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Canonical Vector Heights on Algebraic K3 Surfaces with Picard Number Two

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Abstract. Let V be an algebraic K3 surface defined over a number field K. Suppose V has Picard number two and an infinite group of automorphisms $\mathcal{A} = \operatorname{Aut}(V/K)$. In this paper, we introduce the notion of a vector height $\mathbf{h}\colon V \to \operatorname{Pic}(V) \otimes \mathbb{R}$ and show the existence of a canonical vector height $\widehat{\mathbf{h}}$ with the following properties:

$$\widehat{\mathbf{h}}(\sigma P) = \sigma_* \widehat{\mathbf{h}}(P)$$

$$h_D(P) = \widehat{\mathbf{h}}(P) \cdot D + O(1),$$

where $\sigma \in \mathcal{A}$, σ_* is the pushforward of σ (the pullback of σ^{-1}), and h_D is a Weil height associated to the divisor D. The bounded function implied by the $O(1)$ does not depend on P. This allows us to attack some arithmetic problems. For example, we show that the number of rational points with bounded logarithmic height in an $\mathcal{A}$-orbit satisfies

$$N_{\mathcal{A}(P)}(t, D) = \#\{Q \in \mathcal{A}(P) : h_D(Q) < t\} = \frac{\mu(P)}{s \log \omega} \log t + O\Big(\log\big(\widehat{\mathbf{h}}(P) \cdot D + 2\big)\Big).$$

Here, $\mu(P)$ is a nonnegative integer, s is a positive integer, and ω is a real quadratic fundamental unit.

In [S], Silverman demonstrated the existence of canonical heights on the class of K3 surfaces generated by the smooth intersection of a $(1,1)$ form and a $(2,2)$ form in $\mathbb{P}^2 \times \mathbb{P}^2$ and with Picard number two. These K3 surfaces have an infinite group of automorphisms. In this paper, we generalize his results to all algebraic K3 surfaces defined over a number field K which have Picard number two and an infinite group of automorphisms $\mathcal{A} = \operatorname{Aut}(V/K)$. These are the K3 surfaces V with Picard number two which contain no curves with self-intersection equal to 0 or -2 and for which the number field K is sufficiently large.

Our main results are in Sections 1 and 3. In Section 1, we introduce the notion of a vector height

$$\mathbf{h}\colon V \to \operatorname{Pic}(V) \otimes \mathbb{R}$$

and analyze its properties. In Section 3, we prove the following:

Theorem 3.1 *Let V be a* K3 *surface over a number field K. Suppose that V has Picard number two and an infinite group of automorphisms $\mathcal{A} = \operatorname{Aut}(V/K)$. Then there exists a canonical vector height $\widehat{\mathbf{h}}$ such that*

$$\widehat{\mathbf{h}}(\sigma P) = \sigma_* \widehat{\mathbf{h}}(P)$$

for any $\sigma \in \mathcal{A}$.

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Given a Weil height h_D, we can relate it to the canonical vector height via the relation

$$h_D(P) = \widehat{\mathbf{h}}(P) \cdot D + O(1),$$

where the bound on the function implied by the $O(1)$ is independent of P. We use this and Theorem 3.1 to show the following two arithmetic results:

Theorem 3.2 *Suppose V is a* K3 *surface with Picard number two and an infinite group of automorphisms $\mathcal{A}$. Then there are only a finite number of K-rational points $P \in V$ such that the $\mathcal{A}$-orbit of P is finite.*

Theorem 3.2 *Suppose V is a* K3 *surface with Picard number two and an infinite group of automorphisms $\mathcal{A}$. Let h_D be a Weil height on V associated to an ample divisor D. Then*

$$N_{\mathcal{A}(P)}(t, D) = \#\{Q \in \mathcal{A}(P) : h_D(Q) < t\} = \frac{\mu(P)}{s \log \omega} \log t + O\Big(\log\big(\widehat{\mathbf{h}}(P) \cdot D + 2\big)\Big),$$

where $\mu(P)$ is a nonnegative integer that depends on P, can take on only a finite number of values (for fixed V/K), and is zero if and only if $\mathcal{A}(P)$ is finite; s is a positive integer that depends on V and K; and ω is a real quadratic fundamental unit that depends on V.

Theorem 3.2 can be thought of as an analogue of a result due to Northcott [No]. Northcott showed that, if σ is a morphism from a variety V to itself defined over $\mathbb{P}^n(K)$ for some number field K, and if σ has degree at least 2, then there are only a finite number of points $P \in V$ such that the set $\{\sigma^k(P) : k \in \mathbb{Z}, k \geq 0\}$ is finite. Note that such a σ is not invertible.

Theorem 3.2 can also be thought of as an analogue of the finiteness of the torsion group for an elliptic curve defined over a number field. It is a rather nice application of the existence of a canonical height. The full power of a canonical vector height is not required to prove this type of result. On some K3 surfaces, it is possible to define a Weil height with respect to an ample divisor and an automorphism with infinite order, and to use this height to prove an analogue of Theorem 3.2. This was done, for example, by Wang [W] and Billard [Bi]. We will elaborate on the method in Section 4

There is some hope that an analogue of Theorem 3.2 might be true for all K3 surfaces (all varieties?) that have an infinite group of automorphisms. However, in Section 4, we give an example of a K3 surface with an infinite group $\mathcal{A}$ but for which this technique cannot be applied.

