Q'$ for some polynomial Q', and $(\mathscr{W}Q')(\mathscr{W}G)$ is the Weyl transform of a Schwartz space function, so that it is in $\mathscr{I}_1^E$. By Theorem 2.3, then,

$$\langle Q, P\rangle = (2|\lambda|\,)^{-\kappa}(1 - s)^{n+\kappa} \sum_{N=p'}^{\infty} s^{N-p'}$$
$$\times \sum_{|\alpha|=N} (\,(\mathscr{W}Q)E_\alpha|\,(\mathscr{W}P)E_\alpha).$$

Thus, if $b_M = (\mathscr{W}(Q), \mathscr{W}(P)\,)_M$, we find

$$\sum_{M=0}^{\infty} b_{M+p'}s^M = (2|\lambda|\,)^{\kappa+n-1}\langle Q, P\rangle(1 - s)^{-(n+\kappa)}.$$

The series on the left side converges absolutely for $|s| < 1$, hence is the power series of the function on the right side. Hence

$$b_{M+p'} = (2|\lambda|\,)^{\kappa+n-1}\binom{M + \kappa + n - 1}{M}\langle Q, P\rangle.$$

This, together with the first identity, at once gives the second identity.

Remark. The first identity was proved by Coifman and Weiss [**2**] on $\mathbf{R}^m$, in general, by use of representation theory. The simple proof above adapts at once to the general case. For another proof, see the remark at the end of Section 3.

We have yet to show that

$$\mathscr{O}(\mathscr{V}_N) = \oplus\, \mathscr{W}(\mathscr{H}_{pq})\Big|_{\mathscr{V}_N}$$

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(orthogonal direct sum over all p, q), just as

$$L^2(S_n) = \oplus \mathscr{H}_{pq}\Big|_{S_n}.$$

We wish also to determine $\mathscr{W}(P)$ explicitly, for $\mathscr{P} \in \mathscr{H}_{pq}$. It is convenient to do the latter first.

As in the introduction, if P is as in (2.1), we let

$$\tau_\lambda(P) = \sum a_{\rho\gamma}(W^+)^\gamma W^\rho, \quad \tau'_\lambda(P) = \sum a_{\rho\gamma} W^\rho (W^+)^\gamma,$$

and we assert:

PROPOSITION 2.7. *If P is harmonic, $\mathscr{W}(P) = \tau(P) = \tau'(P)$.*

We begin with a lemma.

LEMMA 2.8. *Let P be any polynomial,* $\deg P = k$. *Then there exist polynomials P_1 and P_2, of degree not exceeding $k - 1$, such that*

$$\mathscr{W}(P) = \tau(P + P_1), \quad \tau(P) = \mathscr{W}(P + P_2);$$

similarly for τ' in place of τ.

Proof. Suppose Q' is any non-commuting polynomial in the W, W^+. We let $\sigma(Q')$ denote the polynomial obtained formally from Q' by replacing W by ζ, W^+ by $\bar{\zeta}$. Suppose that

$$\sigma(Q') = \sum b_{\rho\gamma}\zeta^\rho \bar{\zeta}^\gamma$$

and suppose k' is the degree of $\sigma(Q')$. We then let

$$\sigma_{\text{princ}}(Q') = \sum_{|\rho|+|\gamma|=k'} b_{\rho\gamma}\zeta^\rho \bar{\zeta}^\gamma.$$

For the first statement of the lemma, say

$$P = P_{\text{princ}} + P_{\text{lower}}$$

where P_{princ} is homogeneous of degree k and

$$\deg P_{\text{lower}} \leqq k - 1.$$

By Proposition 2.1 (a), there exists Q such that $\mathscr{W}(P) = Q$ and such that $\sigma(Q) = P$; thus

$$\sigma_{\text{princ}}(Q) = P_{\text{princ}}.$$

In Q, we use the commutation rules to commute all the W^+'s to the left and all the W's to the right, to obtain a non-commuting polynomial Q_1. It is evident that, although $\sigma(Q_1)$ might not be P, we still have

$$\mathscr{W}(P) = Q_1 \quad \text{and} \quad \sigma_{\text{princ}}(Q_1) = P_{\text{princ}}.$$

This, however, is just another way of saying

$$\mathscr{W}(P) = \tau(P + P_1) \quad \text{for some } P,$$

with $\deg P_1 \leqq k - 1$.

For the second statement, we argue by induction on k; it is trivial if $k = 0$. Assume it is known when $k \leqq j - 1$, and suppose $k = j$. By the first statement,

$$\mathscr{W}(P) = \tau(P) + \tau(P_1) \quad \text{for some } P_1, \deg P_1 \leqq j - 1.$$

By the induction hypothesis, $\tau(P_1) = \mathscr{W}(-P_2)$ for some P_2, $\deg P_2 \leqq j - 1$. This proves the second statement.

Similarly for τ'.

Proof of Proposition 2.7. We may assume $P \in \mathscr{H}_{pq}$ for some p, q. We show only $\mathscr{W}(P) = \tau(P)$; the proof that $\mathscr{W}(P) = \tau'(P)$ is similar.

By Lemma 2.8, $\tau(P) = \mathscr{W}(P')$ where $P' = P + P_2$, $\deg P_2 < \kappa = p + q$. It suffices to show that $P' \in \mathscr{P}_{pq}$, so that $P_2 = 0$.

We first show that P' is harmonic. Observe to begin that if $P_0 \in \mathscr{P}$ is arbitrary, then

$$\tau(\partial_l P_0) = D_l \tau(P_0), \quad \tau(\bar{\partial}_l P_0) = \tilde{D}_l \tau(P_0).$$

This is shown immediately by a check using the derivation law and the facts that

$$D_l(W_m) = \tilde{D}_l(W_m^+) = \delta_{lm}, \quad D_l(W_m^+) = \tilde{D}_l(W_m) = 0.$$

Accordingly,

$$0 = \tau(\Delta P) = \Omega\tau(P) = \Omega\mathscr{W}(P'),$$

so that P' is harmonic by Proposition 2.5.

To show $P' \in \mathscr{P}_{pq}$, we need only show

$$(\zeta \cdot \partial)P' = pP', \quad (\bar{\zeta} \cdot \bar{\partial})P' = qP'.$$

Now, if $P_0 \in \mathscr{P}$ is arbitrary, we have for any j that

$$\mathscr{W}(\,(\zeta_j - \lambda\bar{\partial}_j)P_0) = \mathscr{G}[-\tilde{\mathscr{Z}}_j\mathscr{F}'^{-1}P_0] = (\mathscr{W}P_0)W_j,$$

while

$$\mathscr{W}(\,(\bar{\zeta}_j - \lambda\partial_j)P_0)\,) = \mathscr{G}(\mathscr{Z}_j^R\mathscr{F}'^{-1}P_0) = W_j^+(\mathscr{W}P_0).$$

Further,

$$\tau(\zeta_j P_0) = (\mathscr{W}P_0)W_j \quad \text{and} \quad \tau(\bar{\zeta}_j P_0) = W_j^+(\mathscr{W}P_0)$$

by the definition of τ. Now, since P' is harmonic,

$$(\zeta \cdot \partial)P' = [\,(\zeta - \lambda\bar{\partial}) \cdot \partial]P'.$$

Thus, using (2.4), we find

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$$\begin{aligned}\mathscr{W}((\zeta \cdot \partial)P') &= \mathscr{W}((\zeta - \lambda\bar{\partial}) \cdot \partial P') \\ &= \sum \mathscr{W}(\partial_l P') \cdot W_l = \sum [D_l \mathscr{W}(P')] \cdot W_l \\ &= \sum [D_l \tau(P)] \cdot W_l = \sum \tau(\partial_l P) \cdot W_l \\ &= \sum \tau((\zeta \cdot \partial)P) = p\tau(P) = p\mathscr{W}(P');\end{aligned}$$

consequently $(\zeta \cdot \partial)P' = pP'$. Similarly, using the other identities, $(\bar{\zeta} \cdot \bar{\partial})P' = qP'$; this completes the proof.

We next show

$$\mathscr{O}(\mathscr{V}_N) = \oplus\, \mathscr{W}(\mathscr{H}_{pq})\Big|_{\mathscr{V}_N}.$$

Let us first recall the proof of the following proposition.

PROPOSITION 2.9. (a) $\mathscr{P}_{pq} = \oplus\, |z|^{2k}\mathscr{H}_{p-k,q-k}$ (*sum from* $k = 0$ *to* $\min(p, q)$).

(b) $L^2(S_n) = \oplus\, \mathscr{H}_{pq}\Big|_{S_n}$ (*orthogonal direct sum over all* p, q).

(c) *For some* $C \in \mathbf{R}$,

$$\dim \mathscr{H}_{pq} < C[(p + 1)(q + 1)]^{n-2}(p + q + 1) \quad \textit{for all } p, q.$$

Proof. For (a), one need only show

$$\mathscr{P}_{pq} = \mathscr{H}_{pq} \oplus |z|^2\mathscr{P}_{p-1,q-1}.$$

This follows immediately from the fact that $\mathscr{H}_{pq}$ is the orthogonal complement of $|z|^2\mathscr{P}_{p-1,q-1}$ in $\mathscr{P}_{pq}$ under the Fisher inner product $\langle\, ,\, \rangle$. Indeed, if $P \in \mathscr{P}_{pq}$,

$$P \perp |z|^2\mathscr{P}_{p-1,q-1} \Leftrightarrow \Delta P \perp \mathscr{P}_{p-1,q-1} \Leftrightarrow \Delta P = 0.$$

(b) follows from (a), since polynomials are dense in $L^2(S_n)$. For (c), note that by (a)

$$\begin{aligned}\dim \mathscr{H}_{pq} &= \dim \mathscr{P}_{pq} - \dim \mathscr{P}_{p-1,q-1} \\ &= \binom{p+n-1}{n-1}\binom{q+n-1}{n-1} \\ &\quad - \binom{p+n-2}{n-1}\binom{q+n-2}{n-1} \\ &= (n-1)!^{-1}p!^{-1}q!^{-1}(p+n-2)!(q+n-2)! \\ &\quad \times [(p+n-1)(q+n-1) - pq],\end{aligned}$$

so (c) follows.

Here is the analogue.

PROPOSITION 2.10. (a) *For any* N

$$\tau(\mathscr{P}_{pq})\Big|_{\mathscr{V}_N} = \oplus\, \tau(\mathscr{H}_{p-k,q-k})\Big|_{\mathscr{V}_N}$$

(*sum from* $k = 0$ *to* $\min(p, q)$).

(b) $\mathcal{O}(\mathscr{V}_N) = \oplus\, \mathscr{W}(\mathscr{H}_{p-k,q-k})\Big|_{\mathscr{V}_N}$

(*orthogonal direct sum over all* p, q).

(c) $\dim \mathscr{W}(\mathscr{H}_{pq})\Big|_{\mathscr{V}_N} = \dim \mathscr{H}_{pq} \Leftrightarrow N \geqq p'$.

Otherwise, $\mathscr{W}(\mathscr{H}_{pq}) = \{0\}$ *on* $\mathscr{V}_N$.

Remark. In (a), we could replace τ by $\mathscr{W}$; we omit the proof.

Proof. (a) follows at once from Proposition 2.9 (a) and the following fact: Suppose $p, q, k, N \in \mathbf{Z}^+$. Then there exists $c \in \mathbf{R}$ such that for all $P_0 \in \mathscr{P}_{p_1,q_1}$,

$$\tau(\,|z|^{2k}P_0)\Big|_{\mathscr{V}_N} = c\tau(P_0)\Big|_{\mathscr{V}_N}.$$

To see this, observe that we may assume $k = 1$, $P_0(\zeta) = \zeta^\rho\bar{\zeta}^\gamma$. Suppose $\lambda > 0$, $|\alpha| = N$. Note

$$W^\rho E_\alpha = c_{\alpha\rho}E_{\alpha-\rho} \text{ for some } c_{\alpha\rho} \in \mathbf{R};$$

here $E_{\alpha-\rho}$ is defined to be zero if $\alpha - \rho \notin (\mathbf{Z}^+)^n$. Thus

$$\begin{aligned}\tau(\,|z|^2P_0)E_\alpha &= (W^+)^\gamma(W^+ \cdot W)W^\rho E_\alpha \\ &= c_{\alpha\rho}(W^+)^\gamma(W^+ \cdot W)E_{\alpha-\rho} \\ &= c_{\alpha\rho}2|\lambda|\,(W^+)^\gamma(\,|\alpha - \rho|\,)E_{\alpha-\rho} \\ &= c_{\alpha\rho}(2|\lambda|\,)(N - p)(W^+)^\gamma E_{\alpha-\rho} \\ &= (2|\lambda|\,)(N - \rho)\tau(P_0)E_\alpha.\end{aligned}$$

This proves the claim when $\lambda > 0$, with $c = 2(N - p)\,|\lambda|$. Similarly for $\lambda < 0$.

(b) follows from (a) and Proposition 2.7, once one shows that $\tau(P)\Big|_{\mathscr{V}_N}$ is dense in $\mathcal{O}(\mathscr{V}_N)$. If $\lambda > 0$, it suffices to note that if

$$S_{\beta\alpha} = (W^+)^\beta W^\alpha = \tau(z^\alpha\bar{z}^\beta),$$

then $\{S_{\beta\alpha}|\beta \in (\mathbf{Z}^+)^n,\ |\alpha| = N\}$ is an orthogonal basis for $\mathcal{O}(\mathscr{V}_N)$. Indeed,

$$S_{\beta\alpha}E_\gamma = c_{\beta\alpha}\delta_{\alpha\gamma}E_\beta$$

if $|\gamma| = N$, where $c_{\beta\alpha} \neq 0$. Similarly, if $\lambda < 0$.

(c) follows from Theorem 2.6 (a) if one notes

$$a(N, p, q, \lambda) > 0 \Leftrightarrow N \geqq p'.$$

3. An estimate for spherical harmonics. This section and the next are not used in Part B; the reader may proceed there now if he wishes.

In the study of spherical harmonics on $\mathbf{R}^n$, one has the estimate

$$\|P\|_\infty \leqq C(\kappa + 1)^{n/2-1}\|P\|_2. \tag{3.1}$$

Here P is a spherical harmonic of degree κ, the L^∞ and L^2 norms are taken on the unit sphere, and C depends only on n. Such an estimate is essential for characterizing the expansions of C^∞ function on the sphere in spherical harmonics. The estimate is simply a matter of estimating the L^2 norm of the zonal harmonic of degree κ (see the proof of Proposition 2.2). This is done by use of the transitive action of the orthogonal group on the sphere. (See e.g. [**18**], page 144.) In the further study of the action of $\mathcal{G}$ on functions which are C^∞ away from the origin, it is similarly important to have a sharp estimate for the norm of $\mathcal{W}(P)|_{\mathcal{V}_N} \in \mathcal{O}(\mathcal{V}_N)$. To obtain this estimate, in analogy to the Euclidean case, we study more closely the action of $U(n)^\cdot$ on $\mathcal{O}(\mathcal{V}_N)$.

As we observed during the proof of Theorem 2.3, if $V \in U(n)$, then

$$V^\cdot:\mathcal{V}_N \to \mathcal{V}_N,$$

as one sees by checking in the Bargmann representation. One also sees from this representation that $\mathcal{V}_N$ is isomorphic to the spherical harmonic space $\mathcal{H}_{NO}$ with an isomorphic action of $U(n)^\cdot$. Thus $\mathcal{V}_N$ is irreducible under this action.

If $U \in U(n)$ we define

$$\pi_\lambda(U^\cdot):\mathcal{O}(\mathcal{V}_N) \to \mathcal{O}(\mathcal{V}_N)$$

in the same manner as before Proposition 1.3. Namely if $S \in \mathcal{O}(\mathcal{V}_N)$,

$$\pi_\lambda(U^\cdot)S = \bar{U}^\cdot S(\bar{U}^\cdot)^* \text{ if } \lambda > 0;$$

$$\pi_\lambda(U^\cdot)S = U^\cdot S(U^\cdot)^* \text{ if } \lambda < 0.$$

Let $H = U(n)^\cdot$. π is evidently a representation of H on $\mathcal{O}(\mathcal{V}_N)$. Let μ denote Haar measure on H. We then have the following result.

LEMMA 3.1. (a) *π is a unitary representation of H on $\mathcal{O}(\mathcal{V}_N)$.*

(b) *If $S \in \mathcal{O}(\mathcal{V}_N)$, $\pi(u)S = S$ for all $u \in H \Leftrightarrow$ for some $c \in \mathbf{C}$,*

$$S = cI\Big|_{\mathcal{V}_N}.$$

(c) *If $R, S \in \mathcal{O}(\mathcal{V}_N)$, $w \in \mathcal{V}_N$, $\|w\| = 1$, then*

$$(R, S)_N = \binom{N+n-1}{n-1}(2|\lambda|)^{n-1} \int_H (Ruw|Suw)d\mu(u).$$

Proof. For (a), suppose $R, S \in \mathcal{O}(\mathcal{V}_N)$. Note that R^* is a bounded operator from $\mathcal{H}$ to $\mathcal{V}_N$. Suppose $u \in H$, and put $R_1 = \pi(u)R$, $S_1 = \pi(u)S$.

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Then if $\lambda > 0$,

$$(R_1, S_1)_N = (2|\lambda|)^{n-1}\operatorname{tr}(R_1^* S_1)$$
$$= (2|\lambda|)^{n-1}\operatorname{tr}(\bar{u}R^*S\bar{u}^*) = (2|\lambda|)^{n-1}\operatorname{tr}(R^*S)$$
$$= (R, S)_N.$$

(Here the trace is always taken of operators which map $\mathscr{V}_N$ to itself.) Similarly if $\lambda < 0$.

For (b) suppose that $S \in \mathscr{O}(\mathscr{V}_N)$ and $uSu^* = S$ for all $u \in H$. For each $K \in \mathbf{Z}^+$, let Q_k denote the projection of $\mathscr{H}$ onto $\mathscr{V}_K$. Q_K commutes with H, so for each K,

$$uQ_KSu^* = Q_KS \text{ for all } u \in H.$$

By Schur's lemma and the irreducibility of the action of H on each $\mathscr{V}_K$, $Q_KS = 0$ unless $K = N$, and $S = Q_NS$ is a constant multiple of the identity.

For (c) suppose $\lambda > 0$ and let $G = R^*S$. Consider

$$B = \int_H \pi(\bar{v}^*)Gd\mu(v).$$

By (b), $B = cI_N$ for some $c \in \mathbf{C}$, where $I_N = I\Big|_{\mathscr{V}_N}$. Thus, by (a),

$$(2|\lambda|)^{n-1}\binom{N+n-1}{n-1}\int_H (Rvw|Svw)d\mu(w)$$
$$= (2|\lambda|)^{n-1}\binom{N+n-1}{n-1}(w|Bw)$$
$$= (2|\lambda|)^{n-1}\binom{N+n-1}{n-1}c = (I_N, B)_N$$
$$= \int_H (I_N, \pi(\bar{v}^*)G)_N d\mu(v)$$
$$= \int_H (I_N, G)_N d\mu(v) = (I_N, G)_N = (R, S)_N$$

as desired. Similarly, if $\lambda < 0$.

