17

Methods of Scattering Theory

In Chapter 15, the fermionic signature operator and the unregularized fermionic projector were introduced abstractly. In Chapter 16, we computed them in the Minkowski vacuum. It remains to construct them in the presence of an external potential. In order to prove the mass oscillation properties, our task is to analyze the Dirac solutions asymptotically for large times and near spatial infinity. This can be accomplished with methods of scattering theory, which we now briefly introduce. We follow the presentation in [81].

We return to the Cauchy problem in the presence of an external potential,

$$(\mathcal{D} - m)\,\psi_m = 0\,, \qquad \psi_m\big|_{t_0} = \psi_0 \in C^\infty(\mathscr{N}_{t_0} \simeq \mathbb{R}^3, S\mathscr{M})\,, \tag{17.1}$$

with $\mathcal{D}$ as in (1.39). For notational clarity, we shall often denote the objects in the presence of the external potential by a tilde (the "interacting objects"), whereas the objects without tilde refer to the Minkowski vacuum.

17.1 The Lippmann–Schwinger Equation

The Dirac dynamics can be rewritten in terms of a symmetric operator $\tilde{H}$. To this end, we multiply the Dirac equation (1.39) by γ^0 and bring the t-derivative separately on one side of the equation,

$$\mathrm{i}\partial_t \psi_m = \tilde{H}\psi_m\,, \qquad \text{where} \qquad \tilde{H} := -\gamma^0(\mathrm{i}\vec{\gamma}\vec{\nabla} + \mathcal{B} - m), \tag{17.2}$$

note that $\gamma^j \partial_j = \gamma^0 \partial_t + \vec{\gamma}\vec{\nabla}$. We refer to (17.2) as the Dirac equation in *Hamiltonian form*. The fact that the scalar product (15.2) is time independent implies that for any two solutions $\phi_m, \psi_m \in C^\infty_{\mathrm{sc}}(\mathscr{M}, S\mathscr{M}) \cap \mathcal{H}_m$,

$$0 = \partial_t (\phi_m \,|\, \psi_m)_m = \mathrm{i}\big((\tilde{H}\phi_m \,|\, \psi_m)_m - (\phi_m \,|\, \tilde{H}\psi_m)_m \big)\,,$$

showing that the Hamiltonian is a symmetric operator on $\mathcal{H}_m$.

The Lippmann–Schwinger equation can be used to compare the dynamics in the Minkowski vacuum with the dynamics in the presence of an external potential. We denote the time evolution operator in the Minkowski vacuum by U_m^{t,t_0}.

Proposition 17.1.1 *The Cauchy problem* (17.1) *has a solution* ψ_m *which satisfies the equation*

$$\psi_m|_t = U_m^{t,t_0}\psi_0 + \mathrm{i}\int_{t_0}^t U_m^{t,\tau}\,\big(\gamma^0 \mathcal{B}\,\psi_m\big)\big|_\tau \,\mathrm{d}\tau\,, \tag{17.3}$$

referred to as the **Lippmann–Schwinger equation**.

Proof Obviously, the wave function $\psi_m|_t$ given by (17.3) has the correct initial values at $t = t_0$. Thus, it remains to show that this wave function satisfies the Dirac equation. To this end, we rewrite the Dirac equation in the Hamiltonian form (17.2) and separate the vacuum Hamiltonian H from the term involving the external potential,

$$(\mathrm{i}\partial_t - H)\,\psi_m = -\gamma^0 \mathcal{B}\,\psi_m \qquad \text{with} \qquad H = -\mathrm{i}\gamma^0 \vec{\gamma}\vec{\nabla} + \gamma^0 m\,. \tag{17.4}$$

Applying the operator $\mathrm{i}\partial_t - H$ to (17.3) and observing that the time evolution operator maps to solutions of the vacuum Dirac equation, only the derivative of the upper limit of integration contributes,

$$(\mathrm{i}\partial_t - H)\,\psi_m|_t = -U_m^{t,\tau}\,\big(\gamma^0 \mathcal{B}\,\psi_m\big)\big|_{\tau=t} = -\gamma^0 \mathcal{B}\,\psi_m|_t\,, \tag{17.5}$$

so that (17.4) is indeed satisfied. □

We remark that one way of solving the Lippmann–Schwinger equation is to substitute the left-hand side on the right-hand side to obtain

$$\begin{aligned}
\psi_m|_t &= U_m^{t,t_0}\psi_0 + \mathrm{i}\int_{t_0}^t U_m^{t,\tau}\,\big(\gamma^0\mathcal{B}\,\psi_m\big)\big|_\tau\,\mathrm{d}\tau \\
&= U_m^{t,t_0}\psi_0 + \mathrm{i}\int_{t_0}^t U_m^{t,\tau}\,\gamma^0\mathcal{B}|_\tau\,U_m^{\tau,t_0}\psi_0\,\mathrm{d}\tau \\
&\quad - \int_{t_0}^t U_m^{t,\tau}\,\gamma^0\mathcal{B}|_\tau \int_{t_0}^\tau U_m^{\tau,\tau'}\,\big(\gamma^0\mathcal{B}\,\psi_m\big)\big|_{\tau'}\,\mathrm{d}\tau'\,\mathrm{d}\tau\,.
\end{aligned} \tag{17.6}$$

Iterating this procedure, one gets a series of nested integrals referred to as the *Dyson series*, which is commonly used in perturbative quantum field theory (see, e.g., [147, Section 3.5]). The Dyson series can be regarded as an *ordered exponential* (see Exercise 17.1).