In contrast, an analogue of Theorem 3.3 cannot be derived using the canonical heights found in [W] and [Bi]. Such a result would follow from the existence of a canonical vector height, but I am doubtful that these objects even exist on K3 surfaces with Picard number $n \geq 3$, except perhaps in rare cases. We will elaborate on this thought too in Section 4.

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1 Vector Heights on Surfaces

Let V be an algebraic surface defined over a number field K. Then $\mathrm{Pic}(V) \otimes \mathbb{R}$ is a finite dimensional vector space. Let $\mathcal{D} = \{D_1, \dots, D_n\}$ be a basis for $\mathrm{Pic}(V) \otimes \mathbb{R}$, and let the intersection matrix be

$$J = J_{\mathcal{D}} = [D_i \cdot D_j].$$

By the Hodge index theorem, J has signature $(1, n-1)$. That is, J has one positive eigenvalue and $n-1$ negative eigenvalues. Let the *dual basis* of $\mathcal{D}$ be $\mathcal{D}^* = \{D_1^*, \dots, D_n^*\}$ where

$$D_i \cdot D_j^* = \delta_{ij}.$$

That is, $[D_i \cdot D_j^*]$ is the identity matrix. Let A be the change of basis matrix from $\mathcal{D}$ to $\mathcal{D}^*$, so

$$D_i^* = \sum_{k=1}^{n} A_{ik} D_k.$$

For clarity of exposition, let us adopt the Einstein convention that a product is summed over any index that appears twice. For example, using this convention, we would write

$$D_i^* = A_{ik} D_k.$$

Then

$$I = [D_i \cdot D_j^*] = [D_i \cdot (A_{jk} D_k)] = [A_{jk} J_{ik}] = A J_{\mathcal{D}}^t = A J_{\mathcal{D}}.$$

Thus, $A = J_{\mathcal{D}}^{-1}$. Note that $J_{\mathcal{D}}^{-1}$ exists (and hence $\mathcal{D}^*$ exists) since J has no eigenvalues equal to zero.

A morphism σ from V to V acts linearly on $\mathrm{Pic}(V) \otimes \mathbb{R}$ via its pull back σ^*. If σ is invertible, then its push forward $\sigma_* = (\sigma^{-1})^*$ exists. A function

$$\mathbf{h} \colon V \to \mathrm{Pic}(V) \otimes \mathbb{R}$$

is called a *vector height* on V if

$$\mathbf{h}(\sigma P) = \sigma_* \mathbf{h}(P) + \mathbf{O}(1)$$

for every element $\sigma \in \mathcal{A} = \mathrm{Aut}(V/K)$, and

$$h_D(P) = \mathbf{h}(P) \cdot D + O(1)$$

for any divisor D and any Weil height h_D associated to D. The symbol $\mathbf{O}(1)$ represents a vector function that is bounded independent of P, and the $O(1)$ notation represents a function bounded independent of P. In this section, our main result is to show that such vector heights exist and are unique up to bounded vector functions. We begin with a lemma:

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Lemma 1.1 *Let $\mathcal{D} = \{D_1, \ldots, D_n\}$ be a basis of* $\mathrm{Pic}(V) \otimes \mathbb{R}$ *and let $\mathcal{D}^*$ be its dual basis. Let $\sigma \in \mathcal{A}$. Then*

$$\sigma_* D_i^* = \sigma_{ji}^* D_j^*,$$

where the matrix $[\sigma_{ji}^*]$ represents σ^* in the basis $\mathcal{D}$ (i.e., $\sigma^* D_i = \sigma_{ij}^* D_j$).

Proof The action of σ_* on D_i^* is linear, so

$$\sigma_* D_i^* = A_{ij} D_j^*$$

for some matrix $A = [A_{ij}]$. Thus,

$$\sigma_* D_i^* \cdot D_j = A_{ik} D_k^* \cdot D_j = A_{ik} \delta_{kj} = A_{ij}.$$

On the other hand,

$$\sigma_* D_i^* \cdot D_j = D_i^* \cdot \sigma^* D_j = D_i^* \cdot \sigma_{jk}^* D_k = \sigma_{jk}^* \delta_{ik} = \sigma_{ji}^*.$$

