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A SCHUNCK CLASS CONSTRUCTION AND A PROBLEM CONCERNING PRIMITIVE GROUPS

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Abstract

Gaschütz has introduced the concept of a product of a Schunck class and a (saturated) formation (differing from the usual product of classes) and has shown that this product is a Schunck class provided that both of its factors consist of finite soluble groups. We investigate the same question in the context of arbitrary finite groups.

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Given a Schunck class $\mathscr{X}$ and a saturated formation $\mathscr{F}$ (both of them contained in the class of all finite soluble groups), Gaschütz [6] has introduced the homomorph

$$\mathscr{X} * \mathscr{F} = \left\{ G \in \mathscr{E} \middle| G^{\mathscr{F}} \in \mathscr{X} \right\}$$

of all finite groups G whose $\mathscr{F}$-residual $G^{\mathscr{F}}$ belongs to $\mathscr{X}$, and has shown that $\mathscr{X} * \mathscr{F}$ is a Schunck class, provided the classes $\mathscr{X}$ and $\mathscr{F}$ consist of finite soluble groups. The present note is an investigation into the question to which extent this result can be generalized for classes of finite (not necessarily soluble) groups, and is an application of the methods developed in [5]. Therefore the reader is assumed to be familiar with the results of as well as the notation employed in [5]; the same notation and terminology shall be used throughout the present note without further reference. As in [5], all groups considered here are supposed to belong to a fixed but otherwise arbitrary universe $\mathscr{V} \subseteq \mathscr{E}$ closed under taking subgroups and quotients.

Apart from a generalization of Gaschütz's above-mentioned result, we shall derive in a uniform manner several results concerned with products of Schunck classes in the universe of all finite π-soluble groups as well as a result on products of saturated formations, and it shall become clear how to obtain further results of the same type by means of our Theorem below.

Our main result gives a description of the groups in $b(\mathscr{X} * \mathscr{F})$, where $\mathscr{X}$ is a Schunck class and $\mathscr{F}$ a formation. We shall state the Theorem in the most general setting in order to be able to derive various criteria. Quite naturally, the most important special cases lead to considerably shorter statements, mainly because Schunck classes $\mathscr{X}$ with primitive groups of type III in their boundaries are ruled out by the results of [5].

THEOREM. *Let $\mathscr{X}$ be a Schunck class, $\mathscr{F}$ a formation, $G \in b(\mathscr{X} * \mathscr{F})$, and put $N = G^{\mathscr{F}}$. Then precisely one of the following four statements holds*:

(i) *$G \in b(\mathscr{F})$ and $\mathbf{S}(G) \notin \mathscr{X}$;*

(ii) *$\mathbf{S}(G) < N$ is an abelian minimal normal subgroup of G, N splits over $\mathbf{S}(G)$ with any $X \in \mathrm{Proj}_{\mathscr{X}}(N)$ complementing $\mathbf{S}(G)$ in N, and $\mathbf{S}(G)$ is a faithful, completely reducible X-module over some $GF(p)$; if T is an irreducible X-submodule of $\mathbf{S}(G)$, then $\{(X/\mathbf{C}_X(T))T\} = b(\mathscr{X}) \cap Q\{N\} \subseteq \mathscr{P}_{\mathrm{I}}$. Moreover, either*

(a) *$\mathbf{S}(G) \leqslant \Phi(G)$, in which case $(X/\mathbf{C}_X(T))T$ possesses two non-conjugate maximal subgroups with trivial core, or*

(b) *$G \in \mathscr{P}_I$, and then every element of $\mathrm{Proj}_{\mathscr{X} * \mathscr{F}}(G)$ may be written as $\mathbf{N}_G(X)$ for some $X \in \mathrm{Proj}_{\mathscr{X}}(N)$.*

(iii) *$\mathbf{S}(G) < N$ is a non-abelian minimal normal subgroup of G, $\mathbf{S}(G) = T_1 \times \cdots \times T_n$ with $\{T_1, \ldots, T_n\}$ being the set of all minimal normal subgroups of N, $G \in \mathscr{P}_{\mathrm{II}}$, and either*