Here now is the estimate alluded to at the beginning of the section.

THEOREM 3.2. *There is a constant C_n depending only on n such that for all $R \in \mathscr{W}(\mathscr{H}_{pq})$, one has*

$$\|R\|_N^2 \leqq C_n a'(N, \lambda)^{-1}(\kappa + 1)[(p + 1)(q + 1)]^{n-2}(R, R)_N. \tag{3.2}$$

Here $\|R\|_N$ is the norm of $R\Big|_{\mathscr{V}_N} \in \mathscr{O}(\mathscr{V}_N)$, $\kappa = p + q$ and

$$a'(N, \lambda) = \binom{N+n-1}{n-1}(2|\lambda|)^{n-1}$$

is the "area" of $\mathscr{V}_N$, *namely* $(I, I)_N$.

Proof. We denote

$$\mathscr{K}_{pq} = \mathscr{W}(\mathscr{H}_{pq}) \quad \text{and} \quad \mathscr{K}_{pqN} = \mathscr{K}_{pq}\Big|_{\mathscr{V}_N}.$$

Fix $N, p, q, w \in \mathscr{V}_N$ and $w' \in \mathscr{H}$ with $\|w\| = \|w'\| = 1$. We may as well assume $N \geqq p'$, for otherwise $\mathscr{K}_{pqN} = \{0\}$ by Proposition 2.10 (c). It will suffice to estimate $|(w'|\pi(u)Rw)|^2$ for all $u \in H$, $R \in \mathscr{K}_{pqN}$ by the right side of (3.2), with an explicit C_n. For each $v \in H$, there exists a unique $Z_v \in \mathscr{K}_{pqN}$ such that

$$(Z_v, R)_N = (w'|\pi(v)Rw) \quad \text{for all } R \in \mathscr{K}_{pqN}.$$

In fact, if $\{R_k\}$ is an orthonormal basis of $\mathscr{K}_{pqN}$,

$$Z_v = \sum_k (\pi(v)R_k w|w')R_k.$$

We need to estimate $(Z_v, Z_v)_N$. Suppose $u \in H$, $R \in \mathscr{K}_{pqN}$. By Lemma 2.4,

$$\pi(u^*)R \in \mathscr{K}_{pqN}.$$

Thus, by Lemma 3.1 (a),

$$(\pi(u)Z_v, R)_N = (Z_v, \pi(u^*)R)_N = (w'|\pi(vu^*)Rw).$$

Hence $\pi(u)Z_v = Z_{vu^*}$. Now

$$(Z_v, Z_v)_N = (w'|\pi(v)Z_v w) = (w'|\pi(u)Z_u w)$$

(for all $u \in H$)

$$= \sum_k |(w'|\pi(u)R_k w)|^2$$

(for all $u \in H$)

$$= \sum_k \int_H |(w'|\pi(u)R_k w)|^2 d\mu(u)$$

$$\leqq \sum_k \int_H \|\pi(u)R_k w\|^2 d\mu(u)$$

$$= a'(N, \lambda)^{-1} \sum_k (R_k, R_k)_N$$

$$= a'(N, \lambda)^{-1} \dim \mathscr{K}_{pqN};$$

we used Lemma 3.1 (c). So by Proposition 2.10 (c),

$$|(w'|\tau(v)Rw)|^2 \leqq (Z_v, Z_v)_N (R, R)_N$$
$$= a'(N, \lambda)^{-1} \dim \mathscr{H}_{pq}(R, R)_N.$$

The result now follows from Proposition 2.9 (c) if we set $v = I$.

For $N, p, q \in \mathbf{Z}^+$, let

$$b(N, p, q, \lambda) = [a(N, p, q, \lambda)/a'(N, \lambda)]^{\frac{1}{2}}$$

(see (2.5)). Then we have:

COROLLARY 3.3. *Suppose* $P \in \mathscr{H}_{pq}$, $\|P\|_2 = 1$ ($\| \ \|_2 = L^2(S_n)$ *norm*).
(a) *For some* $C_n \in \mathbf{R}$,

$$\|\mathscr{W}(P)\|_N \leqq C_n(\kappa + 1)^{\frac{1}{2}}[(p + 1)(q + 1)]^{(n-2)/2}b(N, p, q, \lambda).$$

(b) *For some* α, $|\alpha| = N$,

$$\|\mathscr{W}(P)E_\alpha\| \geqq b(N, p, q, \lambda).$$

Thus

$$b(N, p, q, \lambda) \leqq \|\mathscr{W}(P)\|_N \leqq C_N(\kappa + 1)^{n-1}b(N, p, q, \lambda).$$

Proof. (a) is immediate from Theorem 2.6 (a) and Theorem 3.2. If (b) were false, we would have

$$(\mathscr{W}(P), \mathscr{W}(P))_N < b(N, p, q, \lambda)^2(I, I)_N = a(N, p, q, \lambda),$$

contradicting Theorem 2.6 (a).

Explicitly,

$$b(N, p, q, \lambda) = (2|\lambda|)^{\kappa/2}\left[(2\pi^n)^{-1}(N + q' + n - 1)^{(\kappa+n-1)}\Big/\binom{N + n - 1}{n - 1}\right]^{\frac{1}{2}}$$

if $N \geqq p'$, and is zero otherwise. Thus

$$b(N, p, q, \lambda) \sim (2N|\lambda|)^{\kappa/2}\omega_n^{-1/2},$$

where $\omega_n = 2\pi^n/(n - 1)!$ is the area of S_n. Recall that $\mathscr{V}_N$ is analogous to rS_n, where $r^2 \sim 2N|\lambda|$. Thus Corollary 3.3 is analogous to (3.1), and the simple fact that if $\|P\|_2 = 1$, there must be a z on rS_n such that

$$|P(z)| \geqq r^\kappa\omega_n^{-1/2}$$

Remark. An alternate proof of Theorem 2.6 could be given on the basis of Lemma 3.1 (a) and the following fact. Suppose that π is an irreducible representation of a group G on a finite-dimensional vector space $\mathscr{K}$ and that $(\, , \,)$, $\langle \, , \, \rangle$ are two different inner products on $\mathscr{K}$ with respect to which π is a unitary representation. Then for some c,

$$(x, y) = c\langle x, y\rangle \text{ for all } x, y \in \mathscr{K}.$$

For the proof, observe that $(x, y) = \langle Ax, y \rangle$ for a positive operator A. A commutes with $\pi(G)$, hence $A = cI$ by Schur's lemma. This argument was shown to us by H. Upmeier, who has proved a deep generalization of the first identity of Theorem 2.6 (a) for bounded symmetric domains.

4. Exact formulae for the Weyl transform. We shall now derive exact formulae for the Weyl transform of special types of functions. These formulae are analogous to the Bessel function formulae one meets in $\mathbf{R}^n$ (see [**18**], Theorem 3.10). We claim originality only for Theorem 4.2 and everything following it (except the well-known (4.6)). The material preceding Theorem 4.2 has been expounded in many forms before, in particular [**14**], [**15**] and [**19**]; our approach and formulation may be original.

This section is not used in Part B.

For $z \in \mathbf{C}^n$, we write

$$|\mathbf{z}| = (|z_1|, \dots , |z_n|),$$
$$|\mathbf{z}|^2 = ((z_1|^2, \dots , |z_n|^2).$$

We set $\sigma_\lambda = \operatorname{sgn} \lambda$. If G is a function on $\mathbf{C}^n$, $m \in \mathbf{Z}^n$, we say that G has F. S. (Fourier Series) type m if

$$G(z) = F(|\mathbf{z}|)e^{im\cdot\theta} \text{ for some } F.$$

(Here $z_k = |z_k|e^{i\theta_k}$.) If $S \in \mathcal{O}(\mathcal{H})$ we say that S has F. S. type m if for all α,

$$SE_\alpha = r_\alpha E_{\alpha - \sigma m} \text{ for some } r_\alpha \in \mathbf{C}.$$

(Here, $E_\beta = 0$ if $\beta \notin (\mathbf{Z}^+)^n$.) Thus, polyradial functions and operators are those with F. S. type 0.

PROPOSITION 4.1. (a) *Suppose $G \in L^1$, G has* F. S. *type m. Then $\mathcal{G}G$ has* F. S. *type m.*

(b) *Suppose $\mathcal{G} \in L^2$. The $\mathcal{G}$ has* F. S. *type m if and only if $\mathcal{G}G$ has* F. S. *type m.*

The analogue for $\mathcal{F}'$ is well known. For the proposition, we assume $\lambda = 1/2$ and compute $V_z E_\alpha$. In the computation, if α, β are two multi-indices, we say $\alpha \leqq \beta$ if $\alpha_k \leqq \beta_k$ for all k. For $m \in \mathbf{Z}^n$, we let

$$m^+ = (1/2)(|m| + m), m^- = m^+ - m.$$

Then

$$e^{|z|^2/2} V_z E_\alpha = \exp(-z \cdot W^+) \exp(\overline{z} \cdot W) E_\alpha$$
$$= \sum_{\beta \leq \alpha} [\alpha!/(\alpha - \beta)!]^{\frac{1}{2}} \beta!^{-1} \overline{z}^\beta \exp(-z \cdot W^+) E_{\alpha - \beta}$$

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$$= \sum_{0 \leqq \beta \leqq \alpha} \sum_{\gamma \geqq \alpha - \beta} [\gamma!/(\alpha - \beta)!]^{\frac{1}{2}} [\alpha!/(\alpha - \beta)!]^{\frac{1}{2}} \beta!^{-1}$$

$$\times (\gamma - \alpha + \beta)!^{-1} \bar{z}^{\beta} (-z)^{\gamma - \alpha + \beta} E_{\gamma}$$

$$= \sum_{m \in \mathbf{Z}^n} [\alpha!(\alpha - m)!]^{\frac{1}{2}} \sum_{m^{+} \leqq \beta \leqq \alpha} [\beta!(\alpha - \beta)!(\beta - m)!]^{-1} \bar{z}^{\beta} (-z)^{\beta - m} E_{\alpha - m}$$

$$= \sum_{m \in \mathbf{Z}^n} [(\alpha - m^{+})!/(\alpha + m^{-})!]^{\frac{1}{2}} \times \sum_{0 \leqq k \leqq \alpha - m^{+}} (-1)^{|k + m^{-}|} (\alpha + m^{-})!$$

$$\times [(\alpha - m^{+} - k)!k!(k + |m|)!]^{-1} |z|^{2k + |m|} e^{-im \cdot \theta} E_{\alpha - m}$$

with all sums converging absolutely. (This is justified in Lemma I.1.3 (a)). Here

$$z_j = |z_j| e^{i\theta_j}.$$

We have set $k = \beta - m^{+}$ and noted that

$$(k + m^{+})!(k + m^{-})! = k!(k + |m|)!$$

Finally

$$(4.1) \qquad V_z E_{\alpha} = \sum_{m \in \mathbf{Z}^n} (-1)^{|m^{-}|} l^{|m|}_{\alpha - m} \cdot (|z|^2) e^{-im \cdot \theta} E_{\alpha - m}.$$

Here $l^{|m|}_{\alpha}$ is a Laguerre function, defined as follows. First, if $x \in \mathbf{R}^{+}$, α, $m \in \mathbf{Z}^{+}$, put

$$L^{m}_{\alpha}(x) = \sum_{k=0}^{\infty} \binom{\alpha + m}{\alpha - k} \frac{(-x)^k}{k!}.$$

These are the Laguerre polynomials. Also put

$$l^{m}_{\alpha}(x) = [\alpha!/(\alpha + m)!]^{\frac{1}{2}} x^{m/2} L^{m}_{\alpha}(x) e^{-x/2}.$$

If instead $x \in (\mathbf{R}^{+})^n$, α, $m \in (\mathbf{Z}^{+})^n$, put

$$L^{m}_{\alpha}(x) = \prod_{i=1}^{n} L^{m_i}_{\alpha_i}(x_i),$$

and let

$$l_\alpha^m(x) = \prod l_{\alpha_i}^{m_i}(x).$$

From (4.1),

$$|l_\alpha^m(x)| \leqq 1 \text{ for all } \alpha, m, x.$$

Now let λ be arbitrary. Suppose $G \in L^1$ has F. S. type m; say

$$G(z) = F(|z|^2)e^{im\cdot\theta}.$$

Let $m^\sigma = m^+$ if $\sigma = 1$ or m^- if $\sigma = -1$. From (4.1) and a Fourier series argument,

$$(\mathscr{G}G)E_\alpha = r_\alpha E_{\alpha - \sigma m}$$

where

$$(4.2) \qquad r_\alpha = (-1)^{|m^\sigma|} \int_{\mathbf{C}^n} l_{\alpha - m^\sigma}^{|m|}(2|\lambda||z|^2)F(|z|^2)dV.$$

Thus G has F. S. type m.

(Remark. In particular, suppose

$$G(z) = e^{-|\lambda|\,|z|^2}.$$

As we remarked during the discussion of Theorem 1.2, $(\mathscr{G}G)E_\alpha = 0$ unless $\alpha = 0$, while $(\mathscr{G}G)E_o = cE_o$ for some c. By (4.2),

$$c = \int_{\mathbf{C}^n} e^{-2|\lambda|\,|z|^2}dV = (2|\lambda|\,)^{-n}\pi^n.$$

This, as we said before, essentially verifies the argument (i) for Theorem 1.2.) A density argument now proves (4.2) for $G \in L^2$. Simple orthogonality arguments now complete the proof of Proposition 4.1. See Lemma I.3.1.

Other formulae follow rapidly. Suppose $S \in \mathscr{I}_1^E$ has F. S. type m, $SE_\alpha = r_\alpha E_{\alpha - \sigma m}$. Then

$$\begin{aligned} \operatorname{tr}(V_{-z}S) &= \sum (E_\alpha|V_{-z}SE_\alpha) \\ &= \sum (V_zE_\alpha|E_{\alpha-\sigma m})(E_{\alpha-\sigma m}|SE_\alpha) \end{aligned}$$

so that

$$(4.3) \qquad (\mathscr{T}S)(z) = \pi^{-n}(2|\lambda|\,)^n \sum_\alpha r_\alpha(-1)^{|m^{-\delta}|}l_{\alpha - m^\rho}^{|m|}(2|\lambda||z|^2)e^{im.\theta}.$$

As a special case, suppose $n = 1$, $\lambda = 1/2$, $m \geqq 0$. If $\beta \in (\mathbf{Z}^+)^n$, define $S \in \mathscr{O}(\mathscr{H})$ by

$$S_\beta E_\alpha = \delta_{\beta\alpha}E_{\alpha+m}.$$

Let $G_\beta = \mathscr{T}S_\beta$. Then

$$G_\beta(z) = \pi^{-1}(-1)^m l_\alpha^m(\,|z|^2)e^{-im.\theta}.$$

We claim that $\mathscr{G}G_\beta = S_\beta$. Indeed, as we said at the beginning of Section 2, $S_\beta = \mathscr{G}F_\beta$ for some $F_\beta \in \mathscr{S}$. By the inversion theorem, then,

$$F_\beta = \mathscr{T}\mathscr{G}F_\beta = G_\beta.$$

Many properties of the Laguerre functions can be read off. By the polarization of the Plancherel theorem, we find that

$$\pi^{-2}\int_{\mathbf{C}} l_\alpha^m(|z|^2)l_\beta^m(|z|^2)dV = \pi^{-1}\operatorname{tr}(S_\alpha^* S_\beta) = \pi^{-1}\delta_{\alpha\beta}.$$

Evaluating the integral in polar coordinates, we see that the $l_\alpha^m(x)$, for $\alpha \in \mathbf{Z}^+$, are orthonormal functions in $L^2(\mathbf{R}^+)$. Indeed, they are a basis. For, suppose $F \in L^2(\mathbf{R}^+)$,

$$\int_0^\infty l_\alpha^m(x)F(x)dx = 0 \text{ for all } \alpha.$$

Let $G(z) = F(|z|^2)e^{-im\theta}$. Let $\mathscr{G}G = S$, and suppose

$$SE_\alpha = r_\alpha E_{\alpha+m}.$$

Then

$$0 = \int_{\mathbf{C}}(-1)^m l_\alpha^m(|z|^2)G(z)dV = \operatorname{tr}(S_\alpha^* S) = r_\alpha.$$

Thus $S = 0$, $G = 0$ and $F = 0$. Thus the formulae (4.2), (4.3) have obvious interpretations in terms of Laguerre series.

We remark that, if G has F. S. type m, the well known analogous formulae for $\mathscr{F}'G$, involving Bessel functions, may be proved in the same way; that is, through use of the power series expansion of $\exp(-z \cdot \bar{\zeta} + \bar{z} \cdot \zeta)$.

Suppose now that G is radial on $\mathbf{C}^n$, that is $G = G(|z|^2)$, while $P \in \mathscr{H}_{pq}$. We next derive the exact formulae for $\mathscr{G}(GP)$ which are analogous to the Bessel function formulae for $\mathscr{F}'(GP)$ (see [**18**], Theorem 3.10).

THEOREM 4.2. *Suppose $GP \in L^2(\mathbf{C}^n)$ or $L^1(\mathbf{C}^n)$ where G is radial and $P \in \mathscr{H}_{pq}$.*

(a) *$\mathscr{G}(GP) = (-1)^q\mathscr{W}(P)S$ where S is radial, $SE_\alpha = c_{|\alpha|}E_\alpha$, where if $N \geqq p'$,*

$$c_N = (n-1)![(N-p')!/(N+q'+n-1)!] \times \int_{\mathbf{C}^n} L_{N-p'}^{n-1+\kappa}(2|\lambda|\,|z|^2)\,G(|z|^2)e^{-|\lambda|\,|z|^2}|z|^{2\kappa}dV.$$

(b) *If $GP \in L^1$ and $\mathscr{W}(P)S \in \mathscr{I}_1^E$, then*

$$G(|z|^2) = \pi^{-n}(2|\lambda|)^{\kappa+n}\sum_{N\geqq p'} c_N L_{N-p'}^{n-1+\kappa}(2|\lambda|\,|z|^2)e^{-|\lambda|\,|z|^2}.$$

Proof. In the situation of (b), let $R = \mathscr{W}(P)S$. Then

https://doi.org/10.4153/CJM-1984-039-0 Published online by Cambridge University Press

$$\begin{aligned}\text{tr}(V_{-z}R) &= (-1)^q \sum (E_\alpha | V_{-z}\mathscr{W}(P)SE_\alpha) \\ &= (-1)^q \sum_{N \geqq p'} c_N \sum_{|\alpha| = N} (V_z E_\alpha | \mathscr{W}(P) E_\alpha) \\ &= (-1)^q \sum_{N \geqq p'} c_N H_{N,P,\lambda},\end{aligned} \tag{4.4}$$

say, with absolute convergence. Suppose $\lambda = 1/2$. By Theorem 2.3 and the inversion theorem,

$$e^{-a|z|^2/2}P(z) = (-1)^q \sum_{N \geqq p} (1 - s)^{n+\kappa} s^{N-p} H_{N,P}(z).$$

Accordingly,

$$\sum_N s^N H_{N+p,P}(z) = (-1)^q (1 - s)^{-n-\kappa} e^{-a|z|^2/2} P(z),$$

whenever $|s| < 1$, if $a = \psi^{-1}(s)$. The series converges absolutely, so if $P(z) \neq 0$ we can divide by $P(z)$ and differentiate with respect to s, to find that

$$H_{N+p,P}(z) = K_{N,P}(|z|^2)P(z)$$

for certain functions $K_{N,P}$; this then holds even if $P(z) = 0$. With $x = |z|^2$, we find that

$$\sum_N s^N K_{N,P}(x) = (-1)^q (1 - s)^{-n-\kappa} e^{-ax/2}. \tag{4.5}$$

If $n = 1$, $m \geqq 0$ and $P = P_m$ where $P_m(\zeta) = \zeta^m$, then by Proposition 2.1 (b),

$$\begin{aligned}H_{N,P_m}(z) &= (V_z E_N, E_{N-m})(E_{N-m} | \mathscr{W}(P_m) E_N) \\ &= l^m_{N-m}(|z|^2) e^{im\theta} [N!/(N - m)!]^{\frac{1}{2}} \\ &= e^{-|z|^2/2} L^m_{N-m}(|z|^2) z^m.\end{aligned}$$

Accordingly,

$$K_{N,P_m}(|z|^2) = e^{-|z|^2/2} L^m_N(|z|^2).$$

We have derived the generating formula for Laguerre functions:

$$\sum_N s^N L^m_N(x) = (1 - s)^{-1-m} e^{-sx/(1-s)}. \tag{4.6}$$

But then by (4.5) and the uniqueness of generating functions we have

$$K_{N,P}(x) = (-1)^q L_N^{n-1+\kappa}(x) e^{-x/2}.$$

Accordingly, for general λ, one finds that

$$(4.7)\qquad \sum_{|\alpha|=N} (V_z E_\alpha | \mathscr{W}(P) E_\alpha) = (-1)^q (2|\lambda|)^\kappa L_{N-p'}^{n-1+\kappa}(2|\lambda|\,|z|^2) \times P(z) e^{-|\lambda|\,|z|^2}.$$

(4.4) and (4.7) yield (b).