Thus, $A_{ij} = \sigma_{ji}^*$, as desired. ■

Theorem 1.2 *Let V be an algebraic surface defined over a number field K. Then there exists a vector height $\mathbf{h}$ on V. Furthermore, if $\mathbf{h}'$ is another vector height on V, then*

$$\mathbf{h}(P) = \mathbf{h}'(P) + \mathbf{O}(1).$$

Proof Let $\mathcal{D} = \{D_1, \ldots, D_n\}$ be a basis of $\mathrm{Pic}(V) \otimes \mathbb{R}$ and let $\mathcal{D}^* = \{D_1^*, \ldots, D_n^*\}$ be its dual basis. For each divisor D_i, let h_{D_i} be a Weil height with respect to D_i, and define

$$\mathbf{h}(P) = h_{D_i}(P) D_i^*.$$

Then

$$\begin{aligned}
\mathbf{h}(\sigma P) &= h_{D_i}(\sigma P) D_i^* \\
&= \left(h_{\sigma^* D_i}(P) + O(1) \right) D_i^* \\
&= \left(h_{\sigma_{ij}^* D_j}(P) + O(1) \right) D_i^* \\
&= \left(\sigma_{ij}^* h_{D_j}(P) + O(1) \right) D_i^* \\
&= \left(h_{D_j}(P) + O(1) \right) \sigma_{ij}^* D_i^* \\
&= \sigma_* \mathbf{h}(P) + \mathbf{O}(1).
\end{aligned}$$

Now, suppose $D = a_i D_i$. Then

$$\begin{aligned}
\mathbf{h}(P) \cdot D &= h_{D_i}(P) D_i^* \cdot a_j D_j \\
&= h_{D_i}(P) a_j \delta_{ij} \\
&= h_{a_i D_i}(P) + O(1) \\
&= h_D(P) + O(1).
\end{aligned}$$

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Thus, $\mathbf{h}$ is a vector height, so vector heights exist. Finally, suppose $\mathbf{h}'$ is an arbitrary vector height. Then

$$h_{D_i}(P) = \mathbf{h}'(P) \cdot D_i + O(1).$$

On the other hand, expressing $\mathbf{h}'$ in terms of the basis $\mathcal{D}$,

$$\mathbf{h}'(P) = h_j(P) D_j^*,$$

we find

$$\mathbf{h}'(P) \cdot D_i = h_j(P) D_j^* \cdot D_i = h_i(P),$$

so

$$\mathbf{h}'(P) = \big(h_{D_i}(P) + O(1)\big) D_i^* = \mathbf{h}(P) + \mathbf{O}(1). \quad \blacksquare$$

2 Background

In this section, we review some results concerning K3 surfaces and the way their automorphisms act on the Picard group. Let V be a K3 surface defined over a number field K. Let V have Picard number n (≤ 20), let $\mathcal{D} = \{D_1, \dots, D_n\}$ be a basis of the lattice $\operatorname{Pic}(V)$, and let J be its intersection matrix with respect to this basis. Let $\mathcal{A} = \operatorname{Aut}(V/K)$ be its group of automorphisms and let

$$\mathcal{O} = \mathcal{O}(\mathbb{Z}) = \{T \in M_{n\times n}(\mathbb{Z}) : T^t J T = J\}.$$

Let D be an ample divisor, so $D \cdot D = k > 0$. Since the signature of J is $(1, n-1)$, the surface

$$\mathbf{x}^t J \mathbf{x} = k$$

is a (hyper)hyperboloid of two sheets, one of which contains D. Let us call that sheet $\mathcal{H}$ and let

$$\mathcal{O}^+ = \mathcal{O}^+(\mathbb{Z}) = \{T \in \mathcal{O} : T\mathcal{H} = \mathcal{H}\}.$$

Let $\mathcal{E}$ be the set of effective divisor classes in $\operatorname{Pic}(V)$. That is, $E \in \mathcal{E}$ if we can write

$$E = \sum_{i=1}^{m} a_i C_i$$

with $a_i \geq 0$ and C_i a divisor class that can be represented with a curve in V. Let

$$W = \{C \in \operatorname{Pic}(V) : C \cdot C \geq 0, C \cdot E \geq 0 \text{ for all } E \in \mathcal{E}\}.$$

It is clear that if $\sigma \in \mathcal{A}$, then $\sigma^*(W) = W$, so let us define

$$\mathcal{O}'' = \mathcal{O}''(\mathbb{Z}) = \{T \in \mathcal{O}^+ : TW = W\}.$$

If there are any -2 curves on V, then there exists a large subset of $\mathcal{O}^+$ that cannot be in $\mathcal{O}''$. For an element $C \in \operatorname{Pic}(V)$ such that $C \cdot C = -2$, the *reflection through* C is the map

$$R_C D = D + (C \cdot D) C.$$

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Note that R_C is in $\mathcal{O}$, since it preserves intersections. The Riemann-Roch theorem for a K3 surface V states

$$l(D) + l(-D) \geq \frac{1}{2} D \cdot D + 2,$$

and D is effective if $l(D) > 0$. Thus, for our divisor C with $C \cdot C = -2$, either C or $-C$ is effective (but not both). Since

$$R_C C = -C,$$

we have that $R_C \notin \mathcal{O}''$. Let $\mathcal{O}' = \mathcal{O}'(\mathbb{Z})$ be the subgroup of $\mathcal{O}$ generated by the reflections through -2 curves. Note that

$$T R_C T^{-1} = R_{TC}.$$

Hence, $\mathcal{O}'$ is a normal subgroup of $\mathcal{O}$. In [PS-S], Pjateckiĭ-Šapiro and Šafarevič show that the pullback map