(a) *$\{N/\mathbf{C}_N(T)\} = b(\mathscr{X}) \cap Q\{N\} \subseteq \mathscr{P}_{\mathrm{II}}$ for each minimal normal subgroup T of N, or*

(b) *$\{(X/\mathbf{C}_X(T))T\} = b(\mathscr{X}) \cap Q\{N\} \subseteq \mathscr{P}_{\mathrm{III}}$ for each minimal normal subgroup T and N and each $X \in \mathrm{Proj}_{\mathscr{X}}(N)$.*

(iv) *$\mathbf{S}(G) = M_1 \times M_1 \leqslant N$, where M_1 and M_2 are the minimal normal subgroups of G, $\mathbf{C}_N(M_i) = M_{3-i}$ $(i = 1, 2)$, each $X \in \mathrm{Proj}_{\mathscr{X}}(N)$ complements both M_1 and M_2 in N, $M_i = T_1^{(i)} \times \cdots \times T_n^{(i)}$ where $\{T_j^{(i)} | i = 1, 2;\ j = 1, \ldots, n\}$ is the set of all minimal normal subgroups of N, $\{(X/\mathbf{C}_X(T))T\} = b(\mathscr{X}) \cap Q\{N\} \subseteq \mathscr{P}_{\mathrm{III}}$ for each minimal normal subgroup T of N and each $X \in \mathrm{Proj}_{\mathscr{X}}(N)$, and either*

a) *$G \in \mathscr{P}_{\mathrm{III}}$, or* b) *$G$ is not primitive.*

PROOF. We subdivide the proof into four parts, each corresponding to one of the above statements.

Case 1. $G \in b(\mathscr{F})$.

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In this case $N = G^{\mathscr{F}}$ coincides with $\mathbf{S}(G)$, since N is the unique minimal normal subgroup of G; and we obviously have $N \notin \mathscr{X}$, whence (i) holds.

As $\mathscr{F} \subseteq \mathscr{X} * \mathscr{F}$, we may assume in the sequel that $M < N$ for each minimal normal subgroup M of G. In fact, $N \neq 1$, and if $M \nleqslant N$, then $N \cong N \times M/M = (G/M)^{\mathscr{F}} \in \mathscr{X}$, yielding the contradiction that $G \in \mathscr{X} * \mathscr{F}$.

Case 2. There exists an abelian minimal normal subgroup M of G.

From $N/M = G^{\mathscr{F}}M/M = (G/M)^{\mathscr{F}} \in \mathscr{X} = E_{\Phi}\mathscr{X}$, we get $\Phi(N) = 1$. Hence there is a complement X of M in N, and $X \cong N/M \in \mathscr{X}$. As $M \leqslant \mathbf{F}(N)$ and $\Phi(N) = 1$, M is a direct product of minimal normal subgroups T_i of N, $i = 1, \ldots, n$. Since $N \notin \mathscr{X}$, we can find an $L \trianglelefteq N$ such that $N/L \in b(\mathscr{X})$, while $N/M \in \mathscr{X}$ forces $M \nleqslant L$. Therefore $T_i \nleqslant L$ for at least one $i \in \{1, \ldots, n\}$. Setting $T = T_i$, $N/L \in \mathscr{P}$ together with the fact that TL/L is an abelian minimal normal subgroup of N/L gives $N/L \in \mathscr{P}_I$ and $\mathbf{S}(N/L) = T \times L/L$. It is easily seen that $N/L \cong \mathrm{Aut}_N(T)T \cong (X/\mathbf{C}_X(T))T$. Consequently,

$$\big(X/\mathbf{C}_X(T^g)\big)T^g \cong N/L^g \cong N/L \in b(\mathscr{X}) \cap \mathscr{P}_{\mathrm{I}} \quad \text{for each } g \in G,$$

and as M is a direct product of G-conjugates of T, we conclude that $X \in \mathrm{Proj}_{\mathscr{X}}(N)$. Having chosen N/L as an arbitrary factor group of N in $b(\mathscr{X})$, we see that to within isomorphism N/L is the unique element of [the isomorphism class] $Q\{N\} \cap b(\mathscr{X})$. Suppose that G possesses a minimal normal subgroup $M^* \neq M$. Then by means of the same arguments as previously (with M^* in place of M), we obtain that X complements M^* in N too. Hence one can easily derive the following contradiction, where $\Phi(X) \cap (M \times M^*) \leqslant \Phi(N) = 1$ has to be taken into account:

$$\mathscr{X} \ni X = Y\left[X \cap (M \times M^*)\right] \cong YM \text{ is a split extension for suitable } Y \leqslant X,$$

but

$$\big(X/\mathbf{C}_X(T)\big)T \cong \big(Y/\mathbf{C}_Y(T)\big)T \in b(\mathscr{X}) \text{ is isomorphic to a factor group of } X.$$