For (a), a simple approximation argument shows we may assume $GP \in L^1$. Suppose $Q \in \mathscr{H}_{p_1 q_1}$ for some p_1, q_1, say $N \in \mathbf{Z}^+$, and observe that

$$\begin{aligned}(\mathscr{W}(Q), \mathscr{G}(GP))_N &= (2|\lambda|)^{n-1} \int_{\mathbf{C}^n} \sum_{|\alpha|=N} (\mathscr{W}(Q) E_\alpha | V_z E_\alpha) G(|z|^2) P(z) dV \\ &= (-1)^q (2|\lambda|)^{\kappa+n-1} \int_{\mathbf{C}^n} L_{N-p'}^{n-1+\kappa}(2|\lambda|\,|z|^2) G(|z|^2) \\ &\quad \times \overline{Q(z)} P(z) e^{-|\lambda|\,|z|^2} dV\end{aligned}$$

by (4.7). Accordingly, if $(Q, P) = 0$ ($L^2(S_n)$ inner product),

$$(\mathscr{W}(Q), \mathscr{G}(GP))_N = 0.$$

By Proposition 2.10 (b), there exist c_N such that

$$\mathscr{G}(GP) E_\alpha = (-1)^q c_N \mathscr{W}(P) E_\alpha \text{ whenever } |\alpha| = N.$$

By Theorem 2.6 (a) and the preceding,

$$\begin{aligned}&c_N (2\pi^n)^{-1} (N + q' + n - 1)^{(\kappa+n-1)} (P, P) \\ &= \int_{\mathbf{C}^n} L_{N-p'}^{n-1+\kappa}(2|\lambda|\,|z|^2) G(|z|^2)\, |P(z)|^2 e^{-|\lambda|\,|z|^2} dV.\end{aligned}$$

This gives (a) at once.

(a) and (b) again have simple interpretations in terms of Laguerre series; in particular they are consistent with the fact that $\{l_M^{n-1+\kappa} | M \in \mathbf{Z}^+\}$ is an orthonormal basis for $L^2(\mathbf{R}^+)$.

Several new special functions formulae follow from the relation between Theorem 4.2 and the notion of F. S. type m. To see this, for $m \in \mathbf{Z}^+$ let

$$\mathscr{P}_m = \{P \in \mathscr{P} | P \text{ has F. S. type } m\}.$$

Note that if $P(\zeta) = \sum a_{\rho\gamma} \zeta^\rho \bar{\zeta}^\gamma$

$$P \in \mathscr{P}_m \Leftrightarrow a_{\rho\gamma} = 0 \text{ unless } \rho - \gamma = m.$$

If $p, q \in \mathbf{Z}^+$, let

$$\mathscr{H}_{pqm} = \mathscr{H}_{pq} \cap \mathscr{P}_m.$$

Since $\Delta:\mathscr{P}_m \to \mathscr{P}_m$ and $\mathscr{P} = \oplus \mathscr{P}_m$, we find $\mathscr{H}_{pq} = \oplus \mathscr{H}_{pqm}$. Note that $\mathscr{H}_{pqm} = \{0\}$ unless

$$p - |m^+| = q - |m^-| \geqq 0.$$

(The equality follows from the above restriction that $\rho - \gamma = m$ for non-zero terms. The inequality is evident.)

In what follows, we shall assume $\lambda = 1/2$. For each p, q, m with $\mathscr{H}_{pqm} \neq \{0\}$ select an orthonormal basis $\mathscr{B}_{pqm}$ for $\mathscr{H}_{pqm}$ (orthonormal in the $L^2(S_n)$ inner product). Let

$$\mathscr{B}_m = \bigcup_{p,q} \mathscr{B}_{pqm}$$

and let

$$\mathscr{B} = \bigcup_m \mathscr{B}_m.$$

Set

$$a(N, p, q) = (2\pi^n)^{-1}(N + q + n - 1)^{(\kappa+n-1)}$$

as before. Let

$$c(N, p, q) = a(N, p, q)^{-1} \text{ if } N \geqq p, 0 \text{ otherwise.}$$

By Proposition 2.10 (b) and Theorem 2.6 (a), if $N \in \mathbf{Z}^+$ and R is in the Hilbert space $\mathscr{O}(\mathscr{V}_N)$, R has the orthogonal expansion

$$R = \sum_{P \in \mathscr{B}} c(N, p, q)(\mathscr{W}(P), R)_N \mathscr{W}(P).$$

(In the sum, of course, $P \in \mathscr{H}_{pq}$.) If $|\alpha| = N$, $\beta \in (\mathbf{Z}^+)^n$, the map $S \to (E_\beta|SE_\alpha)$ is a continuous linear functional from $\mathscr{O}(\mathscr{V}_N)$ to $\mathbf{C}$. Thus

$$(E_{\alpha-m}|RE_\alpha) = \sum_{P \in \mathscr{B}} c(N, p, q)(\mathscr{W}(P), R)_N(E_{\alpha-m}|\mathscr{W}(P)E_\alpha).$$

Note that

$$(E_{\alpha-m}|\mathscr{W}(P)E_\alpha) = 0 \text{ unless } P \in \mathscr{B}_m;$$

thus

$$(4.8) \qquad (E_{\alpha-m}|RE_\alpha) = \sum_{P \in \mathscr{B}_m} c(N, p, q)(\mathscr{W}(P), R)_N(E_{\alpha-m}|\mathscr{W}(P)E_\alpha).$$

Note that all but finitely many terms vanish. Indeed, there are only finitely many pairs (p, q) with

$$p - q = |m^+| - |m^-| \quad \text{and} \quad p \leqq N.$$

If these conditions are not satisfied,

$$(\mathscr{W}(P), R)_N = 0.$$

Putting $R = V_z$ in (4.8) and using (4.1) and (4.7), we find that if $|\alpha| = N$:

$$(4.9) \quad \begin{aligned} &(-1)^{|m|} l^{|m|}_{\alpha - m}(|z|^2)e^{-im\cdot\theta} \\ &= (-1)^q \sum_{P \in \mathscr{B}_m} c(N, p, q) L^{n-1+\kappa}_{N-p}(|z|^2)\overline{P(z)}e^{-|z|^2/2} \\ &\times (E_{\alpha-m}|\mathscr{W}(P)E_\alpha) \end{aligned}$$

or equivalently

$$(4.10) \quad \begin{aligned} &l^{|m|}_{\alpha-m}(|z|^2)e^{-im\cdot\theta}E_{\alpha-m} \\ &= \sum_{P \in \mathscr{B}_m} (-1)^r c(N, p, q) L^{n-1+\kappa}_{N-p}(|z|^2)\overline{P(z)}e^{-|z|^2/2}\mathscr{W}(P)E_\alpha \end{aligned}$$

if $r = r(q, m) = q - |m^-|$. Fix a $P \in \mathscr{B}_{pqm}$. Take the inner product with $\mathscr{W}(P)E_\alpha$ and sum over α:

$$(4.11) \quad \begin{aligned} &\sum_{|\alpha|=N} l^{|m|}_{\alpha-m}(|z|^2)(E_{\alpha-m}|\mathscr{W}(P)E_\alpha) \\ &= (-1)^r L^{n-1+\kappa}_{N-p}(|z|^2)e^{-|z|^2/2}P(|z|). \end{aligned}$$

Here we noted that

$$P(|z|) = P(z)e^{-im\cdot\theta}.$$

(4.11) is true for any $P \in \mathscr{H}_{pqm}$; it is an addition formula, known previously only when $P = P_m$. In this case, (4.11) becomes

$$\sum_{|\alpha|=N} L^{|m|}_\alpha(|z|^2) = L^{n-1+|m|}_\alpha(|z|^2)$$

which is proved easily through (4.6).

Note that $(E_{\alpha-m}|\mathscr{W}(P)E_\alpha)$ is a very simple quantity. If

$$P(z) = \sum a_{\rho\gamma}z^\rho\bar{z}^\gamma,$$

then by Proposition 2.7,

$$(E_{\alpha-m}|\mathscr{W}(P)E_\alpha) = \sum a_{\rho\gamma}[\alpha^{(\rho)}(\alpha - m)^{(\gamma)}]^{\frac{1}{2}}.$$

Thus (4.1) is a completely explicit formula.

We can interpret (4.9) and (4.11) as follows. For $m \in \mathbf{Z}^n$, $\alpha \in (\mathbf{Z}^+)^n$,

$\alpha \geqq m^+$, set

$$e_{\alpha m}(z) = (-1)^{|m|}(2\pi)^{-n/2} L_{\alpha - m^+}^{|m|}(|z|^2) P_m(z) e^{-|z|^2}.$$

For $P \in \mathscr{B}_{pqm}$, $N \in \mathbf{Z}^+$, $N \geqq p$, let

$$f_{N,P}(z) = (-1)^q L_{N-p}^{n-1+\kappa}(|z|^2) P(z) e^{-|z|^2/2}.$$

Then it is easy to see that the sets $S_1 = \{e_{\alpha m}\}$ and $S_2 = \{f_{N,P}\}$ are orthogonal bases for $L^2(\mathbf{C}^n)$. In fact,

$$\|e_{\alpha m}\|^2 = 2^{-n}(\alpha + m^-)!/(\alpha - m^+)! = b_{\alpha m},$$

say, while

$$\|f_{N,P}\|^2 = 2^{-1}(N + q + n - 1)!/(N - p)! = a'(N, p, q),$$

say. (4.9) and (4.11) express one basis in terms of the other. An expansion of an L^2 function using one basis can now easily be replaced by an expansion using the other.

Specifically, let $c'(N, p, q) = a'(N, p, q)^{-1}$ if $N \geqq p$, 0 otherwise. (4.9) reads:

$$(4.12) \quad e_{\alpha m} = \pi^{n/2} \sum_{P \in \mathscr{B}_m} c'(N, p, q) b_{\alpha m}^{1/2} (\mathscr{W}(P) E_\alpha | E_{\alpha - m}) f_{N,P} \text{ if } |\alpha| = N$$

(4.11) reads

$$(4.13) \quad f_{N,P} = \pi^{n/2} \sum_{|\alpha| = N} b_{\alpha m}^{-1/2} (E_{\alpha - m} | \mathscr{W}(P) E_\alpha) e_{\alpha m} \quad \text{if } P \in \mathscr{B}_m.$$

From either:

$$(4.14) \quad (e_{\alpha m}, f_{NP}) = \delta_{|\alpha| N} \pi^{n/2} b_{\alpha m}^{1/2} (E_{\alpha - m} | \mathscr{W}(P) E_\alpha) \quad \text{if } P \in \mathscr{B}_m.$$

(4.12) has a further significance. If one restricts to a sphere centered at 0, (4.12) gives explicitly the spherical harmonic expansion of $e_{\alpha m}$ restricted to that sphere.

Part B. Exact Formulae for the Fourier Transform on the Heisenberg Group

Introduction.

The Heisenberg group $\mathbf{H}^n$ is the Lie group with underlying manifold $\mathbf{R} \times \mathbf{C}^n$ and multiplication

$$(t, z) \cdot (t', z') = (t + t' + \operatorname{Im} z \cdot \bar{z}', z + z'),$$

$$\text{where } z \cdot \bar{z} = \sum_{j=1}^{n} z_j \bar{z}'_j.$$

It is equipped with dilations $D_r (r > 0)$, where

$$D_r(t, z) = (r^2 t, rz).$$

One may then speak of homogeneous distributions. Our main purpose is to derive an exact formula for the group Fourier transform (F. T.) of a class of regular homogeneous distributions (r. h. d.'s). Here "regular" means C^∞ away from 0. An r. h. d. K can be written in the form

$$K(t, z) = \sum_i K_i(t, |z|^2)P_i(z),$$

where $\{P_i\}$ is an orthonormal basis for the bigraded spherical harmonics. Each term in the series is also an r. h. d., and if one wished to find the F. T. of K, it would suffice to determine the F. T. of each term. Thus we restrict attention to K of the form $K'(t, |z|^2)P(z)$. For this purpose, we will make use of the theory of Part A. We shall assume K' has a specific form. A general formula could be given for any K'; we hope to give this in a later paper. The situation considered here, however, gives formulae which are often useful in practice.

The first section is introductory. For proofs, we refer to the first section of [**8**], and [**4**]. The author's paper [**9**] contains relevant material. However, we shall not refer to it for proofs, and we have improved the notation we used there.

In Section 6 we give the main formula. In Section 7 we give some applications, and in particular compute the Fourier transform of the Poisson kernel and its variants. Confluent hypergeometric functions arise here, as in related problems in [**10**].

5. Summary of the basic properties of the F. T. on $\mathbf{H}^n$. On $\mathbf{H}^n$, the differential operators $T = \partial/\partial t$ and $Z_j = \partial/\partial z_j + i\bar{z}_j T$ are left-invariant, and $\{T, Z_j, \bar{Z}_j\}$ $(j = 1, \ldots, n)$ is a basis for left-invariant vector fields on $\mathbf{H}^n$. The only non-trivial commutation relations are

$$[Z_j, \bar{Z}_j] = -2iT.$$

Let $\mathscr{S}(\mathbf{H}^n) = \mathscr{S}(\mathbf{R} \times \mathbf{C}^n)$ denote Schwartz space.

Notation as in Section 1, we let $\mathscr{R}$ denote the set of operator families $\underset{\sim}{R} = (R(\lambda))$ where λ ranges over $\mathbf{R}^*$, where $R(\lambda) \in \mathscr{O}(\mathscr{H}_\lambda)$ for all λ, and where $(E_{\beta,\gamma}|R(\lambda)E_{\alpha,\lambda})$ is a measurable function of λ for all α, β. (We regret that we omitted, though implicitly used, this last condition in [**8**] and in [**9**].) Extend the notion of addition, multiplication and transpose in the obvious manner to R. Let $\underset{\sim}{M}_j = (W_{j\lambda})$, $\underset{\sim}{M} = (\lambda I)$. Let

$$\mathscr{B} = \{\underset{\sim}{S} \in \mathscr{R}| \text{ each } S(\lambda) \in \mathscr{B}(\mathscr{H}_\lambda) \text{ and}$$
$$\|\underset{\sim}{S}\| = \sup\|S(\lambda)\| < \infty\}.$$

The F. T. $\wedge$ is defined as a mapping from $L^1(\mathbf{H}^n)$ to $\mathscr{B}$ in such a way that if $f \in \mathscr{S}(\mathbf{H}^n)$ then

$$T\hat{f} = -i\hat{f}\underset{\sim}{M}, \ Z_j\hat{f} = \hat{f}\underset{\sim}{W}_j^+, \ \bar{Z}_j\hat{f} = -\hat{f}\underset{\sim}{W}_j$$

(more precisely, e.g.,: $Z_j f(\lambda) = \hat{f}(\lambda)W_j^+{}_\lambda$ on $\mathscr{D}_\lambda^j$ for all λ) and in such a way that

$$g * f = \hat{g}\hat{f}(f, g \in L^1).$$

Note that the first three equalities are consistent with the commutation relations.

If we define $\mathscr{F}_c^\lambda: L^1(\mathbf{H}^n) \to L^1(\mathbf{C}^n)$ by

$$(\mathscr{F}_c^\lambda f)(z) = \int_{-\infty}^{\infty} e^{i\lambda t} f(t, z)dt,$$

we have only to set

$$\hat{f}(\lambda) = \mathscr{G}_\lambda \mathscr{F}_c^\lambda f.$$

Indeed, $\mathscr{F}_c^\lambda (Z_j f) = \mathscr{Z}_j \mathscr{F}_c^\lambda f$, etc. Thus

$$\hat{f}(\lambda) = \int_{\mathbf{H}^n} U_u^\lambda f(u)du$$

where

$$U_{(t,z)}^\lambda = e^{i\lambda t} V_z^\lambda.$$

That $g * f = \hat{g}\hat{f}$ is checked with (1.1).

If $f \in L^1$, $g(t, z) = f(-t, -\bar{z})$, $h(u) = \bar{f}(u^{-1})$ then it is easy to see that

$$\hat{g}(\lambda) = \hat{f}(-\lambda), \ \hat{h} = \hat{f}^*.$$

For $\gamma \in \mathbf{C}$, let

$$L_\gamma = -(1/2) \sum (Z_j\bar{Z}_j + \bar{Z}_jZ_j) + i\gamma T.$$

We note that if $f \in \mathscr{S}$, $L_\alpha \hat{f} = \hat{f}\underset{\sim}{A}$ where $\underset{\sim}{A} = (A_\lambda)$.