$$\begin{aligned} \Phi\colon \operatorname{Aut}(V/\mathbb{C}) &\to \mathcal{O}'' \\ \sigma &\mapsto \sigma^* \end{aligned}$$

has a finite kernel and co-kernel, and that

$$\mathcal{O}'' \cong \mathcal{O}^+/\mathcal{O}'.$$

3 K3 Surfaces with Picard Number Two

The surface $\mathcal{H}$ may be thought of as a model of $(n-1)$-dimensional hyperbolic space, and $\mathcal{O}^+$ a discrete group of isometries on $\mathcal{H}$. For $n = 2$, $\mathcal{H}$ is one dimensional, so is isomorphic to the Euclidean line. There are only four possible types of discrete groups of isometries on the line: the trivial group; a group of order two generated by one reflection; an infinite group generated by a translation; and an infinite group generated by a translation and a reflection.

If the bilinear form

$$\mathbf{x}^t J \mathbf{x} = ax^2 + 2bxy + cy^2 = \frac{1}{a}\big((ax+by)^2 - (b^2 - ac)y^2\big)$$

represents zero (for $(x, y) \in \mathbb{Z}^2$), then $b^2 - ac = -\det J$ is a perfect square. In this case, it is clear that $\mathcal{O}^+$ is finite, since there are only a finite number of solutions on $\mathcal{H}$. If there exists a curve C on V with self intersection equal to zero, then this form represents zero. Conversely, if this bilinear form represents zero, then by the Riemann-Roch theorem, there exist curves on V with self intersection equal to zero. If no such curve exists, then $-\det J$ is not a perfect square and the bilinear form may be thought of as a multiple of the norm map over the real quadratic field $L = \mathbb{Q}[\sqrt{-\det J}]$:

$$\mathbf{x}^t J \mathbf{x} = \frac{1}{a} N(ax + by + y\sqrt{-\det J}).$$

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The ring of integers in L has an infinite group of units generated by -1 and its fundamental unit. If the norm of the fundamental unit is one, let ω be this unit; otherwise, let ω be its square. We can represent multiplication by ω by a matrix $T = T_\omega$. We note that $T \in \mathcal{O}^+(\mathbb{Q})$, but may not have integer entries. However, there exists a power m of ω such that ω^m is a unit in the order $\mathbb{Z}[\sqrt{-\det J}]$, so $T^m \in \mathcal{O}^+$. Conjugation is also a linear map and preserves the norm. The matrix R that represents conjugation followed by multiplication by -1 is a reflection in $\mathcal{O}^+(\mathbb{Q})$. Thus, $\mathcal{O}^+$ is a subgroup of finite index in the group $\langle R, T\rangle$.

If the bilinear form represents -2, then $\mathcal{O}^+$ contains a reflection through this -2 curve. By translating this -2 curve, we get another reflection. Their composition is a translation, so $\mathcal{O}'$ is an infinite group of finite index in $\mathcal{O}^+$ and $\mathcal{O}''$ is finite. Furthermore, again by the Riemann-Roch theorem, the bilinear form represents -2 if and only if there is a -2 curve on V. Thus, $\mathcal{O}''$ is infinite if and only if V contains no zero or -2 curves. In light of the result due to Pjateckiĭ-Šapiro and Šafarevič, $\mathcal{A}$ is infinite if K is sufficiently large.

We are now ready to prove the main results:

Theorem 3.1 *Let V be a* K3 *surface over a number field K. Suppose that V has Picard number two and an infinite group of automorphisms $\mathcal{A} = \mathrm{Aut}(V/K)$. Then there exists a canonical vector height $\widehat{\mathbf{h}}$ such that*

$$\widehat{\mathbf{h}}(\sigma P) = \sigma_* \widehat{\mathbf{h}}(P)$$

for any $\sigma \in \mathcal{A}$.

Proof Since $\mathcal{A}$ is infinite, the map

$$\Phi \colon \mathcal{A} \to \mathcal{O}''$$

has a finite kernel and cokernel. We can think of $\mathcal{A}$ as being generated by τ, the elements of the kernel of Φ, and possibly ρ, where τ^* is a translation and ρ^* is a reflection. There exist s and r such that $\tau^* = T^s$ and $\rho^* = T^r R$. Without loss of generality, we may choose $s > 0$, for otherwise, we may replace τ with τ^{-1}.

The eigenvalues of $T = T_\omega$ are ω and ω^{-1}. Let $E_+ \in \mathrm{Pic}(V) \otimes \mathbb{R}$ be an eigenvector with respect to ω. We note that TR is a reflection, so $TRTR = 1$ and hence, $R = TRT$. Applying this to E_+, we get

$$RE_+ = TRTE_+ = TR(\omega E_+) = \omega TRE_+.$$

Thus, RE_+ is an eigenvector of T with respect to the eigenvalue ω^{-1}. Let us set $E_- = RE_+$.