17.2 The Mass Oscillation Property in the Presence of an External Potential

The goal of this section is to prove the following result:

Theorem 17.2.1 *Assume that the external potential* $\mathcal{B}$ *is smooth and for large times decays faster than quadratically in the sense that*

$$|\mathcal{B}(t)|_{C^2} \le \frac{c}{1+|t|^{2+\varepsilon}} \tag{17.7}$$

for suitable constants $\varepsilon, c > 0$*. Then, the strong mass oscillation property holds.*

In words, the condition (17.7) means that the potential and its up to second derivatives must decay faster than quadratically for large times. This condition does not seem to have any physical significance; it is needed in order for our methods to apply.

The C^2-norm in (17.7) is defined as follows. We denote spatial derivatives by ∇ and use the notation with multi-indices; that is, for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_p)$, we set $\nabla^\alpha = \partial_{\alpha_1 \cdots \alpha_p}$ and denote the length of the multi-index by $|\alpha| = p$. Then, the spatial C^k-norms of the potential are defined by

$$|\mathcal{B}(t)|_{C^k} := \max_{|\alpha| \leq k} \sup_{\vec{x} \in \mathbb{R}^3} |\nabla^\alpha \mathcal{B}(t, \vec{x})| \,, \tag{17.8}$$

where $|\,.\,|$ is the sup-norm corresponding to the norm $|\phi|^2 := \prec\!\phi|\gamma^0\phi\!\succ$ on the spinors.

17.2.1 Proof of the Weak Mass Oscillation Property

In this section, we prove the following theorem.

Theorem 17.2.2 *Assume that the time-dependent external potential $\mathcal{B}$ is smooth and decays faster than quadratically for large times in the sense that* (17.7) *holds for suitable constants $c, \varepsilon > 0$. Then, the Dirac operator $\mathcal{D} = i\partial\!\!\!/ + \mathcal{B}$ has the weak mass oscillation property.*

We expect that this theorem could be improved by weakening the decay assumptions on the potential. However, this would require refinements of our methods, which would go beyond the scope of this paper. Also, using that Dirac solutions dissipate, the pointwise decay in time could probably be replaced or partially compensated by suitable spatial decay assumptions. Moreover, one could probably refine the result of Theorem 17.2.2 by working with other norms (like weighted C^k- or Sobolev norms).

The main step is the following basic estimate, which is the analog of Lemma 16.3.1 in the presence of an external potential.

Proposition 17.2.3 *Under the decay assumptions* (17.7) *on the external potential $\mathcal{B}$, there are constants $c, \varepsilon > 0$ such that for every family $\psi \in \mathcal{H}^\infty$ of solutions of the Dirac equation* (1.39) *with varying mass,*

$$\big\|(\mathfrak{p}\psi)|_t\big\|_t \leq \frac{c}{1+|t|^{1+\varepsilon}} \sup_{m \in I} \sum_{b=0}^{2} \big\|(\partial_m^b \psi_m)|_{t=0}\big\|_{W^{2,2}} \,. \tag{17.9}$$

We first show that this proposition implies the weak mass oscillation property.

Proof of Theorem 17.2.2 assuming that Proposition 17.2.3 holds. In order to derive the inequality (15.21), we begin with the estimate

$$\begin{aligned} |{<}\mathfrak{p}\psi|\mathfrak{p}\phi{>}| &\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \Big| \big(\mathfrak{p}\psi|_t \,\big|\, \mathfrak{p}\phi|_t\big)\big|_t \Big| \,\mathrm{d}t \\ &\leq \sup_{t \in \mathbb{R}} \big\|\mathfrak{p}\phi|_t\big\|_t \int_{-\infty}^{\infty} \big\|\mathfrak{p}\psi|_t\big\|_t \,\mathrm{d}t \,. \end{aligned} \tag{17.10}$$

The last integral is finite by Proposition 17.2.3. The supremum can be bounded by the Hilbert space norm using the Hölder inequality,

$$\|\mathfrak{p}\phi|_t\|_t = \Big\| \int_I \phi_m|_t \;\mathrm{d}m \Big\|_t \leq \int_I \big\|\phi_m|_t\big\|_t \;\mathrm{d}m$$
$$\leq \sqrt{|I|} \left(\int_I \|\phi_m|_t\|_t^2 \;\mathrm{d}m \right)^{\frac{1}{2}} = \sqrt{|I|}\, \|\phi\| \,, \tag{17.11}$$

giving (15.21).