Certain subspaces of $\mathscr{R}$ are useful. We set $\mathscr{Q} = \{ (S(\lambda)) \in \mathscr{R} | S_{\beta\alpha}(\lambda) = (E_{\beta,\lambda}|S(\lambda)E_{\alpha,\lambda}) \in C_c^\infty(\mathbf{R}^*)$ (as a function of λ) for each α, β; and for some $N \in \mathbf{Z}^+$, $S_{\beta\alpha}(\lambda) = 0$ for all λ if $|\alpha| + |\beta| > N\}$. Then if $\underset{\sim}{S} \in \mathscr{Q}$, $\underset{\sim}{S} = \hat{f}$ for some $f \in \mathscr{S}(\mathbf{H}^n)$. We set

$$\mathscr{R}_1^E = \{ (S(\lambda)) \in \mathscr{R} | \text{ for a.e. } \lambda,$$

$$||S(\lambda)||_1^E = \sum_\alpha ||S(\lambda)E_\alpha|| > \infty \text{ and}$$

$$\int_{-\infty}^{\infty} ||S(\lambda)||_1^E (2|\lambda|)^n d\lambda < \infty\}.$$

Let

$$\mathscr{R}_2 = \{\underset{\sim}{S} = (S(\lambda)) \in \mathscr{R} | \text{ for a.e. } \lambda,$$
$$\|S(\lambda)\|_2 < \infty \text{ and}$$
$$\|\underset{\sim}{S}\|_2^2 = \int_{-\infty}^{\infty} \|S(\lambda)\|_2^2 (2|\lambda|)^n d\lambda < \infty\}.$$

Here $\|S(\lambda)\|_2$ is the Hilbert-Schmidt norm of $S(\lambda)$. Then

$$^\vee : \mathscr{R}_1^E \to C(\mathbf{H}^n) \cap L^\infty(\mathbf{H}^n)$$

by

$$\overset{\vee}{\underset{\sim}{S}}(u) = \int_{-\infty}^{\infty} \operatorname{tr}(U_{-u}S(\lambda))(2|\lambda|)^n d\lambda$$

is well defined. Here is a version of the inversion and Plancherel theorems for $\wedge$, which may be proved on the basis of Theorem 1.2:

THEOREM 5.1. (a) *If $f \in \mathscr{S}(\mathbf{H}^n)$ then $\hat{f} \in \mathscr{R}_1^E \cap \mathscr{R}_2$.*
(b) *If $f \in L^1(\mathbf{H}^n)$ and $\underset{\sim}{S} = \hat{f} \in \mathscr{R}_1^E$ then*

$$f(u) = c_n' \overset{\vee}{\underset{\sim}{S}}(u) \quad \textit{for a.e. } u;$$

here $c_n' = (2\pi^{n+1})^{-1}$.
(c) *If $f \in \mathscr{S}$ then $\|f\|_2^2 = c_n'\|\hat{f}\|_2^2$. $\wedge$ can then be extended to a constant multiple of a unitary map from $L^2(\mathbf{H}^n)$ onto $\mathscr{R}_2$.*

One thus sees that $\mathscr{Q}$, $\mathscr{R}_1^E$ and $\mathscr{R}_2$ play the role on the Fourier transform side that C_c^∞ L^1, and L^2 play on the Fourier transform side in the usual Fourier analysis on $\mathbf{R}^m$.

One has a natural pairing for elements of $\mathbf{R}$. Namely if $\underset{\sim}{R}$, $\underset{\sim}{S} \in \mathscr{R}$, and if

$$(5.1) \qquad \int_{-\infty}^{\infty} \sum_\alpha |(R(\lambda)E_\alpha|S(\lambda)E_\alpha)| \, |\lambda|^n d\lambda < \infty$$

we set

$$(\underset{\sim}{R}|\underset{\sim}{S}) = \int_{-\infty}^{\infty} \sum_\alpha |(R(\lambda)E_\alpha|S(\lambda)E_\alpha)(2|\lambda|)^n d\lambda.$$

Thus (by polarizing Plancherel) if $f, g \in L^2$ we have

$$(f|g) = c_n(\hat{f}|\hat{g})$$

where the inner product on the left side is the L^2 inner product.

Given $R \in \mathscr{R}$, it is particularly important to be able to tell simply whether $\underset{\sim}{R} \in \mathscr{R}_1^E$ (for example, for Theorem 5.1 (a)). Given $R, S \in \mathscr{R}$, we would also like simple criteria for (5.1) to hold.

Now if f is a measurable function on $\mathbf{R}^n$ such that

$$f(x) = O(|x|^{-m_1}) \text{ as } x \to 0,$$

where $m_1 < n$, while

$$f(x) = O(|x|^{-m_2}) \text{ as } x \to \infty,$$

where $m_2 > n$, then $f \in L^1$. The analogue in the present situation is this:

PROPOSITION 5.2. *Suppose* $F:(\mathbf{Z}^+)^n \times \mathbf{R}^* \to \mathbf{R}^+$ *satisfies*

$$F(\alpha, \lambda) < C(\nu_\alpha|\lambda|)^{-m_1} \text{ for some } m_1 < n + 1 \text{ and}$$

$$F(\alpha, \lambda) < C(\nu_\alpha|\lambda|)^{-m_2} \text{ for some } m_2 > n + 1.$$

Then

$$I = \int_{-\infty}^{\infty} \sum_\alpha F(\alpha, \lambda)\, |\lambda|^n d\lambda < \infty.$$

(*As always*, $\nu_\alpha = |\alpha| + n/2$, *and* F *is measurable in* λ.)

Proof. Indeed, for $N \in \mathbf{Z}^+$, put $N_+ = N + n/2$. Then

$$I < C \sum_{N=0}^{\infty} \binom{N + n - 1}{n - 1} \left[N_+^{-m_1} \int_0^{1/N_+} \lambda^{n-m_1} + N_+^{-m_2} \int_{1/N_+}^{\infty} \lambda^{n-m_2} \right] < C \sum_{N=0}^{\infty} N^{n-1} N_+^{-n-1} < \infty.$$

If we make use of this proposition, we see the importance of the classes

$$\text{Ord}(m) = \{\underset{\sim}{R} \in \mathscr{R} | \sup_{\alpha,\lambda} (2\nu_\alpha|\lambda|)^{-\text{Re}\, m/2} \|R(\lambda)E_\alpha\| = \|\underset{\sim}{R}\|^{\text{Ord}(m)} < \infty\} \text{ for } m \in \mathbf{C}.$$

Let

$$\text{Rap} = \bigcap_{m \leqq 0} \text{Ord}(m),$$

topologized as a Frechet space. (Ord stands for "order"; Rap stands for "rapid decay". In [**9**], Ord(m) was called $\widetilde{\mathscr{C}}(-m/2)$, while Rap was called $\widetilde{\mathscr{C}}$.) These spaces are analogous to the spaces on $\mathbf{R}^n$ defined by

$$\{f | \,|f(x)| < C|x|^m\} \text{ and}$$

$$\{f | \,|f(x)| < C_m|x|^m \text{ for all } m \leqq 0\}$$

respectively. From Proposition 5.2 follows immediately:

PROPOSITION 5.3. (a) Rap $\subset \mathscr{R}_1^E$ *continuously.*

(b) $\mathrm{Ord}(m)$ *is contained in the dual of* Rap (*under the natural pairing*) *if* $m > -2n - 2$.

The analogue of this on $\mathbf{R}^n$ is evident.

Theorem 5.1 (a) is now an easy consequence of the fact that $[\mathscr{S}(\mathbf{H}^n)]^\wedge \subset$ Rap. Indeed, if $f \in \mathscr{S}$,

$$\hat{f}\underset{\sim}{A}^N = [L_o^N f]^\wedge \in \mathscr{B} \text{ for all } N,$$

so this inclusion follows at once.

As a trivial consequence of the above arguments and the dominated convergence theorem, we have:

PROPOSITION 5.4. *Suppose that* $\{R_N | N \in \mathbf{N}\}$ *is a bounded subset of* $\mathrm{Ord}(m)$ *where* $m > -2n - 2$, *that* $\underset{\sim}{R} \in \mathscr{R}$ *and that for all* α, λ *we have*

$$\lim_{N\to\infty} R_N(\lambda)E_\alpha = R(\lambda)E_\alpha.$$

Suppose further that $\underset{\sim}{S} \in$ Rap. *Then*

$$(\underset{\sim}{R}_N | S) \to (\underset{\sim}{R} | \underset{\sim}{S}).$$

Suppose $\underset{\sim}{R} \in \mathscr{R}$ and (5.1) holds for all $\underset{\sim}{S} \in [\mathscr{S}(\mathbf{H}^n)]^\wedge$. (For example, this will hold if $\underset{\sim}{R} \in \mathrm{Ord}(m)$ if $m > -2n - 2$, by Proposition 5.3.) Suppose $F \in \mathscr{S}'(\mathbf{H}^n)$. We say $\hat{F} = \underset{\sim}{R}$ (in the $\mathscr{S}'$ sense) if for all $G \in \mathscr{S}$,

$$(F|G) = c_n'(\underset{\sim}{R}|\hat{G}).$$

We note the following useful proposition.

PROPOSITION 5.5. *Suppose* $\underset{\sim}{R} \in \mathscr{R}_1^E$. *Let*

$$F(u) = c_n' \check{\underset{\sim}{R}}(u) = c_n' \int_{-\infty}^{\infty} \mathrm{tr}(U_{-u}R(\lambda))(2|\lambda|)^n d\lambda.$$

Then $\hat{F} = \underset{\sim}{R}$ *in the* $\mathscr{S}'$ *sense.*

Proof. F is a bounded continuous fraction, hence in $\mathscr{S}'$. Suppose $G \in \mathscr{S}(\mathbf{H}^n)$. Then

$$\begin{aligned}(G|F) &= c_n' \int_{-\infty}^{\infty} \Sigma \left(E_\alpha \middle| \left[\int_{\mathbf{H}^n} U_{-u}\overline{G(u)}du \right] R(\lambda)E_\alpha \right)(2|\lambda|)^n d\lambda \\ &= c_n'(\hat{G}|\underset{\sim}{R})\end{aligned}$$

as desired.

One may now investigate the FTs of regular homogeneous distributions. Recall that these are defined as follows. For $r > 0$, define $D_r:\mathbf{H}^n \to \mathbf{H}^n$ by

$$D_r(t, z) = (r^2t, rz).$$

If f is a function on $\mathbf{H}^n$, $r > 0$, define the new functions D_rf, D^rf by

$$D_r f = f \circ D_r, \quad D^r f = r^{-2n-2} f \circ D_{1/r}.$$

By duality, one defines D_r, $D^r: \mathscr{S}' \to \mathscr{S}'$. If $K \in \mathscr{S}'$, $l \in \mathbf{C}$, K is called homogeneous of degree l if

$$D_r K = r^l K \text{ for all } r > 0.$$

K is called regular if it agrees with a C^∞ function away from the origin.

It is shown in [**9**] that if K is an r. h. d. of degree l, Re $l < 0$, then

$$K \in \mathrm{Ord}(m) \quad \text{where } m = -2n - 2 - l.$$

(Thus Re m exceeds $-2n - 2$. As we said before Proposition 5.5, this is important.) However, no formulae for K were computed in [**9**]; we are about to give some in the next section. We will not be assuming any of the theory of [**9**].

The situation is of course modelled on the well known Euclidean theory. For example, if K is an r. h. d. on $\mathbf{C}^n$, of degree l, where Re $l > -2n$, one may write K as an infinite sum of terms of the form $c|z|^{-2k}P(z)$, where $P \in \mathscr{H}_{pq}$ for some p, q, $c \in \mathbf{C}$; and $l = \kappa - 2k$ ($\kappa = p + q$). To find $\mathscr{F}'K$, one need only find all $\mathscr{F}'(|z|^{-2k}P(z))$. This is done in [**18**], by use of Hecke's identity. The result, placed in our notation, reads that

$$\mathscr{F}'(\Gamma(k)\,|z|^{-2k}P(z)) = (-1)^q \pi^n \Gamma(j)\,|\zeta|^{-2j}P(\zeta)$$

where $j = n + \kappa - k$.

On $\mathbf{H}^n$, suppose $-2n - 2 < \mathrm{Re}\, l < 0$, and that K is an r. h. d. of degree l. It is not difficult to show that one can write K as an infinite sum of terms of the form $G(t, |z|^2)P(z)$, where G is homogeneous of the appropriate degree. (We shall not use this, but state it for motivational purposes.) Thus it is reasonable to seek a formula for the F. T. of such a GP. We do this in the next section, in specific cases. It could then be done in general, but we do not do this here. We shall use Theorem 2.3, the analogue of Hecke's identity for $\mathscr{G}$.

The case where K is an r. h. d. on $\mathbf{C}^n$ of degree $-2n$ is also of particular interest in the Euclidean theory. Away from 0, K can be written as an infinite sum of terms of the form $c|z|^{-2k}P(z)$, where $P \in \mathscr{H}_{pq}$ and $\kappa \geqq 1$. (The last restriction is evidently necessary for the mean value zero condition.) Under these circumstances we have

$$\mathscr{F}'(\mathrm{P.V.}(\Gamma(k)\,|z|^{-2k}P(z))) = (-1)^p \pi^n \Gamma(j)\,|\zeta|^{-2j}P(\zeta)$$

where $j = n + \kappa - k$. Similarly, on $\mathbf{H}^n$, we shall also seek formulae in the case where K is an r. h. d. of degree $-2n - 2$, of the form P.V. $[G(t, |z|^2)P(z)]$ (see Section 8 of [**4**] for the definitions). In the $\mathbf{H}^n$ case, however, one could have $P \equiv 1$. The following proposition may elucidate matters somewhat.

PROPOSITION 5.6. *Suppose* $F(t, z) = G(t, |z|^2)P(z)$ *is smooth on* $\mathbf{H}^n \setminus \{0\}$

(*and not necessarily homogeneous*), *where* $P \in \mathscr{H}_{pq}$, $p + q \geqq 1$. *Then if* $0 < a < b$,

$$\int_{a<|u|<b} F(u)dV(u) = 0.$$

(*Here, if* $u = (t, z)$, $|u| = (|z|^4 + t^2)^{1/4}$.) *In particular, if F is homogeneous of degree* $-2n - 2$, *there exists an* r. h. d. K *of degree* $-2n - 2$ *which equals* F *away from* 0.

Proof. Integrate in the order $dzd\bar{z}dt$.

We further assert, but shall not use, the following fact, which clarifies the situation when $P \equiv 1$. Suppose $G(t, |z|^2)$ is smooth on $\mathbf{H}^n \setminus \{0\}$, is homogeneous of degree $-2n - 2$, and that it has mean value 0. Then there exist an r. h. d. K_1 of degree $-2n$ and $c \in \mathbf{C}$ such that

$$\text{P.V.}(G) = TK_1 + c\delta \quad \text{and} \quad K_1 = G_1(t, |z|^2)$$

for some smooth homogeneous G_1. This fact may be verified without difficulty if one thinks of $\mathbf{H}^n$ as $\mathbf{R}^{2n+1}$ and considers the Euclidean Fourier transform of P.V.(G). Details are left to the interested reader.

6. The main formula. In this section we derive an F. T. formula of considerable applicability (Theorem 6.2). It immediately gives the F. T. of many homogeneous distributions. Lemma 6.1, which contains the heart of the matter, is in part a special case of Theorem 6.2.

In what follows, if $k \in \mathbf{C}$ we write $f(z) = z^k$ to denote the principal branch of this function, defined in the complex plane with the negative real axis removed. Thus if z_1, z_2 are in the open right half plane,

$$z_1^k z_2^k = (z_1 z_2)^k.$$

Also note that $z^k/|z|^{\text{Re}k}$ and $|z|^{\text{Re}k}/z^k$ are bounded functions of z.

If $u = (t, z) \in \mathbf{H}^n$, we set

$$h(u) = |z|^2 - it.$$

ψ will be as before Theorem 2.3.

LEMMA 6.1. *Suppose* $\delta > 0$, $s \in \mathbf{C}$, $|s| < 1 - \delta$, $\epsilon > 0$, $P \in \mathscr{H}_{pq}$, $\eta = \pm 1$. *Let* $H_\eta : \mathbf{R} \to \mathbf{R}$ *be the characteristic function of* $[0, \infty)$ *if* $\eta = 1$, *and of* $(-\infty, 0]$ *if* $\eta = -1$. *Suppose* $j \in \mathbf{C}$, $n + (\kappa/2) + 1 > \text{Re}\, j$, *and* $\underset{\sim}{J}_\epsilon^{\eta,s} = \underset{\sim}{J}_\epsilon \in \mathscr{R}$ *satisfies*

$$J_\epsilon(\lambda)E_\alpha = c_\epsilon(|\alpha|, \lambda)\, |\lambda|^{-j}\mathscr{W}_\lambda(P)E_\alpha$$

where if $N \geqq p'$,

$$c_\epsilon(N, \lambda) = (-1)^q \pi^{n+1} 2^{1-n-\kappa}\Gamma(n + \kappa + 1 - j)^{-1} \times s^{N-p'}(1 - s)^{j-1} \exp(-(\psi^{-1}s)\, |\lambda|\epsilon) H_\eta(\lambda).$$

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(Again, $0^0 = 1$.) *Let* $k = n + \kappa + 1 - j$. *Then* $\underset{\sim}{J}_\epsilon \in \mathscr{R}_1^E$,

$$\|\underset{\sim}{J}_\epsilon\|_1^E \leqq C \text{ independent of } s \text{ for } |s| < 1 - \delta.$$

Further $c_n' \check{\underset{\sim}{J}}_\epsilon = K_\epsilon$ *where*

$$K_\epsilon(u) = G_\epsilon(w(u))P(z) \text{ and}$$

$$G_\epsilon(w) = (s(w + \epsilon) + (\bar{w} + \epsilon))^{-k} \text{ if } \eta = 1,$$

$$G_\epsilon(w) = (s(\bar{w} + \epsilon) + (w + \epsilon))^{-k} \text{ if } \eta = -1.$$

(Observe that if $|s| < 1 - \delta$ *then* $s(w + \epsilon) + (\bar{w} + \epsilon)$ *never lies on the negative real axis if* $w = w(u)$, $u \in \mathbf{H}^n$.) *Thus* $\hat{K}_\epsilon = \underset{\sim}{J}_\epsilon$ *in the* $\mathscr{S}'$ *sense, by Proposition* 5.5.

Proof. We may assume $\eta = 1$. We note the estimate

(6.1) $\quad \|\mathscr{W}(P)E_\alpha\| < C(\nu_\alpha|\lambda|)^{\kappa/2}$

C independent of α, λ, directly from Proposition 2.7. (Corollary 3.3 gives a much sharper estimate.) Now

$$\|J_\epsilon(\lambda)\|_1^E \leqq Ce^{-\lambda\epsilon/C}|\lambda|^{-\mathrm{Re}\,j+\kappa/2} \sum_{N=p'}^{\infty} |s|^{N-p'} N^{\kappa/2} \binom{N+n-1}{n-1},$$

C independent of s, for $|s| < 1 - \delta$. The sum is bounded by

$$C_1 \sum_{M=0}^{\infty} |s|^M M^{n-1+\kappa/2} \leqq C_2,$$

C_2 independent of s, for $|s| < 1 - \delta$. So

$$\|\underset{\sim}{J}_\epsilon\|_1^E \leqq C_3 \int_0^\infty \lambda^r e^{-\lambda\epsilon/C} d\lambda$$

where

$$r = n - \mathrm{Re}\,j + \kappa/2 > -1.$$

Thus $\|J_\epsilon\|_1^E < C_4$ as desired. Finally, if $u = (t, z)$, in the notation of Theorem 1.2 we have

$$c_n' \check{J}_\epsilon(u) = c_n' \int_{-\infty}^{\infty} e^{i\lambda t} \mathscr{T}_\lambda(J_\epsilon(\lambda))|\lambda|^n d\lambda$$

$$= \Gamma(k)^{-1}(1 - s)^{-k} P(z) \int_0^\infty \exp[-\lambda((\psi^{-1}(s))(|z|^2 + \epsilon)$$
$$+ it)]\lambda^{k-1} d\lambda$$
$$= (1 - s)^{-k}[\psi^{-1}(s)(|z|^2 + \epsilon) + it]^{-k} P(z)$$
$$= (s(w + \epsilon) + (\bar{w} + \epsilon))^{-k} P(z),$$

as desired. We used Theorem 2.3, and its notation. To deform the contour, we noted that

$$[\psi^{-1}(s)(|z|^2 + \epsilon) + it]$$

is in the open right half plane.