We define the canonical vector height on V to be

$$\widehat{\mathbf{h}}(P) = \widehat{h}_{E_+}(P)E_+^* + \widehat{h}_{E_-}(P)E_-^*,$$

where we use Tate's averaging method to define the coefficients

$$\widehat{h}_{E_\pm}(P) = \lim_{k\to\infty} \omega^{-sk} h_{E_\pm}(\tau^{\pm k}P).$$

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Here, h_{E_+} and h_{E_-} are any Weil heights associated to the divisors E_+ and E_-.

Let us first show that $\widehat{\mathbf{h}}(\sigma P) = \sigma_* \widehat{\mathbf{h}}(P)$. To do this, we will look at the composition of $\widehat{\mathbf{h}}$ with the generators of $\mathcal{A}$, the elements τ, ρ, and those in the kernel. It will be useful to recall that

$$\sigma_* = (\sigma^{-1})^*,$$
$$\sigma^* D \cdot D' = D \cdot \sigma_* D',$$

so

$$\sigma_* D^* = (\sigma^* D)^*. \tag{1}$$

For the composition of $\widehat{\mathbf{h}}$ with τ, we note that

$$\begin{aligned}\widehat{h}_{E_\pm}(\tau P) &= \lim_{k\to\infty} \omega^{-sk} h_{E_\pm}\big(\tau^{\pm k}(\tau P)\big)\\ &= \omega^{\pm s} \lim_{k\to\infty} \omega^{-s(k\pm 1)} h_{E_\pm}(\tau^{\pm k+1} P)\\ &= \omega^{\pm s} \widehat{h}_{E_\pm}(P),\end{aligned}$$

so

$$\widehat{\mathbf{h}}(\tau P) = \omega^s \widehat{h}_{E_+}(P) E_+^* + \omega^{-s} \widehat{h}_{E_-}(P) E_-^*.$$

By (1),

$$\tau_* E_+^* = (\tau^* E_+)^* = (\omega^s E_+)^* = \omega^s E_+^*.$$

Similarly, $\tau_* E_-^* = \omega^{-s} E_-^*$, so

$$\widehat{\mathbf{h}}(\tau P) = \tau_* \widehat{\mathbf{h}}(P).$$

For the composition of $\widehat{\mathbf{h}}$ with an element κ of $\ker \Phi$, we first observe that $\tau^k \kappa = \kappa' \tau^k$ for some κ' in the kernel which depends on the power k. Thus

$$\begin{aligned}\widehat{h}_{E_\pm}(\kappa P) &= \lim_{k\to\infty} \omega^{-ks} h_{E_\pm}(\tau^{\pm k} \kappa P)\\ &= \lim_{k\to\infty} \omega^{-ks} h_{E_\pm}(\kappa' \tau^{\pm k} P)\\ &= \lim_{k\to\infty} \omega^{-ks} \big(h_{(\kappa')^* E_\pm}(\tau^{\pm k} P) + O(1)\big)\\ &= \lim_{k\to\infty} \omega^{-ks} h_{E_\pm}(\tau^{\pm k} P) + \omega^{-ks} O(1)\\ &= \widehat{h}_{E_\pm}(P) + \lim_{k\to\infty} \omega^{-ks} O(1).\end{aligned}$$

Note that the bound implied by the $O(1)$ depends on κ' and hence on k, but since $\ker \Phi$ is finite, the bound can be chosen independent of k. Thus, the limit of the second term is zero, and hence

$$\widehat{\mathbf{h}}(\kappa P) = \widehat{\mathbf{h}}(P) = \kappa_* \widehat{\mathbf{h}}(P).$$

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For the composition of $\widehat{\mathbf{h}}$ with ρ (if such a ρ exists in $\mathcal{A}$), we first observe that $(\tau^k\rho)_*$ is a reflection, so $\big((\tau^k\rho)^2\big)_* = 1$. Thus

$$\tau^k\rho\tau^k\rho = \kappa,$$

an element of the kernel (which again depends on k). Hence,

$$\tau^k\rho = \kappa\rho^{-1}\tau^{-k},$$

and

$$\begin{aligned}
\widehat{h}_{E_\pm}(\rho P) &= \lim_{k\to\infty} \omega^{-ks} h_{E_\pm}(\tau^{\pm k}\rho P)\\
&= \lim_{k\to\infty} \omega^{-ks} h_{E_\pm}(\kappa\rho^{-1}\tau^{\mp k}P)\\
&= \lim_{k\to\infty} \omega^{-ks}\big(h_{\kappa^* E_\pm}(\rho^{-1}\tau^{\mp k}P) + O(1)\big)\\
&= \lim_{k\to\infty} \omega^{-ks}\big(h_{\rho^* E_\pm}(\tau^{\mp k}P) + O(1)\big)\\
&= \lim_{k\to\infty} \omega^{-ks}\big(h_{T^r R E_\pm}(\tau^{\mp k}P) + O(1)\big)\\
&= \lim_{k\to\infty} \omega^{-ks}\big(h_{T^r E_\mp}(\tau^{\mp k}P) + O(1)\big)\\
&= \lim_{k\to\infty} \omega^{-ks}\big(\omega^{\pm r} h_{E_\mp}(\tau^{\mp k}P) + O(1)\big)\\
&= \omega^{\pm r}\widehat{h}_{E_\mp}(P).
\end{aligned}$$

