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ON THE TOTAL DISTANCE AND DIAMETER OF GRAPHS

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Abstract

The total distance (or Wiener index) of a connected graph G is the sum of all distances between unordered pairs of vertices of G. DeLaViña and Waller ['Spanning trees with many leaves and average distance', *Electron. J. Combin.* **15**(1) (2008), R33, 14 pp.] conjectured in 2008 that if G has diameter $D > 2$ and order $2D + 1$, then the total distance of G is at most the total distance of the cycle of the same order. In this note, we prove that this conjecture is true for 2-connected graphs.

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1. Introduction

Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Denote by $d_G(u, v)$ the distance between vertices u and v in G. The *eccentricity* of a vertex v in a connected graph G is defined to be $\varepsilon_G(v) = \max\{d_G(u, v) \mid u \in V(G)\}$. If $\varepsilon_G(v) = d_G(u, v)$ for some vertex u in a connected graph G, then u is said to be an *eccentric vertex* of v. The *diameter* of a connected graph G is equal to $\max\{\varepsilon_G(v) \mid v \in V(G)\}$. Let u and v be two distinct nonadjacent vertices of a graph G and $S \subseteq V(G) - \{u, v\}$. If u and v belong to different components of $G - S$, then we say that S separates u and v or S is a *vertex-cut* of G. If, for any vertex-cut S in G, we always have $|S| \geq 2$, then G is said to be a 2-connected graph. Other notation and terminology not defined here will conform to [3].

For a connected graph G, the *total distance* or *Wiener index* of G, denoted by $W(G)$, is defined to be

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u, v) = \frac{1}{2} \sum_{v\in V(G)} D_G(v), \tag{1.1}$$

where $D_G(v) = \sum_{u\in V(G)} d_G(u, v)$.

The Wiener index is one of the oldest and best-studied distance-based graph invariants associated with a connected (molecular) graph G and has applications in mathematical chemistry (see [2] and the references cited therein).

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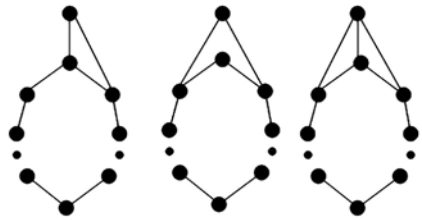

FIGURE 1. Graphs C_n^1, C_n^2 and C_n^3 (n is odd).

The *average distance* of G, denoted by $\overline{D}(G)$, is defined to be

$$\overline{D}(G) = \binom{n}{2}^{-1} \sum_{\{u,v\}\subseteq V(G)} d_G(u, v) = \frac{2}{n(n-1)} W(G).$$

The computer programs Graffiti [5] and AutoGraphiX [1] with the classical 1984 paper by Plesnik [8] are three of the best sources for problems and conjectures related to average distance and total distance (Wiener index). These sources contain some pretty and long-standing problems on this topic (see also [4] and the references cited therein).

One such problem of Plesnik [8] which remains unresolved can be stated as follows.

PROBLEM 1.1. What is the maximum total distance (Wiener index) or average distance among all graphs of order n with diameter D?

To see how hard it is to solve Problem 1.1, consider the following Graffiti conjecture from DeLaViña and Waller [4], which is a special case of Problem 1.1.

CONJECTURE 1.2. *Let G be a connected graph of diameter $D > 2$ and order $2D + 1$. Then $W(G) \leq W(C_{2D+1})$, where C_{2D+1} is the cycle of length $2D + 1$.*

As far as we know, Conjecture 1.2 remains open. In this paper, we give a partial solution to this conjecture. More specifically, we prove that this conjecture is true for 2-connected graphs.

2. Proof of Conjecture 1.2 for 2-connected graphs

Recently, Hua *et al.* proved the following result.

LEMMA 2.1 [6]. *Let G be a 2-connected graph of order n with diameter D and radius r. If $n = 2D + 1$ and $r = D$, then $G \cong C_n$ or C_n^i (see Figure 1) for some i with $1 \leq i \leq 3$.*

Before we proceed any further, we introduce a well-known result on connectivity of a graph due to Menger [7] in 1927.