THEOREM 6.2. *Suppose $k \in \mathbf{C}$, $P \in \mathscr{H}_{pq}$. Suppose μ is a complex measure on $\mathbf{R}^+$, and for some δ, $0 < \delta < 1$, there is a smooth function g on*

$$I_\delta = [1 - \delta, (1 - \delta)^{-1}]$$

such that $d\mu(s) = g(s)ds$ on I_δ. Let μ_1 denote μ restricted to [0, 1], *and on* [0, 1] *assume there is a measure $\mu_{(-1)}$ with*

$$d\mu_{(-1)}(s) = -s^k d\mu(1/s) \text{ for } s \in [0, 1].$$

We assume that μ_1, $\mu_{(-1)}$ are finite measures.

Suppose $\epsilon > 0$. For each $u = (t, z) \in \mathbf{H}^n$, set

$$K_\epsilon(u) = G_\epsilon(w(u))P(z) \tag{6.2}$$

where

$$G_\epsilon(w) = \int_0^\infty (s(w + \epsilon) + (\bar{w} + \epsilon))^{-k} d\mu(s) \tag{6.3}$$

$$= \int_0^1 (s(w + \epsilon) + (\bar{w} + \epsilon))^{-k} d\mu_1(s)$$

$$+ \int_0^1 (s(\bar{w} + \epsilon) + (w + \epsilon))^{-k} d\mu_{(-1)}(s). \tag{6.4}$$

(*Note that if* $0 \leqq s \leqq 1$, $|s(w + \epsilon) + (\bar{w} + \epsilon)| \geqq \epsilon$, *so the integrals are well defined.*)

Next, suppose $\epsilon \geqq 0$, $j \in \mathbf{C}$ and $n + (\kappa/2) + 1 >$ Re $j > 0$. Define $\underset{\sim}{J_\epsilon} \in \mathscr{R}$ by

$$J_\epsilon(\lambda)E_\alpha = (-1)^q \pi^{n+1} 2^{1-n-\kappa} c_\epsilon(|\alpha|, \lambda) \mathscr{W}_\lambda(P) E_\alpha \tag{6.5}$$

where, if $N \geqq p'$,

$$c_\epsilon(N, \lambda) = |\lambda|^{-j}\Gamma(n + \kappa + 1 - j)^{-1} \int_0^1 s^{N-p'}(1 - s)^{j-1} \times \exp(-(\psi^{-1}(s))|\lambda|\epsilon) d\mu_\sigma(s)$$

($\sigma = \operatorname{sgn} \lambda$). *Then*

(a) $K_0 = \lim_{\epsilon \to 0} K_\epsilon$ *exists in* $C^\infty(\mathbf{H}^n - \{0\})$.

(b) *Suppose*

$$\text{Re } k < n + 1 + (\kappa/2) \tag{6.7}$$

https://doi.org/10.4153/CJM-1984-039-0 Published online by Cambridge University Press

or

(6.8) $k = n + 1 + (\kappa/2), \kappa \geqq 1.$

If (6.7) *holds, then* $K = K_0$ *is an* r. h. d. *on* $\mathbf{H}^n$. *If* (6.8) *holds,* $K = PV(K_0)$ *is an* r. h. d. *on* $\mathbf{H}^n$.

(c) $K = \lim_{\epsilon \to 0} K_\epsilon$ *in* $\mathscr{S}'(\mathbf{H}^n)$.

(d) $\underset{\sim}{J}_\epsilon \in \mathrm{Ord}(\kappa - 2j)$ *and*

$$\lim_{\epsilon \to 0} \underset{\sim}{J}_\epsilon = \underset{\sim}{J}_0 \text{ in } [\hat{\mathscr{S}}(\mathbf{H}^n)]^*.$$

(e) *If* $\epsilon > 0, j = n + \kappa + 1 - k$, *then* $\hat{K}_\epsilon = \underset{\sim}{J}_\epsilon$ *in the* $\mathscr{S}'$ *sense. If* $\epsilon = 0$ *and* (6.7) *or* (6.8) *holds,* $\hat{K} = \underset{\sim}{J}_0$ *in the* $\mathscr{S}'$ *sense.*

Remark. The theorem is, on a formal level, an immediate consequence of Lemma 6.1. The proof involves a series of limiting processes. Before we give it, we derive the most important special cases.

If $\gamma \in \mathbf{C}$, $\mathrm{Re}\, \gamma > 0$, $\mathrm{Re}(k - \gamma) > 0$, $d\mu(s) = s^{\gamma - 1}ds$, then

$$d\mu_{(-1)}(s) = s^{k - \gamma - 1}ds.$$

If $\epsilon > 0$ and $v = w + \epsilon$ then

$$\begin{aligned} G_\epsilon(w) &= \int_0^\infty (sv + \bar{v})s^{\gamma - 1}ds \\ &= v^{-\gamma}\bar{v}^{-(k-\gamma)} \int_0^\infty [(v/\bar{v})s + 1]^{-k}[(v/\bar{v})s]^{\gamma - 1}(v/\bar{v})ds \\ &= v^{-\gamma}\bar{v}^{-(k-\gamma)} \int_0^\infty (s + 1)^{-k}s^{\gamma - 1}ds \\ &= \Gamma(\gamma)\Gamma(k - \gamma)\Gamma(k)^{-1}(w + \epsilon)^{-\gamma}(\bar{w} + \epsilon)^{-(k-\gamma)} \end{aligned}$$

so certainly (a) is true in this case. In addition (b) and (c) are apparent if (6.7) holds. In addition,

$$\begin{aligned} c_0(N, \lambda) = |\lambda|^{-j}\Gamma(n + \kappa + 1 - j)^{-1}\Gamma(N - p' + \gamma') \\ \times \Gamma(j)\Gamma(N - p' + \gamma' + j)^{-1} \end{aligned}$$

where $\gamma' = \gamma'(\lambda) = \gamma$ if $\lambda > 0$, $\gamma' = k - \gamma$ if $\lambda < 0$. We also note the more special case $P \equiv 1$, $k = n, j = 1$, in which

$$c_0(N, \lambda) = |\lambda|^{-1}\Gamma(n)^{-1}(N + \gamma')^{-1}.$$

Thus if $\varphi_\gamma = w^{-\gamma}\bar{w}^{-(n-\gamma)}$, (c) of the theorem implies

(6.9) $$\hat{\varphi}_\gamma E_\alpha = c_\gamma[(2N + 2\gamma')|\lambda|]^{-1}E_\alpha \quad \text{if } |\alpha| = N$$

where

$$c_\gamma = \pi^{n+1}2^{2-n}\Gamma(\gamma)^{-1}\Gamma(n - \gamma)^{-1}.$$

If $f \in \mathcal{S}(\mathbf{H}^n)$ then, setting $\eta = 2\gamma - n$,

$$(L_\eta f)^\wedge(\lambda)E_\alpha = (2N + 2\gamma')|\lambda|\hat{f}(\lambda)E_\alpha \quad \text{if } |\alpha| = N.$$

It follows easily that

(6.10) $\quad L_\eta(\varphi_\gamma) = c_\gamma\delta$

which is Theorem 6.2 of [**4**] if $0 < \text{Re}\,\gamma < n$. (The volume element of [**4**] is 2^{-n} times ours; see 5.1 of [**4**].) To extend (6.10) and hence (6.9) to general $\gamma \in \mathbf{C}$, simply observe that if $f \in \mathcal{S}(\mathbf{H}^n)$, both $(L_{2\bar{\gamma}-n}\bar{f}|\varphi_\gamma)$ and $c_\gamma f(0)$ are analytic functions of γ, so they coincide.

Proof of Theorem 6.2. For (a), if w lies in the open right half plane we may use (6.3) with $\epsilon = 0$ to define $G_0(w)$ since then $sw + \bar{w}$ will also lie in the open right half plane if $0 \leqq s \leqq 1$. Let us first show that G_0 can be smoothly extended to

$$D = \{w \in \mathbf{C}|\ \text{Re}\, w \geqq 0\} - \{0\}.$$

G_0 is evidently C^∞ where it is defined. It suffices to prove that if $w_0 \in D$ has the form $w_0 = (0, t)$, $t \neq 0$ and if $M, N \in \mathbf{Z}^+$, then

$$\lim(\partial/\partial\bar{w})^M(\partial/\partial\ \bar{w})^N G_0$$

exists as $w \to w_0$ through int D. For, by the homogeneity of G_0, it will then be clear that these limits will exist uniformly at $(0, t')$ for t' near t, and the smoothness of G_0 will follow readily. Using (6.3), we reduce at once to the case $M = N = 0$ by allowing k to vary. We may assume Re $k > 0$, since otherwise the conclusion is immediate. Let $\ell = [\text{Re}\, k]$ ([ ] = greatest integer function). Let $c = 0$ if $\ell \neq \text{Re}\, k$; otherwise, let $c = 1/2$. Write

$$s^c g(s) = p(s) + v(s) \text{ in } I_\delta$$

where p is a polynomial of degree not more than $\ell - 1$ and

$$v(s) = (1 - s)^\ell v_0(s) \text{ where } v_0 \text{ is } C^\infty \text{ in } I_\delta.$$

Let

$$d\mu'(s) = s^{-c}p(s)ds;$$

$$d\mu''(s) = s^{-c}(1-s)^\ell v_0(s)ds \text{ if } s \in I_\delta,$$

$$d\mu''(s) = 0 \text{ otherwise};$$

$$d\mu''' = d\mu - d\mu' - d\mu''.$$

We verify the conclusion with each of $d\mu'$, $d\mu''$, $d\mu'''$ in place of $d\mu$. For $d\mu'$, use the special case we computed in the Remark. For $d\mu'''$, if

$$F_1(w) = \int_0^1 [s + (\bar{w}/w)]^{-k} d\mu'''(s),$$

then $F_1(w)$ converges absolutely and uniformly for $w \in D$ since

$$|s + \zeta| \geqq \delta \text{ if } 0 \leqq s \leqq 1 - \delta, |\zeta| = 1.$$

Thus $d\mu'''$ is easily dealt with. For $d\mu''$, we need only examine

$$\int_{1-\delta}^{(1-\delta)^{-1}} [s + (\bar{w}/w)]^{-k} s^{-c}(1 - s)^f v_0(s) ds.$$

Now

$$|[s + \bar{w}/w]^{-k}| < C|s + \bar{w}/w|^{-\text{Re}k} \leqq C|s - 1|^{-\text{Re}k}$$

so that the integral converges absolutely and uniformly for $w \in D$, and the desired limit exists.

Thus G_0 has a smooth extension, which we also call G_0, and $G_0(w(u))$ is smooth for $u \in \mathbf{H}^n \setminus \{0\}$. As $\epsilon \to 0$,

$$G_\epsilon(w) = G_0(w + \epsilon) \to G_0(w) \text{ in } C^\infty(D).$$

(a) therefore follows. For (b), if (6.7) holds, note that

$$K_0(r^2 t, rz) = r^{-2k+\kappa} K_0(t, z),$$

first if $z \neq 0$, then if $(t, z) \neq 0$ by the continuity of K_0. If instead (6.8) holds, use the same fact together with Proposition 5.6. For (c), we reduce again to using $d\mu'$, $d\mu''$, $d\mu'''$ in place of $d\mu$. The above analysis showed that in each case $|G_0(w + \epsilon)|$ is bounded by

$$C|w + \epsilon|^{-\text{Re}k} \leqq C|w|^{-\text{Re}k}.$$

Thus, if (6.7) holds, the dominated convergence theorem may be applied, and (c) holds. If (6.8) holds, and $f \in \mathcal{S}$, by Proposition 5.6

$$(K_\epsilon|f) = \int_{|u| \leqq 1} \overline{K_\epsilon(u)}[f(u) - f(0)] dV(u) + \int_{|u|>1} \overline{K_\epsilon(u)} f(u) dV(u).$$

Again

$$|G_0(w + \epsilon)| < C|w|^{-\text{Re}k}$$

independent of ϵ so that the dominated convergence theorem still applies. This concludes the proof of (c).

For the first conclusion of (d), we can split up $d\mu_\sigma$ into two parts and study two cases: (i) $d\mu_\sigma(s)$ vanishes if $1 - \delta < s < 1$.

(ii) $d\mu_\sigma(s) = f(s)ds$ where $f \in L^\infty(0, 1)$.

In case (i), we have

$$|c_\epsilon(N, \lambda)| < C_1(1 - \delta)^N |\lambda|^{-j}$$

where C_1 is independent of λ, N or ϵ. In case (ii)

$$|c_\epsilon(N, \lambda)| \leqq C_2|\lambda|^{-\mathrm{Re} j} \|f\|_\infty \int_0^1 s^{N-p'}(1 - s)^{\mathrm{Re} j - 1} ds$$

$$= C_2|\lambda|^{-\mathrm{Re} j}\|f\|_\infty \Gamma(N - p' + 1)\Gamma(\mathrm{Re}\, j)\Gamma(N - p' + \mathrm{Re}\, j + 1)^{-1}$$

so

$$(6.11) \quad |c_\epsilon(N, \lambda)| \leqq C_3\|f\|_\infty(N_+)^{-\mathrm{Re} j}.$$

Here $N_+ = N + (n/2)$ and C_2, C_3 are independent of λ, N, ϵ or f. So in both cases

$$|c_\epsilon(N, \lambda)| \leqq C(N_+|\lambda|)^{-\mathrm{Re} j},$$

whence, by (6.1),

$$\|J_\epsilon(\lambda)E_\alpha\| \leqq C(N_+|\lambda|)^{\kappa/2 - \mathrm{Re} j},$$

where C is independent of λ, N, ϵ, as desired. For the second conclusion, it suffices to use this last fact, the fact that

$$J_\epsilon(\lambda)E_\alpha \to J(\lambda)E_\alpha$$

for each α, λ, and Proposition 5.4.

For (e), we may assume by the results of (a), (b), (c) and (d) that $\epsilon > 0$. We fix ϵ and drop it. For each $l \in \mathbf{Z}^+$ with $1/l < \delta$, define $\varphi_l \in L^\infty(\mathbf{R})$ by

$$\varphi_l(x) = 1 - \chi_l(x)$$

where χ_l is the characteristic function of $(1 - 1/l, (1 - 1/l)^{-1})$. Let

$$d\mu_l = \varphi_l d\mu$$

and let K^l, $\underset{\sim}{J}^l$ be obtained from $d\mu^l$ in the same way that K, $\underset{\sim}{J}$, were obtained from $d\mu$. As $l \to \infty$, $K^l \to K$ pointwise and

$$J^l(\lambda)E_\alpha \to J(\lambda)E_\alpha \text{ for each } \alpha, \lambda;$$

we claim $K^l \to K$ in $\mathscr{S}'$, $\underset{\sim}{J}^l \to \underset{\sim}{J}$ in $[\hat{\mathscr{S}}]^*$. For, from (6.3),

$$\|K^l\|_\infty \leqq \epsilon^{-\mathrm{Re} k}(\|\mu_1^l\| + \|\mu_2^l\|)$$
$$\leqq \epsilon^{-\mathrm{Re} k}(\|\mu_1\| + \|\mu_2\|),$$

so dominated convergence shows $K^l \to K$ in $\mathscr{S}'$. To show $J^l \to J$, we reduce to case (ii) of the proof of (b) and use the estimate (6.11) and Proposition 5.4.

Thus, to prove (e) we may assume that $\epsilon > 0$, $g \equiv 0$ (changing δ if necessary). At this point our only restrictions on j and k will be

$$\operatorname{Re} j < n + 1 + (\kappa/2).$$

Notation as in Lemma 6.1, we have

$$J_\epsilon(\lambda) = \int_0^{1-\delta} J_\epsilon^{1,s}(\lambda) d\mu_1(s) + \int_0^{1-\delta} J_\epsilon^{-1,s}(\lambda) d\mu_{(-1)}(s)$$

and we know

$$\| \underset{\sim}{J}_\epsilon^{n,s} \|_1^E \leqq C$$

independent of $s \in [0, 1 - \delta]$; so

$$\underset{\sim}{J} \in \mathscr{R}_1^E \quad \text{and} \quad c_n \hat{\underset{\sim}{J}} = K.$$

By Proposition 5.5, the proof is concluded.

For a different method of proving (a), see the proof of Lemma 7.5.

The interesting case $k = n + 1$, $P \equiv 1$ is not covered by Theorem 6.2, and we treat it separately now.

THEOREM 6.3. *Hypotheses as in Theorem* 6.2, *up to and including equation* (6.4), *but now assuming* $k = n + 1$, $P \equiv 1$. *Then*:

(a) K_0 *has mean value* 0 *if and only if* $g(1) = 0$.

Suppose K_0 *has mean value* 0. *Let* $K = \text{P.V.}(K_0)$. *Then*

(b) *With* $\epsilon = 0$, $j = 0$, *the integral in* (6.6) *is absolutely convergent. We may therefore define* $\underset{\sim}{J}_0$ *by* (6.5). *Then* $\underset{\sim}{J}_0 \in \text{Ord}(0)$.

(c) *There exists* $c \in \mathbf{C}$ *such that* $(K + c\delta)^\wedge = \underset{\sim}{J}_0$.

Proof. For (a), first suppose $g(1) = 0$. Define a measure β on $\mathbf{R}^+$ by

$$d\beta(s) = -[ni(1 - s)]^{-1} d\mu(s).$$

Because g is smooth, β satisfies all the same conditions that μ does. We may therefore put

$$H_\epsilon(w) = \int_0^\infty (s(w + \epsilon) + (\bar{w} + \epsilon))^{-n} d\beta(s).$$

Let $K'_\epsilon(u) = H_\epsilon(w(u))$. The analogue of (6.4) for H_ϵ shows that $TK'_\epsilon = K_\epsilon$. Since $K'_\epsilon \to K'_0$ in $\mathscr{S}'$, $TK'_\epsilon \to TK'_0$ in $\mathscr{S}'$; but $TK'_\epsilon = K_\epsilon \to K_0$ in $C^\infty(\mathbf{H}^n - \{0\})$ by Theorem 6.2 (a). Thus $K_0 = TK'_0$ away from 0. TK'_0 is an r. h. d. of degree $-2n - 2$, so that K_0 must have mean value zero. (Note for later purposes, then, that $TK'_0 = K + c\delta$ for some $c \in \mathbf{R}$.)