Thus,

$$\begin{aligned}
\widehat{\mathbf{h}}(\rho P) &= \omega^r \widehat{h}_{E_-}(P)E_+^* + \omega^{-r}\widehat{h}_{E_+}(P)E_-^*\\
&= T^r R\widehat{\mathbf{h}}(P)\\
&= \rho_*\widehat{\mathbf{h}}(P).
\end{aligned}$$

To verify that $\widehat{\mathbf{h}}$ is a vector height, we must show

$$h_D(P) = \widehat{\mathbf{h}}(P)\cdot D + O(1)$$

for any Weil height h_D. We do this by comparing $\widehat{\mathbf{h}}$ with the vector height $\mathbf{h}$ where

$$\mathbf{h}(P) = h_{E_+}(P)E_+^* + h_{E_-}(P)E_-^*.$$

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Observe that

$$\begin{aligned}\widehat{h}_{E_\pm}(P) &= \lim_{k\to\infty} \omega^{-ks} h_{E_\pm}(\tau^{\pm k}P)\\ &= \lim_{k\to\infty} \omega^{-ks}\big(h_{(\tau^{\pm 1})^* E_\pm}(\tau^{\pm(k-1)}P) + O(1)\big)\\ &= \lim_{k\to\infty} \omega^{-ks}\big(\omega^{s} h_{E_\pm}(\tau^{\pm(k-1)}P) + O(1)\big)\\ &= \lim_{k\to\infty} \omega^{-ks}\big(\omega^{2s} h_{E_\pm}(\tau^{\pm(k-2)}P) + \omega^{s}O(1) + O(1)\big)\\ &= \lim_{k\to\infty} \omega^{-ks}\big(\omega^{ks} h_{E_\pm}(P) + \omega^{(k-1)s}O(1) + \cdots + O(1)\big).\end{aligned}$$

The functions implied by the $O(1)$ at each step are all the same, except that they are evaluated at different points. Since the bound on these functions are independent of the point of evaluation, these terms can be combined to give

$$\begin{aligned}\widehat{h}_{E_\pm}(P) &= \lim_{k\to\infty}\left(h_{E_\pm}(P) + \frac{\omega^{sk}-1}{\omega^{sk}(\omega^{s}-1)}O(1)\right)\\ &= h_{E_\pm}(P) + O(1).\end{aligned}$$

Thus,

$$\widehat{\mathbf{h}}(P) = \mathbf{h}(P) + \mathbf{O}(1),$$

and for any Weil height h_D, we have

$$h_D(P) = \mathbf{h}(P)\cdot D + O(1) = \big(\widehat{\mathbf{h}}(P) + \mathbf{O}(1)\big)\cdot D + O(1) = \widehat{\mathbf{h}}(P)\cdot D + O(1). \qquad \blacksquare$$

As mentioned in the introduction, an obvious application of the existence of a canonical height is the following result:

Theorem 3.2 *Suppose V is a* K3 *surface with Picard number two and an infinite group of automorphisms $\mathcal{A}$. Then there are only a finite number of K-rational points $P \in V$ such that the $\mathcal{A}$-orbit of P is finite.*

Proof Suppose $\widehat{\mathbf{h}}(P) \neq \mathbf{0}$. Then the elements $\{T^{sk}\widehat{\mathbf{h}}(P)\}$ for $k \in \mathbb{Z}$ are all distinct (recall, T is a translation on $\mathcal{H}$). But $\{T^{sk}\widehat{\mathbf{h}}(P)\} = \{\widehat{\mathbf{h}}(\tau^k P)\}$, so the set $\{\tau^k P\}$ is an infinite set. Thus, the $\mathcal{A}$-orbit of P is infinite. On the other hand, if $\widehat{\mathbf{h}}(P) = \mathbf{0}$, then $\widehat{\mathbf{h}}(\sigma P) = \sigma_*\widehat{\mathbf{h}}(P) = \mathbf{0}$ for all $\sigma \in \mathcal{A}$. Hence, for any ample divisor D, $h_D(\sigma P)$ is bounded. There are only a finite number of points with bounded height, so the $\mathcal{A}$-orbit of P must be finite. $\blacksquare$

We also have the following application of the existence of a canonical vector height:

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Theorem 3.3 *Suppose V is a* K3 *surface with Picard number two and an infinite group of automorphisms $\mathcal{A}$. Let h_D be a Weil height on V associated to an ample divisor D. Then*

$$N_{\mathcal{A}(P)}(t, D) = \#\{Q \in \mathcal{A}(P) : h_D(Q) < t\} = \frac{\mu(P)}{s \log \omega} \log t + O\Big(\log\big(\widehat{\mathbf{h}}(P) \cdot D + 2\big)\Big),$$

where $\mu(P)$ is a nonnegative integer that depends on P, can take on only a finite number of values (for fixed V/K), and is zero if and only if $\mathcal{A}(P)$ is finite; s is a positive integer that depends on V and K; and ω is a real quadratic fundamental unit that depends on V.