We have shown that $\mathbf{S}(G) = M$ is minimal normal in G, and we infer that $\mathrm{Core}_G(X) = 1$. In particular, $\mathbf{C}_X(M) = 1$: otherwise from $\mathbf{C}_X(M) \trianglelefteq XM = N$ and the fact that $\mathbf{C}_X(M)$ does not contain a characteristic subgroup of N one would get that $\mathbf{C}_N(M) = \mathbf{C}_X(M) \times M \trianglelefteq G$ is a direct product of minimal normal subgroups of N N-isomorphic to $\mathrm{Aut}(N)$-conjugates of T, which is readily seen to contradict $X \in \mathrm{Proj}_{\mathscr{X}}(N)$. This completes the proof of the first part of statement (ii), whereas the proof of the second part, being an easy consequence of the Frattini argument and the results of Section 4 of [5], is left to the reader.

For the remaining part of the proof we are permitted to assume that $\mathbf{F}(G) = 1$.

Case 3. $\mathbf{S}(G)$ is a non-abelian minimal normal subgroup of G.

Proceeding as in Case 2, we choose an arbitrary $L \trianglelefteq N$ with $N/L \in b(\mathfrak{X})$ and an arbitrary minimal normal subgroup T of N. Since $\Phi(N) = 1$ and $\mathbf{S}(G) \leqslant N$, we get $T \leqslant \mathbf{S}(N) = \mathbf{S}(G)$, and we note that G permutes the elements of $\{T_1, \ldots, T_n\}$ (that is to say, the minimal normal subgroups of N) transitively. If $N/L \in \mathscr{P}_{\mathrm{I}}$, an argument as above yields that $\mathbf{S}(N/L) = T^g \times L/L$ for some $g \in G$, whence $L = \mathbf{C}_N(T^g)$. Now it is readily seen that the situation is as described in (iii.a). If $N/L \in \mathscr{P}_{\mathrm{III}}$, a similar argument shows that $N/L = (XL/L)(T^g \times L/L)$ is a split extension for each $X \in \mathrm{Proj}_{\mathfrak{X}}(N)$, giving rise to the situation described in (iii.b). Finally, we observe that $N/L \in \mathscr{P}_{\mathrm{I}}$ is impossible, as $T^g \leqslant \mathbf{S}(G)$ (with suitable $g \in G$) is isomorphic to a minimal normal subgroup of N/L and is non-abelian; and $G \in \mathscr{P}_{\mathrm{II}}$ follows from the fact that $\mathbf{S}(G)$ is a non-abelian minimal normal subgroup of G.

To complete the proof we have to handle the following case:

Case 4. $\mathbf{F}(G) = 1$ and $\mathbf{S}(G) = M_1 \times \cdots \times M_m$ $(m \geqslant 2)$, where $M_1, \ldots, M_m$ denote the minimal normal subgroups of G.