Suppose next $g(1) = a \neq 0$. If $d\mu(s)$ were equal to $as^{[(n+1)/2-1]}ds$, the computation in the remark would show that

$$K_0(u) = ab|u|^{-(2n+2)}$$

where

$$b = [\Gamma((n + 1)/2)]^2/\Gamma(n + 1).$$

This does not have mean value zero. If $d\mu$ is general, consider

$$K_0(u) - ab|u|^{-(2n+2)}$$
$$= \lim_{\epsilon \to 0} \int_0^\infty [s(w(u) + \epsilon) + (\bar{w}(u) + \epsilon)]^{-n-1}$$
$$\times [d\mu(s) - as^{[(n+1)/2-1]}ds].$$

By the first part of the proof, $K_0 - ab|u|^{-(2n+2)}$ does have mean value zero; consequently, K_0 does not. This proves (a). If K_0 has mean value 0, we return to the notation of the first paragraph. It is easy to show, from Theorem 6.2, that $\hat{K}'_0 = \underset{\sim}{J}'$ where

$$J'(\lambda) = (-i\lambda)^{-1} J_0(\lambda),$$

$\underset{\sim}{J}_0$ as in (b). Thus $(TK'_0)^\wedge = \underset{\sim}{J}_0$, and (c) follows. Further we conclude

$$\underset{\sim}{J}' \in \mathrm{Ord}(-2),$$

so it follows easily that

$$\underset{\sim}{J}_0 \in \mathrm{Ord}(0);$$

thus (b) also follows.

Because of the arguments in the preceding proof, and also the considerations mentioned at the end of Section 5, the following question is of some interest. Suppose K' is an r. h. d. of degree $-2n$; $K'(t, z) = G(t, |z|^2)$ for some G; $TK' = K_0$ away from 0, where $K_0 \in C^\infty(\mathbf{H}^n)\setminus\{0\}$; and $K = \mathrm{P.V.}(K_0)$. Then certainly $TK' = K + c\delta$ for some $c \in \mathbf{C}$. One would like a simple means of determining c; in particular, this would make Theorem 6.3 (c) more explicit.

For this purpose, we introduce "polar coordinates" on the Heisenberg group. Suppose $f(t, z) \in L^1(\mathbf{H}^n)$. Define ρ, ξ by $\rho = |z|^2$, $\xi = z/|z|$ and suppose $f(t, z) = g(t, \rho, \xi)$. Then

$$\int_{\mathbf{H}^n} f dV = (1/2) \int_{S_n} \int_{\mathbf{R}} \int_0^\infty g(t, \rho, \xi)\rho^{n-1} d\rho dt d\xi.$$

We now let

$$R = (\rho^2 + t^2)^{1/2} = (|z|^4 + t^2)^{1/2},$$
$$\varphi = \arctan(t/\rho)$$

so that

$$|z|^2 + it = \rho + it = \mathrm{Re}^{i\varphi}.$$

Suppose $(t, \rho, \xi) = F(R, \varphi, \xi)$; then

$$\int_{\mathbf{H}^n} f dV = (1/2) \int_{S_n} \int_{-\pi/2}^{\pi/2} \int_0^\infty F(R, \varphi, \xi)$$

$$\times (\cos \varphi)^{n-1} R^n dR d\varphi d\xi.$$

We call (R, φ, ξ) polar coordinates on $\mathbf{H}^n$. We note that, in polar coordinates,

$$T = (\sin \varphi)\partial/\partial R + [(\cos \varphi)/R]\partial/\partial\varphi,$$

as one sees at once from the change of variables $(t, \rho, \xi) \to (R, \varphi, \xi)$.

In the problem under consideration, we can write $K' = R^{-n}h(\varphi)$ for some bounded function h. We wish to determine c such that

$$K(F) = TK'(F) - cF(0) \text{ for all } F \in \mathscr{S}(\mathbf{H}^n).$$

We may assume

$$\operatorname{supp} F \subset \{R < 1\}.$$

Writing $F = F(R, \varphi, \xi)$, we have

$$2K(F) = \int_{S_n} \int_{-\pi/2}^{\pi/2} \int_0^1 \sin \varphi[\partial/\partial R(h(\varphi)R^{-n})] \times [F(R, \varphi, \xi) - F(0)]R^n dR \cos^{n-1}\varphi d\varphi d\xi$$
$$+ \int_{S_n} \int_0^1 \int_{-\pi/2}^{\pi/2} \cos \varphi[\partial/\partial\varphi(h(\varphi)R^{-n})] \times [F(R, \varphi, \xi) - F(0)] \cos^{n-1}\varphi d\varphi R^{n-1} dR d\xi.$$

In these inner integrals we integrate by parts, to find

$$2K(F) = -2K'(TF) - \int_{S_n} \int_{-\pi/2}^{\pi/2} F(0)h(\varphi) \sin \varphi \cos^{n-1} \varphi d\varphi d\xi.$$

Accordingly, $K(F) = TK'(F) - cF(0)$ where

$$c = \pi^n(n-1)!^{-1} \int_{-\pi/2}^{\pi/2} h(\varphi) \sin \varphi \cos^{n-1} \varphi d\varphi,$$

and $TK' = K + c\delta$.

7. Applications. In the remark of Theorem 6.2, we gave the most important cases. Let us begin by giving some very simple extensions of these. Specifically, suppose the r. h. d. $K = \bar{w}^a w^b P(z)$ where $a, b \in \mathbf{C}$ and

$$-2n - 2 < \operatorname{Re}(\deg K) < 0.$$

We compute $\hat{K}$.

PROPOSITION 7.1. *Suppose* $k \in \mathbf{C}$, $P \in \mathscr{H}_{pq}$, $\kappa = p + q$. *Suppose* γ *and* $k - \gamma$ *are not nonnegative integers, and that*

$$-2n - 2 < \kappa - 2 \operatorname{Re} k < 0,$$

or that

$$\kappa \geqq 1, \kappa - 2k = -2n - 2.$$

Define

$$G_{k\gamma}(w) = \Gamma(\gamma)\Gamma(k - \gamma)\bar{w}^{\gamma - k}w^{-\gamma}, \text{ and}$$

$$K_{k\gamma P}(u) = G_{k\gamma}(w(u))P(z).$$

Let $j = n + \kappa + 1 - k$. *Then* $K_{k\gamma P} = \underset{\sim}{J}_{j\gamma P}$, *an element of* $\text{Ord}(\kappa - 2j)$, *defined as follows.*

$$J_{j\gamma P}(\lambda)E_\alpha = (-1)^q \pi^{n+1} 2^{1-n-\kappa} c_{j\gamma}(|\alpha|, \lambda)\mathscr{W}_\lambda(P)E_\alpha,$$

where if $M = N - p' \geqq 0$, $\gamma' = \gamma$ *if* $\lambda > 0$ *and* $\gamma' = k - \gamma$ *if* $\lambda < 0$, *then*

$$c_{j\gamma}(N, \lambda) = |\lambda|^{-j}\Gamma(M + \gamma')\Gamma(j)\Gamma(M + \gamma' + j)^{-1}.$$

Proof. This has already been verified in the remark and in the proof of Theorem 6.2 if $\text{Re}\,\gamma > 0$, $\text{Re}(k - \gamma) > 0$. It follows in general by analytic continuation. Indeed, for fixed k, P, let

$$S = \{\gamma | \gamma, k - \gamma \notin \mathbf{Z}^-\}.$$

It suffices to show that if $f \in \mathscr{S}$, $(f|K_{k\gamma P})$ and $(f|J_{j\gamma P})$ are analytic functions of $\gamma \in S$. Indeed, $\text{Re}\,k > 0$, so that there does exist an open set of γ in S with $\text{Re}\,\gamma > 0$, $\text{Re}(k - \gamma) > 0$. That $(f|K_{k\gamma P})$ is analytic is immediate. Now

$$\kappa - 2\,\text{Re}\,j = -2n - 2 + 2\,\text{Re}\,k - \kappa > -2n - 2.$$

Thus, by Proposition 5.3 (b),

$$|(f|J_{j\gamma P})| < C\|J_{j\gamma P}\|$$

where the norm is taken in $\text{Ord}(\kappa - 2j)$. It suffices then to show that $\|J_{j\gamma P}\|$ remains bounded if γ varies through compact subsets of S. But this is easy. Note first that if $a, b \in \mathbf{C}$, $a' = \text{Re}\,a > 0$, $b' = \text{Re}\,b > 0$, then

$$\begin{aligned} |\Gamma(a)\Gamma(b)/\Gamma(a + b)| &= \left|\int_0^1 t^{a-1}(1 - t)^{b-1}dt\right| \\ &\leqq \Gamma(a')\Gamma(b')/\Gamma(a' + b'). \end{aligned}$$

So if $M = N - p' > -\text{Re}\,\gamma'$,

$$(7.1) \qquad |c_{j\gamma}(N, \lambda)| \leqq c_{\text{Re}j,\text{Re}\gamma}(N, \lambda) \leqq C(N_+|\lambda|)^{-\text{Re}j}.$$

Here $N_+ = N + n/2$. This estimate then holds for all N. If we recall (6.1), the proof is concluded.

The formula has a simple generalization to the case where γ or $k - \gamma$ is a negative integer.

PROPOSITION 7.2. *Suppose* $k \in \mathbf{C}$, $P \in \mathscr{H}_{pq}$, $p + q = \kappa$. *Suppose*

$$-2n - 2 < \kappa - 2 \operatorname{Re} k < 0$$

or that

$$\kappa \geqq 1, \kappa - 2 \operatorname{Re} k = -2n - 2.$$

Let $j = n + \kappa + 1 - k$. *Suppose* (i) $\gamma = -l \in \mathbf{Z}^-$ *or* (ii) $k - \gamma = -l \in \mathbf{Z}^-$. (*Both cannot occur simultaneously, since* $\operatorname{Re} k > 0$.) *Define*

$$G_{k\gamma}(w) = (-1)^l \Gamma(k + l) l!^{-1} \overline{w}^{\gamma - k} w^{-\gamma}.$$

Let

$$K_{k\gamma P}(u) = G_{k\gamma}(w(u))P(z).$$

Then $\hat{K}_{k\gamma P}, = \underset{\sim}{J}_{j\gamma P}$, *an element of* $\operatorname{Ord}(\kappa - 2j)$, *defined as follows.*

$$J_{j\gamma P}(\lambda)E_\alpha = (-1)^q \pi^{n+1} 2^{1-n-\kappa} c_{j\gamma}(|\alpha|, \lambda) \mathscr{W}_\lambda(P) E_\alpha.$$

Here, if $M = N - p' \geqq 0$, $M \leqq l$, *then*

$$c_{j\gamma}(N, \lambda) = |\lambda|^{-j}(-1)^{l-M}\Gamma(j)[\Gamma(M + j - l)(l - M)!]^{-1},$$

provided $\lambda > 0$ *in case* (i) *and provided* $\lambda < 0$ *in case* (ii). *For all other values of* N, λ, $c_{j\gamma}(N, \lambda) = 0$.

Proof. Say we are in case (i); case (ii) is handled similarly, or through use of the symmetry of the F. T. under $(t, z, \lambda) \to (-t, -\bar{z}, -\lambda)$. We need only show that

$$\lim_{\epsilon \to 0} \epsilon K_{k,\gamma+\epsilon,P} \to K_{k\gamma P} \text{ in } \mathscr{S}' \quad \text{and}$$

$$\lim_{\epsilon \to 0} J_{j,\gamma+\epsilon,P} = J_{j\gamma P} \quad \text{in } \operatorname{Ord}(\kappa - 2j).$$

(Here $K_{k,\gamma+\epsilon,P}$ and $J_{j,\gamma+\epsilon,P}$ are as in Proposition 7.1.) The first is easy by dominated convergence, since if $\gamma = -l$,

$$\lim_{\epsilon \to = 0} \epsilon\Gamma(\gamma + \epsilon) = (-1)^l l\,!^{-1}.$$

For the second, note that if $M > l$, or $\lambda < 0$,

$$\epsilon c_{j,\gamma+\epsilon}(N, \lambda) \to 0$$

trivially. If $M \leqq l$,

$$\lim_{\epsilon \to 0} \epsilon\Gamma(M + \gamma + \epsilon) = (-1)^{l-M}(l - M)!^{-1},$$

so

https://doi.org/10.4153/CJM-1984-039-0 Published online by Cambridge University Press

$$\epsilon c_{j,\gamma+\epsilon}(N, \lambda) \to c_{j,\gamma}(N, \lambda)$$

for all λ. We still have the bound (7.1) for $M > l$ and $\lambda > 0$, or for $\lambda < 0$, with $\gamma + \epsilon$ replacing γ. By Proposition 5.4, we are done.

We shall not give explicitly any analogous formulae in the case $P \equiv 1$, $-2 \operatorname{Re} k = -2n - 2$, since by the proof of Theorem 6.3 the only examples can be read off by considering TK', K' homogeneous of degree $-2n$. One case of this, however, is quite important; the Cauchy-Szegö kernel. We discuss this, and its application to H^2 theory, now.

Let

$$\mathbf{U}^{n+1} = \{ [z_o, z] \in \mathbf{C} \times \mathbf{C}^n | h = \operatorname{Im} z_0 - |z|^2 > 0\}.$$

This is the Siegel upper half space of type II. Frequently we use instead the coordinates (h, u) where $u = (t, z)$, $t = \operatorname{Re} z_0$. In these coordinates, one thinks of $\mathbf{U}^{n+1}$ as $\mathbf{R}^+ \times \mathbf{H}^n$. The reason: if $u \in \mathbf{H}^n$, the "left translation" $T_u:\mathbf{U}^{n+1} \to \mathbf{U}^{n+1}$ by $T_u(h, v) = (h, uv)$ is then easily seen to be a holomorphic homeomorphism of $\mathbf{U}^{n+1}$. Thus one thinks of $\mathbf{H}^n$ as $\partial\mathbf{U}^{n+1}$. We write $H = \partial/\partial h$. We record the transformation law from $[z_0, z]$ to (h, u) coordinates:

$$T - iH = 2\partial/\partial z_0, \quad Z_j = \partial/\partial z_j + 2i\bar{z}_j\partial/\partial z_0.$$

Thus a function f on $\mathbf{U}^{n+1}$ is holomorphic if and only if

$$(7.2) \qquad (T + iH)f = \bar{Z}_j f = 0, \quad \text{for all } j.$$

In particular, if f is analytic, then Tf is analytic. $\mathbf{U}^{n+1}$ is biholomorphic to the unit ball under a "Cayley transform." (See [**6**] for this and further information, and [**16**] for H^2 theory for the unit ball.) If F is a function on $\mathbf{U}^{n+1}$, we define, for $h > 0$, the function F_h on $\mathbf{H}^n$ by $F_h(u) = F(h, u)$. We let

$$H^2(\mathbf{U}^{n+1}) = \{F \text{ holomorphic on } \mathbf{U}^{n+1} \Big| \sup_{h>0} \|F_h\|_2 < \infty\}.$$

Here $\| \|_2$ denotes $L^2(\mathbf{H}^n)$ norm. Define $\underset{\sim}{P} \in \mathscr{R}$ by $P(\lambda)E_\alpha = 0$ unless $\alpha = 0$, $\lambda < 0$, in which case $P(\lambda)E_0 = E_0$. Let

$$V = \{f \in L^2(\mathbf{H}^n) \,|\, \hat{f}\underset{\sim}{P} = \hat{f}\},$$

topologized as a subspace of L^2. The following result is then known; see [**13**]. We give the natural proof in our context.

PROPOSITION 7.3. (a) *If* $F \in H^2(\mathbf{U}^{n+1})$, *there exists* $F_0 \in V$ *such that* $F_h \to F_0(L^2)$ *as* $h \to 0$. *The map* $B:H^2(\mathbf{U}^{n+1}) \to V$ *defined by* $BF = F_0$ *is an isomorphism.*

(b) *The projection* $C_b:L^2 \to V$ *is given by*

$$C_b f = c_n'(\hat{f}\underset{\sim}{P})^\vee.$$

If $f \in L^2$, *let*

$$C_b^h f = (B^{-1} C_b f)_h.$$

Then

$$C_b^h f = c_n'(\underset{\sim}{E}_h \hat{f} \underset{\sim}{P})^{\vee} \quad \text{where } E_h(\lambda) = e^{-|\lambda| h} I.$$

(c) *Let*

$$S(z_0, z) = n! 2^{n-1} \pi^{-(n+1)} (-iz_0)^{-(n+1)},$$

$$K'(u) = -i(n-1)! 2^{n-1} \pi^{-(n+1)} [w(u)]^{-n},$$

$$K(u) = n! 2^{n-1} \pi^{-(n+1)} w^{-(n+1)} \quad \text{for } u \in \mathbf{H}^n \setminus \{0\}.$$

Let $S_0 = TK'$ *in the* $\mathcal{S}'$ *sense. Then* $S_0 =$ P. V. $K + (1/2)\delta$. *Further, if* $f \in L^2(\mathbf{H}^n)$,

$$C_b^h f = f * S_h, \quad C_b f = f * S_0.$$

Proof. Suppose $F \in \mathbf{H}^2$. The condition $\bar{Z}_j F_h = 0$ holds in the sense of distributions, hence in the sense of tempered distributions since C_0^∞ is dense in $\mathcal{S}$. Accordingly

$$(\underset{\sim}{R} \underset{\sim}{W}_j | \underset{\sim}{\hat{F}}_h) = 0 \quad \text{for all } \underset{\sim}{R} \in \mathcal{Q}.$$

(See the beginning of Section 5 for $\mathcal{Q}$.) This easily gives $\hat{F}_h \underset{\sim}{P} = \hat{F}_h$. If we knew that F, $TF \in H^1$, and hence $HF \in H^1$, we could conclude by Fubini that $H[\hat{F}_h(\lambda)]$ existed and equalled $\lambda \hat{F}_h(\lambda)$, whence

$$e^{-|\lambda| h} \hat{F}_h(\lambda) = e^{-|\lambda| k} \hat{F}_k(\lambda) \quad \text{for all } h, k > 0. \tag{7.3}$$

To obtain (7.3) for general $F \in H^2$, regularize: select $N > n + 1$, $\varphi \in C_c^\infty(H^n)$, $\int \varphi = 1$. For $\epsilon, \eta > 0$, put

$$\psi_\eta(z_0, z) = i^N (\eta z_0 + i)^{-N},$$

$$\varphi_\epsilon = D^\epsilon \varphi$$

(a dilate of φ),

$$G^\eta = \psi_\eta F,$$

$$B^{\eta\epsilon} = \varphi_\epsilon * G^\eta.$$

Since $\psi_\eta \in H^2 \cap L^\infty$, $G^\eta \in H^1 \cap H^2$. Since $\varphi \in L^1$ and φ has compact support, $B^{\eta\epsilon} \in H^1 \cap H^2$, (7.2) being checked at once. Also

$$TB^{\eta\epsilon} = T\varphi_\epsilon * G^\eta \in H^1 \cap H^2,$$

so we have (7.3) for $B^{\eta,\epsilon}$ in place of F. Now

$$\hat{B}^{\eta,\epsilon} = \hat{\varphi}_\epsilon \hat{G}^\eta,$$

$$\hat{\varphi}_\epsilon(\lambda) = \hat{\varphi}(\epsilon^2 \lambda) \to I \quad \text{strongly as } \epsilon \to 0,$$

so that (7.3) holds for G^η in place of F. As $\eta \to 0$, $G_h^\eta \to F_h$ in L^2, so we find, by Plancherel, (7.3) for F for a.e. λ. So for some $\underset{\sim}{R} = (R(\lambda))$ with $\underset{\sim}{R}\underset{\sim}{P} = \underset{\sim}{R}$,

$$\hat{F}_h(\lambda) = e^{-|\lambda|h}R(\lambda).$$

By Plancherel and monotone convergence, $\underset{\sim}{R} \in \mathscr{R}_2$ so $\underset{\sim}{R} = \hat{F}_0$ for some $F_0 \in L^2$. By Plancherel, $F_h \to F_0(L^2)$ as $h \to 0$.