We can be a bit more precise about the values of $\mu(P)$, s, and ω. Recall, ω^s is the largest eigenvalue of τ^* where τ generates the torsion-free portion of $\mathcal{A}$. Let the stabilizer of P be

$$\operatorname{Stab}(P) = \{\sigma \in \mathcal{A} : \sigma P = P\}.$$

Then, $\mu(P) = 0$ if $\widehat{\mathbf{h}}(P) = \mathbf{0}$, and

$$\mu(P) = \frac{2\delta|\ker \Phi|}{\operatorname{Stab}(P)}$$

otherwise. Here, $|\ker \Phi|$ is the order of the kernel of Φ, and $\delta = 2$ or 1, depending on, respectively, whether there does or does not exist a $\rho \in \mathcal{A}$ such that ρ^* is a reflection.

Proof If $\widehat{\mathbf{h}}(P) = \mathbf{0}$ then $\mathcal{A}(P)$ is finite by Theorem 3.2. Thus, $\mu(P) = 0$ and $N_{\mathcal{A}(P)}(t, D) = O(1) = O\big(\log(2)\big)$.

For $\widehat{\mathbf{h}}(P) \neq \mathbf{0}$, let us first suppose that $\sigma \in \operatorname{Stab}(P)$. Then $\widehat{\mathbf{h}}(P) = \widehat{\mathbf{h}}(\sigma P) = \sigma_* \widehat{\mathbf{h}}(P)$, so 1 is an eigenvalue of σ_*. Hence, σ_* is either the identity or a reflection. Note that $\widehat{\mathbf{h}}(P)$ can be an eigenvector of at most one reflection, so the order of $\operatorname{Stab}(P)$ divides $2|\ker \Phi|$. Note that

$$\operatorname{Stab}(\sigma P) = \sigma \operatorname{Stab}(P) \sigma^{-1},$$

so the stabilizers of every element in the $\mathcal{A}$-orbit of P all have the same number of elements.

Let us now count the number of $\sigma \in \mathcal{A}$ such that

$$h_D(\sigma P) < t.$$

We can write $\sigma = \tau^k \rho^\epsilon \kappa$, where $k \in \mathbb{Z}$, $\epsilon = 0$ or 1 (if such a ρ is in $\mathcal{A}$), and $\kappa \in \ker \Phi$. Let us begin with $\epsilon = 0$ and $\kappa = 1$. Then

$$\begin{aligned} h_D(\tau^k P) &= \widehat{\mathbf{h}}(\tau^k P) \cdot D + O(1) \\ &= \tau_*^k \widehat{\mathbf{h}}(P) \cdot D + O(1) \\ &= \omega^{sk} \widehat{h}_+(P) E_+ \cdot D + \omega^{-sk} \widehat{h}_-(P) E_- \cdot D + O(1). \end{aligned}$$

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If $k > 0$, then
$$h_D(\tau^k P) = w^{sk}\widehat{h}_+(P)(E_+ \cdot D) + O(1).$$
Note that $E_+ \cdot D > 0$, since D is ample. Thus, if $h_D(\tau^k P) < t$, then

$$\omega^{sk}\widehat{h}_+(P)(E_+ \cdot D) + O(1) < t$$
$$sk\log\omega + O\Big(\log\big(\widehat{h}_+(P)E_+ \cdot D\big)\Big) + O(1) < \log t$$
$$k < \frac{\log t}{s\log\omega} + O\Big(\log\big(\widehat{h}_+(P)E_+ \cdot D\big)\Big) + O(1).$$

Since D is ample, and $\widehat{h}_\pm(P) \geq 0$ for all P, we can replace $O\Big(\log\big(\widehat{h}_+(P)E_+ \cdot D\big)\Big) + O(1)$ with $O\Big(\log\big(\widehat{\mathbf{h}}(P) \cdot D\big)\Big)$. Similarly, if $k < 0$, then

$$-k < \frac{\log t}{s\log\omega} + O\Big(\log\big(\widehat{\mathbf{h}}(P) \cdot D\big)\Big).$$

For $\epsilon \neq 0$ and/or $\kappa \neq 1$, we can replace P with $\rho^\epsilon\kappa P$ in the above calculation. Thus, there are
$$\frac{2\delta|\ker\Phi|\log t}{s\log\omega} + O\Big(\log\big(\widehat{\mathbf{h}}(P) \cdot D + 2\big)\Big)$$
elements $\sigma \in \mathcal{A}$ such that $h_D(\sigma P) < t$. Here, $\delta = 2$ or 1, depending on whether there is such a ρ in $\mathcal{A}$ or not. Finally, we divide through by the order of the stabilizer of P to get
$$N_P(t, D) = \frac{\mu(P)\log t}{s\log\omega} + O\Big(\log\big(\widehat{\mathbf{h}}(P) \cdot D\big)\Big),$$
where, in this case,
$$\mu(P) = \frac{2\delta|\ker\Phi|}{|\operatorname{Stab}(P)|}.$$