Again we consider an arbitrary $N/L \in b(\mathfrak{X})$, and we recall that $1 \neq M_iL/L \leqslant \mathbf{S}(N/L)$, which is a product of not more than two minimal normal subgroups of N/L. We cannot possibly have $\mathbf{S}(N/L) = M_iL/L$, since then $M_j \leqslant \mathbf{C}_N(M_i) \leqslant \mathbf{C}_N(\mathbf{S}(N/L)) = L$ for any $j \neq i$, a contradiction. Therefore M_iL/L is minimal normal in N/L. If $j \neq i$, then $M_j \leqslant \mathbf{C}_N(M_i) \leqslant \mathbf{C}_N(M_iL/L)$, the latter group being just the other minimal normal subgroup of $N/L \in \mathscr{P}_{\mathrm{III}}$, when taken modulo L. In particular, from $M_j \nleqslant L$ we infer that $\{M_iL/L, M_jL/L\}$ is the set of all minimal subgroups of N/L whenever $i, j \in \{1, \ldots, m\}$ are such that $i \neq j$. We may conclude that firstly, $m = 2$, and secondly, $\mathbf{S}(N/L) = (T_1 \times L/L) \times (T_2 \times L/L)$, where T_i is a minimal normal subgroup of N contained in M_i; indeed, as previously, $\mathbf{S}(G) = \mathbf{S}(N)$, and M_i is a direct product of minimal normal subgroups $T_j^{(i)}$ of N, $i = 1, 2$, $j = 1, \ldots, n_i$. Moreover, each $T_j^{(i)}$ is N-isomorphic to a minimal normal subgroup of some $N/L \in b(\mathfrak{X})$, and for any $X \in \mathrm{Proj}_{\mathfrak{X}}(N)$ we have $\{(X/\mathbf{C}_X(T_j^{(i)}))T_j^{(i)}\} = Q\{N\} \cap b(\mathfrak{X}) \subseteq \mathscr{P}_{\mathrm{III}}$ $(i = 1,2;\ j = 1, \ldots, n_i)$; here it is to be observed that a primitive group of type III in the boundary of a Schunck class $\mathfrak{X}$ is isomorphic to the split extension of any one of its $\mathfrak{X}$-projectors and of its minimal normal subgroups. Finally, from $N/M_i \in \mathfrak{X}$ $(i = 1, 2)$ we get $N = XM_i$ for each $X \in \mathrm{Proj}_{\mathfrak{X}}(N)$ and, in consequence, $X \cap M_i \trianglelefteq XM_{3-i} = N$. Since for each $T_i^{(i)}$ we can find $L_{ij} \trianglelefteq N$ such that $N/L_{ij} \in b(\mathfrak{X})$ and $T_i^{(j)} \times L_{ij}/L_{ij}$ is minimal normal in N/L_{ij}, $T_i^{(j)} \nleqslant X \cap M_i$: otherwise we should have $N/L_{ij} = (XL_{ij}/L_{ij})(T_i^{(j)} \times L_{ij})/L_{ij} = XL_{ij}/L_{ij} \in Q\{X\} \subseteq \mathfrak{X}$. Therefore we deduce that $X \cap M_i = 1$ $(i = 1, 2)$. This eventually leads to the conclusion $\mathbf{C}_N(M_i) = \mathbf{C}_N(M_i) \cap XM_{3-i} = (\mathbf{C}_N(M_i) \cap X)M_{3-i} = M_{3-i}$.

REMARK. *Requiring that* $b(\mathscr{X}) \cap \mathscr{P}_{\mathrm{III}} = \varnothing$ *excludes cases* (iii.b, iv.a, iv.b). *If, in addition,* $\mathscr{F}$ *is saturated, then G is primitive unless* (ii.a) *applies.*

Applying the main result of Section 6 of [5], we obtain the promised generalization of results of Gaschütz [6] (dealing with a Schunck class $\mathscr{X}$ and a formation $\mathscr{X}$ in the universe of all finite soluble groups) and Erickson [4] (where $\mathscr{X}$ and $\mathscr{F}$ are saturated formations in the universe of all finite groups); Erickson's result also appears in the as yet unpublished manuscript of the book by Doerk and Hawkes on "Finite Soluble Groups". Hypothesis (1) of the following Corollary 1 is satisfied in case of Gaschütz's result, whereas hypothesis (2) is satisfied by saturated formations (as is obvious from the well-known structure of primitive groups of type I in the boundary of a saturated and thus local formation).

COROLLARY 1. *Let* $\mathscr{X}$ *be a Schunck class such that either*

(1) $\mathscr{X}$*-projectors form a set of conjugate subgroups in every group and* $\mathscr{V}$ *satisfies the hypothesis of* [5; 6.1], *or*

(2) $b(\mathscr{X}) \cap \mathscr{P}_{\mathrm{III}} = \varnothing$ *and* $\mathscr{X}$*-projectors form a set of conjugate subgroups in each group from* $b(\mathscr{X}) \cap \mathscr{P}_{\mathrm{I}}$.