Conversely, given $\underset{\sim}{R} \in \mathscr{R}_2$ with $\underset{\sim}{R}\underset{\sim}{P} = \underset{\sim}{R}$, put $\underset{\sim}{R}_h = \underset{\sim}{E}_h\underset{\sim}{R}$. Then $\underset{\sim}{R}_h \in \mathscr{R}_1^E$. Define $F_0 = \check{\underset{\sim}{R}}$, $F_h = \check{\underset{\sim}{R}}_h$, $F(h, u) = F_h(u)$. To complete the proof of (a), it will suffice to show F is analytic on $\mathbf{U}^{n+1}$. Let us begin with the observation that if $f, g \in L^2$, then $(f * g)^\wedge = \hat{f}\hat{g}$ in the $\mathscr{S}'$ sense. Indeed, one easily sees that if $\varphi \in \mathscr{S}$,

$$(7.4) \qquad (\varphi | f * g) = (\tilde{f} * \varphi | g) = c_n'(\hat{\tilde{f}}\hat{\varphi}|\hat{g}) = c_n'(\hat{\varphi}|\hat{\tilde{f}}^* \hat{g}) = c_n'(\hat{\varphi}|\hat{f}\hat{g})$$

where $\tilde{f}(u) = \overline{f(u^{-1})}$. Next, we apply Lemma 6.1 with $P \equiv 1$, $\epsilon = h$, $j = 0$, $s = 0$, $\eta = -1$ to find $\check{E}_h = S_h$, (S is as in (c); note that $w(u) + h = -iz_0$ if $[z_0, z] = (h, u)$.) From these two facts,

$$F_h = F_0 * S_h.$$

(7.2) for F is now immediately checked. This proves (a). (b) follows also, the first part being a consequence of Plancherel.

For (c), the arguments of the last paragraph show that

$$C_b^h f = f * S_h \text{ for } f \in L^2.$$

By the Proposition 7.2 case (ii) with $k = \gamma = n$, $l = 0$, $P \equiv 1$, we have

$$c_n'[(-i\underset{\sim}{M})^{-1}\underset{\sim}{P}]^\vee = K'.$$

Thus

$$c_n'\check{\underset{\sim}{P}} = TK' = S_0$$

by the discussion at the end of Section 6. Indeed, one has only to note that

$$\int_{-\pi/2}^{\pi/2} e^{im\varphi} \sin\varphi \cos^{n-1}\varphi d\varphi = -[2^n i]^{-1}\pi.$$

This last formula is easily proved if one writes sin and cos as linear combinations of exponentials and notes

$$\int_{-\pi/2}^{\pi/2} e^{2ik\varphi}d\varphi = \pi\delta_{k0}, \quad \text{for } k \in \mathbf{Z}.$$

From this it follows easily that $C_b f = f * S_0$. Indeed, (7.4) with $f \in \mathscr{S}$, $g = S_0$ shows that

$$(f * S_0)^\wedge = f\underset{\sim}{P},$$

and this extends at once to $f \in L^2$. This completes the proof.

As another application of Theorem 6.2, we turn to the Poisson-Szegö kernel and its variants on $\mathbf{U}^{n+1}$. In [**6**] we explained that the Laplace-Beltrami operator on functions on $\mathbf{U}^{n+1}$ is

$$\Delta_{00} = 4h(hH^2 - nH + hT^2 - L_0).$$

We defined the variants

$$\Delta_{\alpha\beta} = \Delta - 4h[(\alpha + \beta)H + i(\alpha - \beta)T].$$

These operators occur as part of the Laplace Beltrami operator on forms. For $u \in \mathbf{H}^n$, we set

$$(7.5) \quad K^h_{\alpha\beta}(u) = c_{\alpha\beta}(2h)^{n+\alpha+\beta+1}(w(u) + h)^{-(n+\alpha+1)} \times (\bar{w}(u) + h)^{-(n+\beta+1)}$$

where

$$c_{\alpha\beta} = \pi^{-(n+1)}2^{n-1}\Gamma(n + \alpha + 1)\,\Gamma(n + \beta + 1) \times \Gamma(n + \alpha + \beta + 1)^{-1}.$$

Suppose $\operatorname{Re}(\alpha + \beta) > -(n + 1)$ and $n + \alpha + 1, n + \beta + 1 \notin \mathbf{Z}^-$. In [**6**] we asserted that

$$K^h_{\alpha\beta} \in L^1(\mathbf{H}^n) \text{ and } \int K^h_{\alpha\beta} = 1.$$

If $f \in L^p(\mathbf{H}^n)(1 \leqq p \leqq \infty)$ and $g(h, u) = f * K^h_{\alpha\beta}(u)$, we asserted that

$$\Delta_{\alpha\beta}g = 0.$$

Thus $K^h_{\alpha\beta}$ acts as a "Poisson kernel" for $\Delta_{\alpha\beta}$. (Since $K^h_{\alpha\beta} = D_{\bar{h}}K^1_{\alpha\beta}$, $K^h_{\alpha\beta} \to \delta$ as $h \to 0$.) We claimed that these assertions could be verified directly, but could be better motivated by use of the group Fourier transform. We do this now; for applications, see [**6**] and [**11**].

It is easy to see directly that

$$K^h_{\alpha\beta} \in L^q(\mathbf{H}^n) \text{ for } 1 \leqq q \leqq \infty,$$

and that this is also true for all of its derivatives. It suffices, then, to check that

$$\int K^h_{\alpha\beta} = 1 \text{ and } \Delta_{\alpha\beta}K_{\alpha\beta} = 0 \text{ if } K_{\alpha\beta}(h, u) = K^h_{\alpha\beta}(u).$$

Formally taking the F. T., we begin by seeking a solution of

$$(hH^2 - (n + \alpha + \beta)H - h\lambda^2 - (\alpha - \beta)\lambda)J(h, \lambda) - J(h, \lambda)A_\lambda = 0$$

where

$$J_h(\lambda) = J(h, \lambda) \in \mathcal{O}(\mathcal{H}_\lambda),$$

and where

$$J_h(\lambda) \to I \text{ strongly as } h \to 0.$$

J_h is to be the F. T. of $K^h_{\alpha\beta}$. Assume $J_h(\lambda)$ is radial, so that

$$J_h(\lambda)E_\gamma = g(h, |\gamma|, \lambda)E_\gamma$$

for some function $g(h, N, \lambda)$ with

$$g(h, N, \lambda) \to 1 \text{ as } h \to 0.$$

Let

$$g_{N\lambda}(h) = g(h, N, \lambda);$$

then we want

$$hg''_{N\lambda} - (n + \alpha + \beta)g'_{N\lambda}$$
$$-[(2N + n + \alpha - \beta)|\lambda| + h\lambda^2]g_{N\lambda} = 0 \text{ if } \lambda > 0;$$

α and β should be reversed if $\lambda < 0$. Let

$$g_{N+} = g_{N,1/2}, \quad g_{N-} = g_{N,-1/2}.$$

If we can find g_{N+}, g_{N-}, we can therefore obtain $g_{N\lambda}$ by putting

$$g_{N\lambda}(h) = g_{N+}(2\lambda h) \text{ for } \lambda > 0,$$

$$g_{N\lambda}(h) = g_{N-}(2|\lambda|h) \text{ for } \lambda < 0.$$

(This is as it should be, since $K^h_{\alpha\beta} = D^{\sqrt{h}} K^1_{\alpha\beta}$.) Let us find g_{N+}; we can find g_{N-} later by interchanging α and β. Put $g_N = g_{N+}$. Let

$$g_N = e^{h/2}G_N;$$

the differential equation then becomes

$$hG''_N - (n + \alpha + \beta + h)G'_n - (N - \beta)G_N = 0.$$

(We also need that $G_N(h) \to 1$ as $h \to 0$). This is a confluent hypergeometric equation ([**3**], page 248). The general such equation has the form

$$L_{a,c}f = [hD^2 + (c - h)D - a]f = 0;$$

here $D = d/dh$, $f = f(h)$. It is easy to see that

$$L_{a,c}f = 0 \Rightarrow L_{a-c+1,2-c}f_1 = 0 \text{ if } f_1(h) = h^{c-1}f.$$

We propose, then, to seek G_N in the form

$$G_N(h) = h^{n+\alpha+\beta+1}\psi_N(h)$$

where

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$$L_{a,c}\psi_N = 0, a = N + n + \alpha + 1, c = n + \alpha + \beta + 2.$$

If Re $a > 0$, one has the solution ([**3**], page 255, equation 2):

$$\psi_N = [\Gamma(a)/\Gamma(c - 1)]\Psi(a, c; h)$$
$$= \Gamma(c - 1)^{-1} \int_0^\infty e^{-\tau h}\tau^{a-1}(1 + \tau)^{c-a-1}d\tau.$$

(This is evident if one notes that

$$L_{a,c}(e^{-\tau h}\tau^{a-1}(1 + \tau)^{c-a-1}) = -(d/d\tau)(e^{-\tau h}\tau^{a}(1 + \tau)^{c-a}).)$$

With $G_N = h^{c-1}\Psi_N$, as above, one does have $G_N(h) \to 1$ as $h \to 0$, since

$$G_N(h) = \Gamma(c - 1)^{-1} \int_0^\infty e^{-\sigma}\sigma^{a-1}(h + \sigma)^{c-a-1}d\sigma.$$

If, then, $\mathrm{Re}(n + \alpha + 1) > 0$, $\mathrm{Re}(n + \beta + 1) > 0$, we obtain a solution

$$g(h, N, \lambda) = \Gamma(n + \alpha + \beta + 1)^{-1}(2|\lambda|h)^{n+\alpha+\beta+1} \times \int_0^\infty e^{-(2\tau+1)|\lambda|h}\tau^{N+n+\alpha}(1 + \tau)^{-(N-\beta)}d\tau$$

for $\lambda > 0$. If $\lambda < 0$, one must interchange α and β. If we make the substitution $s = \tau/(1 + \tau)$, we find for $\lambda > 0$,

$$g(h, N, \lambda) = \Gamma(n + \alpha + \beta + 1)^{-1}(2|\lambda|h)^{n+\alpha+\beta+1} \times \int_0^1 s^{N+n+\alpha}(1 - s)^{-(n+\alpha+\beta+2)} \exp(-(\psi^{-1}(s))|\lambda|h)ds \tag{7.6}$$

(where $\psi^{-1}(s) = (1 + s)/(1 - s)$ as in (6.6)). If $\lambda < 0$, α and β are to be reversed. Now, for $0 < \delta < 1$, $\lambda > 0$, define $g_\delta(h, N, \lambda)$ by formula (7.6) with $\int_0^1$ replaced by $\int_0^{1-\delta}$. Define $g_\delta(h, N, \lambda)$ similarly for $\lambda < 0$. Define $J_{h,\delta}(\lambda)$ by

$$J_{h,\delta}(\lambda)E_\alpha = g_\delta(h, N, \lambda)E_\alpha;$$

also $J_h(\lambda)E_\alpha = g(h, N, \lambda)E_\alpha$.

Then Theorem 6.2 applies with $j = -(n + \alpha + \beta + 1)$, $k = 2n + \alpha + \beta + 2$ to show that if

$$c'_{\alpha\beta} = \pi^{-(n+1)}2^{n-1}\Gamma(2n + \alpha + \beta + 2)\,\Gamma(n + \alpha + \beta + 1)^{-1},$$

and if

$$K^h_{\alpha\beta\delta}(u) = c'_{\alpha\beta}(2h)^{n+\alpha+\beta+1} \times \left[\int_0^{1-\delta} (s(w(\mathrm{u}) + h) \right.$$

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$$+ (\bar{w}(u) + h))^{-(2n+\alpha+\beta+2)} s^{n+\alpha} ds$$

$$+ \int_0^{1-\delta} (s(\bar{w}(u) + h)$$

$$+ (w(u) + h))^{-(2n+\alpha+\beta+2)} s^{n+\beta} ds \Big],$$

then $(K^h_{\alpha\beta\delta})^\frown = \underset{\sim}{J}_{h,\delta}$. Now, as $\delta \to 0$, $K^h_{\alpha\beta\delta} \to K^h_{\alpha\beta}$ in L^1, where

$$K^h_{\alpha\beta}(u) = c'_{\alpha\beta}(2h)^{n+\alpha+\beta+1} \times \int_0^\infty (s(w(u) + h) + (\bar{w}(u) + h))^{-(2n+\alpha+\beta+2)} s^{n+\alpha} ds.$$

One sees that $K^h_{\alpha\beta}$ is as in (7.5), just as in the remark of Theorem 6.2. To see that $(K^h_{\alpha\beta})^\frown = \underset{\sim}{J}_h$, it suffices then to show that $\underset{\sim}{J}_{h,\delta} \to \underset{\sim}{J}_h$ in $[\mathcal{S}]^*$. By Proposition 5.4 we need only show that $\{\underset{\sim}{J}_{h,\delta} | 0 < \delta < 1\}$ is a bounded subset of Ord(-2). Note, however, that for fixed $h > 0$, there exists $C > 0$ with

$$|g_\delta(h, N, \lambda)| < C|\lambda|^{-1} \int_0^1 s^N [\psi^{-1}(s) |\lambda| h]^{n+\alpha+\beta+2} \times \exp(-\psi^{-1}(s) |\lambda| h) ds$$

for $\lambda > 0$. However, for any $b > 0$, the function $x^b e^{-x}$ is bounded for $x \geqq 0$. Thus, for some $C' > 0$,

$$|g_\delta(h, N, \lambda)| < C'|\lambda|^{-1} \int_0^1 s^N ds = C'[(N + 1)|\lambda|]^{-1},$$

if $\lambda > 0$. Similarly for $\lambda < 0$, and the boundness of $\{\underset{\sim}{J}_{h,\delta}\}$ is established. Accordingly,

$$(K^h_{\alpha\beta})^\frown = \underset{\sim}{J}_h.$$

Since $K^h_{\alpha\beta} \in L^1$,

$$J_h(\lambda) \to \left(\int K^h_{\alpha\beta}\right) I \quad \text{strongly as } \lambda \to 0;$$

thus $\int K^h_{\alpha\beta} = 1$. Since also all derivatives of $K^h_{\alpha\beta}$ are in L^1,

$$\begin{aligned}(\Delta_{\alpha\beta} K_{\alpha\beta})^\frown &= [hH^2 - (n + \alpha + \beta)H - h\lambda^2 \\ &\quad - (\alpha - \beta)\lambda] J(h, \lambda) - J(h, \lambda) A_\lambda \\ &= 0, \text{ with } J(h, \lambda) = J_h(\lambda).\end{aligned}$$

This verifies all our claims.

We leave as an exercise to the reader to use Theorem 2.3 to find the known formula (**[5]**, and the author's thesis) for the Fourier transform in t

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of the "heat kernel" for L_0, that is, the kernel for $(\partial/\partial\tau + L_0)K = 0$. One wants

$$K(\lambda)E_\alpha = e^{-(2|\alpha|+n)|\lambda|\tau}E_\alpha.$$

Theorem 2.3, with $s = e^{-2|\lambda|\tau}$, then makes it easy to find $\mathcal{F}_t K$. ($\mathcal{F}_t$ = F. T. in t.)

We wish to study briefly the question of which functions can be represented essentially in the form $\int_0^\infty (sw + \bar{w})^{-k} d\mu(s)$ (as in (6.3) when $\epsilon \to 0$) for w in the closed right half plane. We would like to indicate a class of functions for which this is possible, which includes the functions

$$\Gamma(\gamma)\Gamma(k - \gamma)\Gamma(k)^{-1}w^{-\gamma}\bar{w}^{-(k-\gamma)}$$

($\operatorname{Re}\gamma > 0$, $\operatorname{Re}(k - \gamma) > 0$) of the remark of Theorem 6.2. We put $\zeta = \bar{w}/w$, so that we may as well consider

$$\int_0^\infty (s + \zeta)^{-k} d\mu(s).$$

This integral will evidently exist under weak assumptions on $d\mu$ if $\zeta \in \mathbf{C}\setminus\mathbf{R}^-$ where

$$\mathbf{R}^- = \{\zeta | \operatorname{Re}\zeta \leqq 0, \operatorname{Im}\zeta = 0\}.$$

The problem comes when ζ approaches $\mathbf{R}^- \setminus \{0\}$ as happened in the proof of Theorem 6.2 in the corresponding case when w was pure imaginary. We shall assume

$$d\mu(s) = g(s)ds \text{ for } g \in C^\infty(\mathbf{R}^+).$$

More precisely, let $S = \{g \in C^\infty(\mathbf{R}^+) \mid$ there exists $\epsilon > 0$ such that for all $l \in \mathbf{Z}^+$

$$|g^{(l)}(s)| = O(s^{\epsilon-l-1}) \text{ as } s \to 0^+,$$

while

$$|g^{(l)}(s)| = O(s^{\operatorname{Re}k-l-\epsilon-1}) \text{ as } s \to \infty\}.$$

Let $T = \{f$ analytic on $\mathbf{C}\setminus\mathbf{R}^- \mid f|_{\{\operatorname{Im}\zeta>0\}}$ can be extended smoothly to $\{\operatorname{Im}\zeta \geqq 0\}$; $f|_{\{\operatorname{Im}\zeta<0\}}$ can be extended smoothly to $\{\operatorname{Im}\zeta \leqq 0)\}$; and there exists $\epsilon > 0$ such that for all $l \in \mathbf{Z}^+$, $|f(\zeta)| = O(|\zeta|^{\epsilon-\operatorname{Re}k-l})$ as $\zeta \to 0$, while $|f(\zeta)| = O(|\zeta|^{-l-\epsilon})$ as $\zeta \to \infty$.$\}$ We then have:

THEOREM 7.4. *Suppose* $\operatorname{Re} k > 1$ *or* $k = 1$. *For* $g \in S$, *put*

$$(7.7) \quad (Ag)(\zeta) = \int_0^\infty (s + \zeta)^{-k} g(s)ds.$$

Then $A:S \to T$, *and* A *is a vector space isomorphism. In fact, for* $f \in T$

put

(7.8) $\quad (Bf)(s) = (k - 1)(2\pi i)^{-1} \int_{\gamma_s} f(\zeta)(\zeta + s)^{k-2} d\zeta \quad$ *if* $\operatorname{Re} k > 1.$

(7.8) $\quad (Bf)(s) = \lim_{\theta \to 0^+} (2\pi i)^{-1}[f(-se^{i\theta}) - f(-se^{-i\theta})] \quad$ *if* $k = 1.$

Here γ_s *denotes the circle of radius* s *centered at* 0, *traversed counterclockwise, beginning and ending at* $-s$. *Then* $B:T \to S$, *and* B *is the inverse of* A.