THEOREM 2.2 (Menger [7]). *Let G be a graph and u, v be two distinct nonadjacent vertices of G. Then the maximum number of pairwise internally vertex disjoint paths connecting u and v is equal to the minimum number of vertices in a vertex-cut set that separates u and v.*

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Now we are in a position to prove that Conjecture 1.2 holds for 2-connected graphs.

THEOREM 2.3. *Let G be a 2-connected graph of diameter $D > 2$ and order $2D + 1$. Then*

$$W(G) \leq \frac{2D^3 + 3D^2 + D}{2}$$

with equality if and only if $G \cong C_{2D+1}$.

PROOF. For any vertex $v \in V(G)$, let u be one of its eccentric vertices, that is, $\varepsilon_G(v) = d_G(v, u)$. Since G is a 2-connected graph, by Theorem 2.2, G has two internally vertex disjoint paths connecting v and u. Write $X_v(i) = \{w \in V(G) \mid d_G(v, w) = i\}$ for $i = 1, \ldots, \varepsilon_G(v)$.

Since G has two internally vertex disjoint paths connecting v and u, we have $|X_v(i)| \geq 2$ for each $i = 1, \ldots, (\varepsilon_G(v) - 1)$ and $|X_v(\varepsilon_G(v))| \geq 1$. Therefore,

$$\begin{aligned} D_G(v) &\leq 2[1 + \cdots + (\varepsilon_G(v) - 1)] + [(2D + 1) - 1 - 2(\varepsilon_G(v) - 1)]\varepsilon_G(v) \\ &= -(\varepsilon_G(v))^2 + (2D + 1)\varepsilon_G(v) \end{aligned} \tag{2.1}$$

and the equality holds only if $|X_v(i)| = 2$ for each i with $i = 1, \ldots, \varepsilon_G(v) - 1$ and $|X_v(\varepsilon_G(v))| = 2D + 2 - 2\varepsilon_G(v)$.

Let $f(x) = -x^2 + (2D + 1)x$. Observe that $f(x)$ is increasing on the interval $(-\infty, \frac{1}{2}(2D + 1)]$. Note that $\varepsilon_G(v) \leq D < \frac{1}{2}(2D + 1)$. Thus,

$$-(\varepsilon_G(v))^2 + (2D + 1)\varepsilon_G(v) = f(\varepsilon_G(v)) \leq f(D) = D^2 + D \tag{2.2}$$

with equality only if $\varepsilon_G(v) = D$.

By (1.1), (2.1) and (2.2),

$$W(G) = \frac{1}{2} \sum_{v \in V(G)} D_G(v) \leq \frac{1}{2} \cdot (2D + 1) \cdot (D^2 + D) = \frac{2D^3 + 3D^2 + D}{2} \tag{2.3}$$

with equality only if $D_G(v) = D^2 + D$ for each v in G, that is, $D_G(v)$ is a constant.

Now we check the equality case. If $W(G) = \frac{1}{2}(2D^3 + 3D^2 + D)$, then all equalities in (2.1)–(2.3) hold together. From this, we conclude that for each v in G, we have $\varepsilon_G(v) = D$, that is, $r = D$, where r is the radius of G. Moreover, $D_G(v)$ is a constant. Now G is a 2-connected graph of order $2D + 1$ satisfying $r = D$. By Lemma 2.1, we must have $G \cong C_{2D+1}$ or C^i_{2D+1} (see Figure 1) for some i with $1 \leq i \leq 3$. But, if $G \cong C^i_{2D+1}$ for some i with $1 \leq i \leq 3$, then $D_G(v)$ cannot be a constant in G, which is a contradiction. Thus, $G \cong C_{2D+1}$. Conversely, if $G \cong C_{2D+1}$, then we clearly have $W(G) = \frac{1}{2}(2D^3 + 3D^2 + D)$.

This completes the proof. □

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