Suppose further that $\mathscr{F}$ *is a formation which is either saturated or satisfies* $\mathscr{E}_p\mathscr{F} = \mathscr{F}$ *for all primes p such that* $\mathscr{E}_p \not\subseteq \mathscr{X}$. *Then* $\mathscr{X} * \mathscr{F}$ *is a Schunck class.*

PROOF. As $\mathscr{X} * \mathscr{F}$ is a homomorph, it is a Schunck class if and only if $b(\mathscr{X} * \mathscr{F}) \subseteq \mathscr{P}$. Therefore we consider $G \in b(\mathscr{X} * \mathscr{F})$ and apply our Theorem together with the above Remark. We are done unless case (i) or (ii.a) applies. In case (i), however, $G \in b(\mathscr{F})$ is primitive: If $\mathscr{F}$ is a saturated formation, then it is a Schunck class; if $\mathscr{E}_p\mathscr{F} = \mathscr{F}$ for all primes p such that $\mathscr{E}_p \not\subseteq \mathscr{X}$, then the unique minimal normal subgroup of $G \in b(\mathscr{X} * \mathscr{F}) \cap b(\mathscr{F})$ is certainly not abelian, whence $G \in \mathscr{P}_{\mathrm{II}}$ is primitive. Finally, case (ii.a) is excluded by (1) or (2), as the maximal subgroups of $(X/\mathrm{C}_X(T))T$ with trivial core are precisely the $\mathscr{X}$-projectors of this group from $b(\mathscr{X})$—here the notation is as in (ii.a) of the Theorem.

We would point out that the question of whether $(\mathscr{X} * \mathscr{F})$-projectors form a single conjugacy class of subgroups in every group (given that a corresponding property is enjoyed by both $\mathscr{X}$ and $\mathscr{F}$) seems to be rather intractable. More specifically, it is not quite clear in general, which subgroups of a group $G \in b(\mathscr{X} * \mathscr{F}) \cap \mathscr{P}_{\mathrm{II}}$ might belong to $\mathscr{X} * \mathscr{F}$ and supplement $\mathrm{S}(G)$. The situation is different in case that $\mathscr{X} * \mathscr{F}$ coincides with $\mathscr{X}\mathscr{F}$, the class of all groups possessing a normal subgroup in $\mathscr{X}$ such that the corresponding factor group is in $\mathscr{F}$. (The latter class, however, is not necessarily a Schunck class, and may contain $\mathscr{X} * \mathscr{F}$

properly.) The next corollaries, generalizing results of Lausch [8, 9], Beidleman and Brewster [2, 3], exploit this observation.

COROLLARY 2. *Let $\mathscr{V}$ be the class of π-separated groups, $\mathscr{X}$ a Schunck class of π-groups and $\mathscr{Y}$ a Schunck class of π'-groups such that* $\mathrm{Proj}_{\mathscr{X}}(G)$ *and* $\mathrm{Proj}_{\mathscr{Y}}(G)$, *respectively, form a set of conjugate subgroups of each π-group and π'-group G. Then $\mathscr{X}\mathscr{Y}$ is a Schunck class with the same property in $\mathscr{U}$, the class of all π-soluble groups.*

PROOF. The proof is a simple application of our Theorem and the following elementary observations. (*) $\mathscr{X}\mathscr{Y} = (\mathscr{X} * \mathscr{E}_{\pi'}) \cap \mathscr{G}$, where $\mathscr{E}_{\pi'}$ is the class of all π'-groups and $\mathscr{G}$ is the class of all groups G such that $G/G^{\mathscr{E}_{\pi'}} \in \mathscr{Y}$. (In fact, the class $\mathscr{G}$ of all groups such that $G/G^{\mathscr{F}} \in \mathscr{H}$, $\mathscr{F} \supseteq \mathscr{H}$ a formation and $\mathscr{H}$ a homomorph, is a homomorph satisfying $\mathscr{G} \cap \mathscr{F} = \mathscr{H}$ and $b(\mathscr{G}) = b(\mathscr{H}) \cap \mathscr{F}$, and is a Schunck class if $\mathscr{H}$ is.)