The equivalence of (7.7) and (7.8) is implicitly in [**12**], chapter IX, especially page 235. Our formulation and the following simple proof may be original.

LEMMA 7.5. $A:S \to T$.

To avoid tiring the reader, we delay the proof of this until after Theorem 7.4.

Proof of Theorem 7.4, *assuming Lemma* 7.5. It is evident that $B:T \to S$. Indeed, this is transparent when $k = 1$. If $k \neq 1$,

$$(Bf)(s) = (k - 1)(2\pi)^{-1}s^{k-1} \int_{-\pi}^{\pi} f(se^{i\theta})(e^{i\theta} + 1)^{k-2}e^{i\theta}d\theta,$$

so that $B:T \to S$ in this case as well.

As in Theorem 6.2, we begin by noting the special case

$$g_\gamma(s) = s^{\gamma-1},$$

$$f_\gamma(\zeta) = \Gamma(\gamma)\Gamma(k - \gamma)\Gamma(k)^{-1}\zeta^{\gamma-k},$$

for $0 < \operatorname{Re} \gamma < \operatorname{Re} k$. Then $g_\gamma \in S$, $f_\gamma \in T$ and $Ag_\gamma = f_\gamma$, just as in the remark of Theorem 6.2. Now we study the cases $k = 1$ and $k \neq 1$ separately.

Case 1. $k = 1$. Say $f \in T$; then $ABf = f$ by the Cauchy integral formula. Ag is called the Stieltjes transform of g, in this case. Thus we need only check $BAg = g$ for all $g \in S$.

First, we claim $BAg_\gamma = g_\gamma$. Indeed, $Bf_\gamma = cg_\gamma$, where

$$c = \pi^{-1}\Gamma(\gamma)\Gamma(1 - \gamma) \sin \pi\gamma.$$

Thus

$$f_\gamma = ABf_\gamma = cAg_\gamma = cf_\gamma.$$

This provides a simple proof that $c = 1$. Also, then,

$$BAg_\gamma = g_\gamma.$$

We prove that $BAg = g$ for all $g \in S$. A dilation argument reduces us to showing that $(BAg)(1) = g(1)$. Further, we may suppose $g(1) = 0$. Indeed, if we can do this case, in general let

$$G(s) = g(s) - g(1)g_\gamma$$

for any fixed γ, $0 < \gamma < 1$. Then $(BAG)(1) = 0$ implies that

$$(BAg)(1) = g(1),$$

since $BAg_\gamma = g_\gamma$. Now assume $g(1) = 0$; observe

$$(Ag)(\zeta) = \int_0^1 (s + \zeta)^{-1}\, \tilde{g}(s)ds + \zeta^{-1} \int_0^1 (s + \zeta^{-1})^{-1}\, \tilde{\tilde{g}}(s)ds$$

$$= \tilde{f}(\zeta) + \tilde{\tilde{f}}(\zeta)$$

where $\tilde{g}(s) = g(s)$, $\tilde{\tilde{g}}(s) = s^{-1}g(s^{-1})$. For any $\epsilon > 1/2$,

$$f(-e^{i\varphi}) - \tilde{f}(-e^{-i\varphi}) = h(\varphi) +$$

$$\int_{1-\epsilon}^1 (s - e^{i\varphi})^{-1}\tilde{g}(s)ds - \int_{1-\epsilon}^1 (s - e^{-i\varphi})^{-1}\tilde{g}(s)ds$$

where $h(\varphi) \to 0$ as $\varphi \to 0$. Since $\tilde{g}(1) = 0$ and $\tilde{g}$ is smooth,

$$|\tilde{g}(s)| < C(1 - s) \text{ for } 1/2 \leqq s \leqq 1;$$

Thus

$$|\tilde{f}(-e^{i\varphi}) - \tilde{f}(-e^{-i\varphi})| < |h(\varphi)| + 2C\epsilon.$$

Thus

$$\tilde{f}(-e^{i\varphi}) - \tilde{f}(-e^{-i\varphi}) \to 0 \text{ as } \varphi \to 0;$$

similarly for $\tilde{\tilde{f}}$; so $(BAg)(1) = 0$.

Case 2. $k \neq 1$. First suppose $f \in T$, and in addition that for some $\epsilon > 0$,

$$|f(z)| = O(|z|^{\epsilon - 1}) \text{ as } z \to 0;$$

we show $ABf = f$. Now, by the Cauchy integral formula,

$$(Bf)(s) = (k - 1)(2\pi i)^{-1} \int_0^s [f_+(-t) - f_-(-t)] \times (s - t)^{k-2}dt,$$

where

$$f_+(-t) = \lim_{\theta \to 0} f(-te^{i\theta}),$$

$$f_-(-t) = \lim_{\theta \to 0} f(-te^{i\theta}).$$

Note that as $s \to 0^+$,

$$\int_0^s |f_+(-t) - f_-(-t)|\,|(s - t)^{k-2}|dt$$

$$< Cs^{\mathrm{Re}\,k-2}\int_0^s t^{\epsilon-1}dt = O(s^{\epsilon-1}),$$

while as $s \to \infty$,

$$\int_0^s |f_+(-t) - f_-(-t)|\,|(s - t)^{k-2}|dt$$

$$< O(s^{\mathrm{Re}\,k-2}) + C\int_1^s t^{-\epsilon}(s - t)^{k-2}dt$$

$$= O(s^{\mathrm{Re}\,k-2}) + Cs^{\mathrm{Re}\,k-\epsilon-1}\int_{1/s}^1 t^{-\epsilon}(1 - t)^{k-2}dt$$

$$= O(s^{\mathrm{Re}\,k-\epsilon-1}).$$

Thus we can substitute the expression for $g = Bf$ into (7.7), interchange integrals and find

$$ABf(\zeta) = (k - 1)(2\pi i)^{-1}\int_0^\infty [f_+(-t) - f_-(-t)] \times \left[\int_t^\infty (s - t)^{k-2}(s + \zeta)^{-k}ds\right]dt.$$

The inner integral is

$$\int_0^\infty s^{k-2}(s + t + \zeta)^{-k}ds = \Gamma(k - 1)\Gamma(1)\Gamma(k)^{-1}(t + \zeta)^{-1} = (k - 1)^{-1}(t + \zeta)^{-1}$$

since $t + \zeta \in \mathbf{C}\setminus\mathbf{R}^-$. Just as in the case $k = 1$, by the Cauchy integral formula we now find $ABf = f$. These considerations apply to f_γ if $\mathrm{Re}\,k - 1 < \mathrm{Re}\,\gamma < \mathrm{Re}\,k$. Further, it is apparent from (7.8) that for all γ (with $0 < \mathrm{Re}\,\gamma < \mathrm{Re}\,k$), $Bf_\gamma = c_\gamma g_\gamma$ for some c_γ. But now, if $\mathrm{Re}\,k - 1 < \mathrm{Re}\,\gamma < \mathrm{Re}\,k$,

$$f_\gamma = ABf_\gamma = c_\gamma Ag_\gamma = c_\gamma f_\gamma,$$

so that $c_\gamma = 1$. By analytic continuation, then, $c_\gamma = 1$ for all $\gamma(0 < \mathrm{Re}\,\gamma < \mathrm{Re}\,k)$, and we always have $Bf_\gamma = g_\gamma$, $BAg_\gamma = g_\gamma$. This proves the identity

$$(7.9)\qquad \int_{-\pi}^{\pi} e^{i(\gamma-k+1)\theta}(e^{i\theta} + 1)^{k-2}d\theta = 2\pi\Gamma(k - 1)[\Gamma(\gamma)\Gamma(k - \gamma)]^{-1}$$

for $0 < \mathrm{Re}\,\gamma < \mathrm{Re}\,k$, and hence for all $\gamma \in \mathbf{C}$. ((7.9) is actually the same as [**4**] equation (6.5), after a simple change of variables. The expression for $Bf(s)$, that was used at the beginning of the paragraph, specialized to the case $f = f_\gamma(\text{Re } k - 1 < \text{Re } \gamma < \text{Re } k)$, could also be used directly to prove (7.9).)

Next, let us show that $BAg = g$ for all $g \in S$. We need only show $(BAg)(1) = g(1)$; further we can assume that g vanishes at 1 to order at least $l = [\text{Re } k]$. Indeed, let $c = 0$ unless $k \in \mathbf{N}$, in which case put $c = 1/2$. Write

$$s^c g(s) = p(s) + v(s)$$

where p is a polynomial of degree no more than $l - 1$ and

$$v(s) = (1 - s)^l v_0(s),$$

where $v_0 \in C^\infty(\mathbf{R}^*)$. Then we could always let

$$G(s) = g(s) - s^{-c}p(s) = s^{-c}v(s).$$

Now $G \in S$, so if we can show

$$(BAG)(1) = G(1) = 0,$$

we find $(BAg)(1) = g(1)$ from the corresponding facts for the g_γ. Now say that g vanishes at 1 to order at least l. Then if $|\zeta| = 1$,

$$(Ag)(\zeta) = \int_0^1 (s + \zeta)^{-k}\tilde{g}(s)ds$$
$$+ \zeta^{-k}\int_0^1 (s + \zeta^{-1})^{-1}\tilde{\tilde{g}}(s)ds = \tilde{f}(\zeta) + \tilde{\tilde{f}}(\zeta),$$

where

$$\tilde{g}(s) = g(s), \tilde{\tilde{g}}(s) = s^{k-2}g(s^{-1}).$$

Because of the behavior of g at 1, the integrals converge, absolutely and uniformly, for $|\zeta| = 1$. To show $(BAg)(1) = 0$, it suffices then to show that if $0 < s < 1$,

$$\int_\gamma (s + \zeta)^{-k}(\zeta + 1)^{k-2}d\zeta = 0 \quad \text{and}$$

$$\int_\gamma (s + \zeta^{-1})^{-k}(\zeta + 1)^{k-2}\zeta^{-k}d\zeta = 0,$$

where $\gamma = \gamma_1$. It is easy to see that if $\zeta \in \gamma$,

$$(s + \zeta)^{-k} = \zeta^{-k}(1 + s\zeta^{-1})^{-k}, \quad \text{and}$$
$$(\zeta + 1)^{k-2} = \zeta^{k-2}(1 + \zeta^{-1})^{k-2}.$$

Writing $z = \zeta^{-1}$, we see that we must show

$$\int_\gamma (1 + sz)^{-k}(1 + z)^{k-2}dz = \int_\gamma (s\zeta + 1)^{-k}(\zeta + 1)^{k-2}d\zeta$$
$$= 0.$$

This is evident, since the integrand is analytic inside γ. Thus

$$BAg = g \text{ for all } g \in S.$$

Finally, we must show $ABf = f$ for all $f \in T$. Let

$$I(s) = \int_{-\pi}^{\pi} |f(se^{i\theta})| \, |(se^{i\theta} + s)^{k-2}| s d\theta.$$

As $s \to 0^+$,

$$I(s) = O(s^{\epsilon - 1});$$

as $s \to \infty$,

$$I(s) = O(s^{\mathrm{Re}k - \epsilon - 1}).$$

Thus we can substitute $g = Bf$, as given in (7.8), into (7.7), and interchange the order of integration. We find

$$(7.10) \quad ABf(\zeta) = (k - 1)(2\pi)^{-1} \int_{-\pi}^{\pi} F(\theta, \zeta)(e^{i\theta} + 1)^{k-2} e^{i\theta} d\theta$$

where, if $-\pi < \theta < \pi$,

$$F(\theta, \zeta) = \int_0^\infty f(se^{i\theta}) s^{k-1} (s + \zeta)^{-k} ds$$

$$= \int_0^\infty f(se^{i\theta})(se^{i\theta})^{k-1}(e^{i\theta})^{-k}(s + \zeta)^{-k} e^{i\theta} ds.$$

Let us be very careful. Up to now we have used the principal branches of the power functions. Right now, though, we must distinguish several cases. Let $\zeta = re^{i\varphi}$, and define a function $G(z)$ as follows. If $\operatorname{Im} e^{i\theta}$ and $\operatorname{Im} e^{i\varphi}$ have opposite signs, or if $e^{i\theta} = 1$ or $e^{i\varphi} = 1$, let $G(z)$ be the principal branch of z^{-k}. If $\operatorname{Im} e^{i\theta} > 0$, $\operatorname{Im} e^{i\varphi} > 0$, let $G(z)$ be the branch of z^{-k} which is analytic away from $\mathbf{R}^+$ and agrees with the principal branch in the upper half plane. Finally, if $\operatorname{Im} e^{i\theta} < 0$, $\operatorname{Im} e^{i\varphi} < 0$, let $G(z)$ be the branch of z^{-k} which is analytic away from $\mathbf{R}^+$ and agrees with the principal branch in the lower half plane. Then we can say

$$F(\theta, \zeta) = \int_0^\infty f(se^{i\theta})(se^{i\theta})^{k-1} G(se^{i\theta} + re^{i(\theta+\varphi)}) e^{i\theta} ds$$

$$= \int_\beta f(z) z^{k-1} G(z + re^{i(\theta+\varphi)}) dz,$$

where β is the path $z = se^{i\theta} (0 < s < \infty)$. Deforming the contour of integration to the path β' on which $z = se^{i\varphi}$, we find

$$F(\theta, \zeta) = \int_0^\infty f(se^{i\varphi})(se^{i\theta})^{k-1} G(se^{i\varphi} + re^{i(\theta+\varphi)}) e^{i\varphi} ds$$

$$= \int_0 f(se^{i\varphi}) s^{k-1} (s + re^{i\theta})^{-k} ds = (Ag_\varphi)(re^{i\theta})$$

https://doi.org/10.4153/CJM-1984-039-0 Published online by Cambridge University Press

where

$$g_{\varphi}(s) = f(se^{i\varphi})s^{k-1} \in S.$$

Then, from (7.10),

$$\begin{aligned} ABf(\zeta) &= [B(Ag_{\varphi})](r)\, r^{-(k-1)} \\ &= g_{\varphi}(r)r^{-(k-1)} = f(re^{i\varphi}) = f(\zeta), \end{aligned}$$

as desired. This completes the proof.

Proof of Lemma 7.5. We show that $Ag|_{\{\mathrm{Im}\zeta>0\}}$ has a smooth extension to a segment with -1 in its interior. A dilation argument shows that -1 could be replaced by any point $-s < 0$. The proof that $Ag|_{\{\mathrm{Im}\zeta<0\}}$ can be extended smoothly to $\{\mathrm{Im}\ \zeta \leqq 0\}\diagdown\{0\}$ will be similar.

Note that the function $F_0(\zeta) = \zeta^{-k}$ can be smoothly extended to $\{\mathrm{Im}\ \zeta \geqq 0\}\diagdown\{0\}$, and further that for every $n \in \mathbf{Z}^+$ there exists $F_n(\zeta)$ which is smooth on $\{\mathrm{Im}\ \zeta \geqq 0\}\diagdown\{0\}$, analytic on the interior, and such that $F_n^{(n)} = F_0$. (F_n will be a constant times ζ^{n-k} if $k \notin \mathbf{Z}^-$, while if $n - k \in \mathbf{Z}^+$, F_n will be of the form $p(\zeta) \log \zeta + q(\zeta)$ for certain polynomials p, q.) Now, if $\mathrm{Im}\ \zeta > 0$, we can write, for any δ with $0 < \delta < 1$,

$$\begin{aligned} (Ag)(\zeta) &= \int_0^{1-\delta} (s + \zeta)^{-k}g(s)ds + \int_{1-\delta}^{(1-\delta)^{-1}} (s + \zeta)^{-k}g(s)ds \\ &\quad + \zeta^{-k} \int_0^{1-\delta} (s + \zeta^{-1})^{-1k}\, \tilde{\tilde{g}}(s)ds \\ &= I_1(\zeta) + I_2(\zeta) + I_3(\zeta) \end{aligned}$$

(where $\tilde{\tilde{g}}(s) = s^{k-2}g(s^{-1})$). $I_1(\zeta)$ and $I_3(\zeta)$ can clearly be extended smoothly to a short segment with -1 in its interior. As for I_2, take any $N > \mathrm{Re}\ k$. For $\mathrm{Im}\ \zeta > 0$, we integrate by parts N times to see that

$$I_2(\zeta) = G_N(\zeta) + \int_{1-\delta}^{(1-\delta)^{-1}} F_N(s + \zeta)g^{(N)}(s)ds$$

where $G_N(\zeta)$ can certainly be extended smoothly in the desired way. However,

$$\int_{1-\delta}^{(1-\delta)^{-1}} F_N(s + \zeta)g^{(N)}(s)ds$$

can clearly be extended to a C^M function to a segment including -1, for any $M < N - \mathrm{Re}\ k$. Since N is arbitrary, the extendability property of Ag is established.

Let $f = Ag$; we must still show that for all l

$$|f^{(l)}(\zeta)| = O(|\zeta|^{\epsilon-\mathrm{Re}k-l}) \quad \text{as } \zeta \to 0,$$

while

$$|f^{(l)}(\zeta)| = O(|\zeta|^{-l-\epsilon}) \quad \text{as } \zeta \to \infty.$$

Since $f(\zeta) = \zeta^{-k}(A\tilde{\tilde{g}})(\zeta^{-1})$ where $\tilde{\tilde{g}}(s) = s^{k-2}g(s^{-1}) \in S$, we need only show the first of these. Now, if $|\zeta| = 1$, note

$$|f^{(l)}(\zeta)| < C_l \left[\int_0^{1-\delta} (|g(s)| + |\tilde{\tilde{g}}(s)|)ds + \left| \int_{1-\delta}^{(1-\delta)^{-1}} (s + \zeta)^{-k-l}g(s)ds \right| \right]$$

$$< C_l' \left[\int_0^1 (|g(s)| + \tilde{\tilde{g}}(s)|)ds + \max\{|g^{(N)}(s)| : 1 - \delta \leqq s \leqq (1-\delta)^{-1}, 0 \leqq N \leqq k + l + 1\} \right].$$

Here C_l, C_l' depend only on l and not on g, and we have again integrated by parts. Now suppose instead $\zeta = r\zeta'$, $|\zeta'| = 1$, $r > 0$. Then

$$f^{(l)}(r\zeta') = r^{-k-l+1}(Ag_r)^{(l)}(\zeta'),$$

where $g_r(s) = g(rs)$. Now

$$\tilde{\tilde{g}}_r = r^{k-2}(\tilde{\tilde{g}})_{1/r}.$$

Thus

$$|f^{(l)}(r\zeta')| < C_l' r^{-\mathrm{Re}k-l+1} \left[r^{-1} \int_0^1 |g(s)|ds + r^{\mathrm{Re}k-1} \int_0^{1/r} |\tilde{\tilde{g}}(s)|ds + r^N \max\{|g^{(N)}(s)| : (1-\delta)r \leqq s \leqq (1-\delta)^{-1}r, 0 \leqq N \leqq k + l + 1\} \right].$$

From this, the desired property is apparent, and the proof is complete.

We leave to the reader to verify that $f|_{\{\mathrm{Im}\zeta>0\}}$ and $f|_{\{\mathrm{Im}\zeta<0\}}$ can be extended analytically past $\mathbf{R}^- \setminus \{0\}$ if and only if g is real analytic. For an application of this idea, see [**17**].

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