Since $|\operatorname{Stab}(P)|$ divides $2|\ker\Phi|$ when $\widehat{\mathbf{h}}(P) \neq \mathbf{0}$, and $\mu(P) = 0$ if $\widehat{\mathbf{h}}(P) = \mathbf{0}$, we have $\mu(P)$ is an integer. Finally, since $\widehat{\mathbf{h}}(P) \cdot D$ is a Weil height, there are only a finite number of points P such that $0 < \widehat{\mathbf{h}}(P) \cdot D < 2$, so

$$O\Big(\log\big(\widehat{\mathbf{h}}(P) \cdot D\big)\Big) = O\Big(\log\big(\widehat{\mathbf{h}}(P) \cdot D + 2\big)\Big).$$

We use this error term since $\log\big(\widehat{\mathbf{h}}(P) \cdot D\big)$ is not defined when $\widehat{\mathbf{h}}(P) = \mathbf{0}$. ■

4 Concluding Remarks

The obvious next step in this research is to construct canonical vector heights on K3 surfaces with Picard number 3. It is not clear to me, though, that such objects exist. Suppose that V is a K3 surface with Picard number 3 on which there exists a canonical vector height $\widehat{\mathbf{h}}$. Suppose there exists a $\tau \in \mathcal{A}$ such that τ^* has an eigenvalue greater

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than one. Then we may use this technique to compute $\widehat{\mathbf{h}}$ in two dimensions—the subspace spanned by the eigenvectors E_+ and E_- associated to the largest eigenvalues of τ^* and τ_*. If there exists another automorphism $\sigma \in \mathcal{A}$ such that σ^* has an eigenvalue larger than one and its associated eigenvector is not in this two dimensional subspace, then we can use this to piece together the rest of $\widehat{\mathbf{h}}$. But it seems doubtful that those pieces should fit, so it is not clear that such an $\widehat{\mathbf{h}}$ should exist.

Nevertheless, using Silverman's [S] technique (the technique used in this paper), we can construct the heights $\widehat{h}_{E_+}$ and $\widehat{h}_{E_-}$, which are canonical with respect the automorphism τ. Taking linear combinations, we get heights $\widehat{h}_{aE_++bE_-} = a\widehat{h}_{E_+} + b\widehat{h}_{E_-}$ that are also canonical with respect to τ. Precisely, we have

$$\widehat{h}_{aE_++bE_-}(\tau P) = a\omega\widehat{h}_{E_+}(P) + b\omega^{-1}\widehat{h}_{E_-}(P).$$

Though the divisors E_+ and E_- are never ample, there is some hope that $aE_+ + bE_-$ is ample for some $a, b \in \mathbb{R}$. As was pointed out in [C-S1], the existence of such a height is enough to prove an analogue of Theorem 3.2, and this was done in [W] and [Bi].

There are, though, K3 surfaces for which this technique cannot be applied. We conclude this paper with an example of such a surface.

In [Ni], Nikulin describes the group structure of $\mathcal{O}''$ for many K3 surfaces, including an example (or class of examples) with Picard number three and $\mathcal{O}''$ isomorphic to $\mathbb{Z}$, up to finite groups. These K3 surfaces are described via their intersection matrix

$$J = \begin{bmatrix} -4 & 4 & 0 \\ 4 & -4 & 2 \\ 0 & 2 & -2 \end{bmatrix}.$$

The group $\mathcal{O}$ includes several obvious elements, including -1 and the maps

$$T_1 = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad T_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix}.$$

The map T_1 can be derived as follows: Think of $\mathbf{x}^t J\mathbf{x} = k$ as a quadratic in x_1 while holding x_2 and x_3 fixed. This quadratic includes two roots, one of which is x_1, and the sum of which is $2x_2$. Thus, the other root is $x_1' = 2x_2 - x_1$. Sending x_1 to x_1' gives T_1. The maps T_2 and T_3 are derived in a similar fashion.

By applying a method of descent to a -2 curve, one can show that $\mathcal{O}^+ = \langle T_1, T_2, T_3\rangle$. All three of these generators are reflections, but only T_3 is a reflection through a -2 curve. Thus, $\mathcal{O}'' = \langle T_1, T_2\rangle$. The torsion-free part of $\mathcal{O}''$ is generated by T_1T_2, whose eigenvalues are all equal to 1. Thus, though the torsion-free part of $\mathcal{O}''$ is isomorphic to $\mathbb{Z}$, Tate's averaging method cannot be applied to derive a canonical height on this K3 surface.

Geometrically, the map T_1T_2 is a parabolic translation in $\mathcal{H}$—the set of points in an orbit all lie on a horocycle. This is clear, since T_1T_2 is a proper isometry of $\mathcal{H}$, so must be a rotation, hyperbolic translation, or parabolic translation. It cannot be a rotation for then it would have a finite order ($\mathcal{O}''$ is discrete, so all rotations in $\mathcal{O}''$ have finite order), and it cannot be a hyperbolic translation for then it would have an eigenvalue larger than one.

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