(**) $\qquad b(\mathscr{H} \cap \mathscr{K}) \subseteq b(\mathscr{H}) \cup b(\mathscr{K})$ for all homomorphs $\mathscr{H}$ and $\mathscr{K}$.

Of course, apart from (*) and (**), the results of [5] (in particular, the main result of Section 5 is important in this context) are to be used here; the Frattini argument is useful, too. The (rather difficult) proof of the statement related to conjugacy of projectors relies on the results and methods contained in the sequel to [5], and will not be given here.

By means of Corollary 2 we get that the following class, being an intersection of two Schunck classes, is a Schunck class whenever $\mathscr{X}$ and $\mathscr{Y}$ satisfy the hypotheses of Corollary 2:

$$\begin{aligned}\mathscr{X} \times \mathscr{Y} &= \{G \in \mathscr{E} \mid G = \mathbf{O}_\pi(G) \times \mathbf{O}_{\pi'}(G), \mathbf{O}_\pi(G) \in \mathscr{X}, \mathbf{O}_{\pi'}(G) \in Y\} \\ &= \mathscr{E}_{\pi'}\mathscr{X} \cap \mathscr{E}_\pi \mathscr{Y}.\end{aligned}$$

Moreover, a brief glance at the structure of groups in $b(\mathscr{X} \times \mathscr{Y})$ shows that we get conjugacy of $(\mathscr{X} \times \mathscr{Y})$-projectors in all $G \in \mathscr{V}$, that is, we do not require G to be π-soluble (or π'-soluble). (Clearly, by restricting attention to $\mathscr{U}$, it is easier to ensure that the hypotheses of the following Corollary 3 are satisfied, a fact used in the papers of Lausch and Beidleman.)

COROLLARY 3. *Let $\mathscr{X}$, $\mathscr{Y}$, and $\mathscr{V}$ be as in Corollary 2. Then $\mathscr{X} \times \mathscr{Y}$ is a Schunck class with* $\mathrm{Proj}_{\mathscr{X}\times\mathscr{Y}}(\mathscr{G}) = \{Z^g \mid g \in G\}$ *(Z any $\mathscr{X} \times \mathscr{Y}$-projector of G) for each $G \in \mathscr{V}$. In addition,* $\mathrm{Proj}_{\mathscr{X}\times\mathscr{Y}} = \mathrm{Cov}_{\mathscr{X}\times\mathscr{Y}}$, *provided that* $\mathrm{Proj}_{\mathscr{Z}} = \mathrm{Cov}_{\mathscr{Z}}$ *for $\mathscr{Z} = \mathscr{X}, \mathscr{Y}$.*

Finally, it is easy to derive various descriptions of projectors and/or covering subgroups for the Schunck classes discussed here by means of the results in Section 3 of [5].

By inspection of the conclusions of our Theorem, further modifications on the hypotheses of $\mathscr{H}$ and $\mathscr{F}$ ensuring $\mathscr{H} * \mathscr{F}$ to be a Schunck class are readily found. We shall conclude the discussion of the above Theorem by exhibiting examples to show that each of the cases listed in the Theorem can occur.

EXAMPLES. (i) $\mathscr{F}$ any formation, $\mathscr{X} = \{1\}$; $G \in b(\mathscr{F})$.

(ii.a) $\mathscr{F} = \mathscr{E}_p$, the class of all p-groups (or $\mathscr{F} = \mathscr{N}$, the class of all nilpotent groups), $\mathscr{X} = \mathscr{N}' = \{G \in \mathscr{E} | \mathbf{F}'(G) = G\}$, the class of "generalized nilpotent groups" (this class was shown to be a Schunck class in 2.5 of [5]—recall that $\mathbf{F}'(G)$, by definition, is $\mathbf{S}(G \bmod \Phi(G))$; $G = EV$, a split extension, where E is a non-abelian simple group, and V is a $GF(p)[E]$-module (for a prime p dividing $|E|$) such that $V/\mathrm{Rad}(V)$ is an irreducible, trivial E-module and $\mathrm{Rad}(V)$ is an irreducible, faithful E-module. Observe that $G^{\mathscr{F}} = \mathbf{O}^p(G) = E\,\mathrm{Rad}(V)$, $\Phi(G) = \mathrm{Rad}(V)$, $G^{\mathscr{F}}/\Phi(G) \cong E \in \mathscr{X}$, $E\Phi(G) \notin \mathscr{X}$, and $\Phi(G)$ is the unique minimal normal subgroup of G contained in V. (An example of such a group G is discussed in 5.7 of [5]; in fact, it is easy to see that such a module V over $GF(p)$ exists for each non-abelian simple group E and each of the prime divisors p of $|E|$.)

(ii.b) $\mathscr{F} = \{1\}$, $\mathscr{X}$ any Schunck class with $b(\mathscr{X}) \cap \mathscr{P}_{\mathrm{I}} \neq \varnothing$; $G \in b(\mathscr{X}) \cap \mathscr{P}_{\mathrm{I}}$.

(iii.a) $\mathscr{F} = \{1\}$, $\mathscr{X}$ any Schunck class with $b(\mathscr{X}) \cap \mathscr{P}_{\mathrm{II}} \neq \varnothing$; $G \in b(\mathscr{X}) \cap \mathscr{P}_{\mathrm{II}}$.

(iii.b) $\mathscr{F} = \mathscr{E}_p$, $\mathscr{X}$ the Schunck class defined by $b(\mathscr{X}) = \{E \times E\}$ for a non-abelian simple group E (i.e., $\mathscr{X} = h(E \times E) = \{G \in \mathscr{E} | Q\{G\} \cap \{E \times E\} = \varnothing\}$); $G = E \mathrm{wr} C_p$, the regular wreath product of E and a cyclic group of order p.

(iv.a) $\mathscr{F} = \{1\}$, $\mathscr{X}$ any Schunck class with $b(\mathscr{X}) \cap \mathscr{P}_{\mathrm{III}} \neq \varnothing$; $G \in b(\mathscr{X}) \cap \mathscr{P}_{\mathrm{III}}$.

(iv.b) $\mathscr{F} = \mathscr{S}$, $\mathscr{X} = h(E \times E)$ for some non-abelian simple group E such that $\mathrm{Inn}(E) < \mathrm{Aut}(E)$; $G = H_1 \times H_2$ where $\mathrm{Inn}(E) < H_i \leqslant \mathrm{Aut}(E)$ (and $H_i/\mathrm{Inn}(E) \in \mathscr{S}$).

Observe that in the last example $G^{\mathscr{F}} = E \times E = \mathbf{S}(G)$ and $\mathbf{C}_G(E \times 1) = 1 \times H_2$, $\mathbf{C}_G(1 \times E) = H_1 \times 1$, whence none of the centralizers of the minimal normal subgroups of G is contained in $G^{\mathscr{F}}$. We are not aware of any example for case (iv.b) satisfying $\mathbf{C}_G(M_i) = M_{3-i}$ for $i = 1, 2$, where M_1 and M_2 denote the minimal normal subgroups of G. Further, we do not know whether this assumption, forcing $G/M_i \in \mathscr{P}_{\mathrm{II}}$ for $i = 1, 2$, leads to the conclusion that $G/M_1 \cong G/M_2$; note that $G^{\mathscr{F}}/M_1 \cong G^{\mathscr{F}}/M_2 \cong X$ whenever $X \in \mathrm{Proj}_{\mathscr{X}}(G^{\mathscr{F}})$. Anyhow, even if both of these assertions are assumed to hold, we are not able to show that $G \in \mathscr{P}$ (that is to say, $G \in \mathscr{P}_{\mathrm{III}}$) follows, thus ruling out case (iv.b) when this special situation occurs. This last question, however, is related to a seemingly interesting problem concerning the primitivity of certain groups—a problem not referring to the notion of a Schunck class (albeit an affirmative solution could be applied to the theory of Schunck classes; cf. Chapter I of [1] or [7]).

Problem. *Let G be a group with* $\mathrm{S}(G) = M_1 \times M_2$, *$M_i$ minimal normal in G, such that $G/M_1 \cong G/M_2 \in \mathscr{P}_{\mathrm{II}}$. Is G necessarily primitive* (*of type* III)?

ADDED IN PROOF. After this paper had been submitted, L. G. Kovács showed us a negative solution to the problem discussed above; an example will appear in Chapter I of [1].

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