Using (1.38), the Dirac operator $i\partial\!\!\!/ + \mathcal{B}$ is symmetric with respect to the inner product $<.|.>$. Therefore, the identity (15.22) can be obtained just as in (16.26) by integrating the Dirac operator in spacetime by parts, noting that we do not get boundary terms in view of the time decay in Proposition 17.2.3. □

The remainder of this section is devoted to the proof of Proposition 17.2.3. We make use of the Lippmann–Schwinger equation (17.3),

$$\psi_m|_t = U_m^{t,0}\, \psi_m|_{t=0} + \mathrm{i} \int_0^t U_m^{t,\tau} \big(\gamma^0 \mathcal{B}\, \psi_m\big)\big|_\tau \;\mathrm{d}\tau \,. \tag{17.12}$$

Since the first summand of this equation is controlled by Lemma 16.3.1, it remains to estimate the second summand. Again using (16.43) and integrating by parts with respect to the mass, we obtain

$$\begin{aligned} &\int_I U_m^{t,\tau} \big(\gamma^0 \mathcal{B}\, \psi_m\big)\big|_\tau \;\mathrm{d}m \\ &= \frac{1}{(t-\tau)^2} \int_I \big(A_m^{t,\tau}\, \partial_m^2 - B_m^{t,\tau} \partial_m + C_m^{t,\tau} \big) \big(\gamma^0 \mathcal{B}\, \psi_m\big)\big|_\tau \;\mathrm{d}m, \end{aligned} \tag{17.13}$$

where I is again the interval (15.8) and thus

$$\begin{aligned} \Big\| \int_I U_m^{t,\tau} \big(\gamma^0 \mathcal{B}\, \psi_m\big)\big|_\tau \;\mathrm{d}m \Big\|_t &\leq \frac{c\,|I|}{(t-\tau)^2} \sup_{m\in I} \sum_{b=0}^2 \big\| \mathcal{B}(\tau)\, (\partial_m^b \psi_m)|_\tau \big\|_{W^{2,2}} \\ &\leq \frac{c\,|I|}{(t-\tau)^2}\, |\mathcal{B}(\tau)|_{C^2} \sup_{m\in I} \sum_{b=0}^2 \big\| \partial_m^b \psi_m|_\tau \big\|_{W^{2,2}} \,. \end{aligned} \tag{17.14}$$

We now bound $\mathcal{B}(\tau)$ with the help of (17.7) and estimate the Sobolev norm $\big\|\partial_m^b \psi_m|_\tau\big\|_{W^{2,2}}$ at time τ by means of Lemma 13.5.1. This gives rise to the inequality

$$\Big\| \int_I U_m^{t,\tau} \big(\gamma^0 \mathcal{B}\, \psi_m\big)\big|_\tau \;\mathrm{d}m \Big\|_t \leq \frac{c^2\, C\, |I|}{(t-\tau)^2}\, \frac{1+|\tau|^2}{1+|\tau|^{2+\varepsilon}} \sup_{m\in I} \sum_{b=0}^2 \big\| \partial_m^b \psi_m|_{t=0} \big\|_{W^{2,2}} \,, \tag{17.15}$$

which yields the desired decay provided that τ and t are not too close to each other. More precisely, we shall apply this inequality in the case $|\tau| \leq |t|/2$. Then, the estimate simplifies to

$$\begin{aligned} &\Big\| \int_I U_m^{t,\tau} \big(\gamma^0 \mathcal{B}\, \psi_m\big)\big|_\tau \;\mathrm{d}m \Big\|_t \\ &\leq \frac{\tilde{C}}{t^2\, (1+|\tau|^\varepsilon)} \sup_{m\in I} \sum_{b=0}^2 \big\| \partial_m^b \psi_m|_{t=0} \big\|_{W^{2,2}} \qquad \text{if } |\tau| \leq |t|/2, \end{aligned} \tag{17.16}$$

with a new constant $\tilde{C} > 0$. In the remaining case $|\tau| > |t|/2$, we use the unitarity of $U_m^{t,\tau}$ to obtain

$$\left\| \int_I U_m^{t,\tau} \left(\gamma^0 \mathcal{B}\, \psi_m \right) |_\tau \; \mathrm{d}m \right\|_t \leq |I| \, |\mathcal{B}(\tau)|_{C^0} \sup_{m \in I} \|\psi_m\| \,. \tag{17.17}$$

Applying (17.7) together with the inequality $|\tau| > |t|/2$, this gives

$$\left\| \int_I U_m^{t,\tau} \left(\gamma^0 \mathcal{B}\, \psi_m \right) |_\tau \; \mathrm{d}m \right\|_t \leq \frac{\tilde{C}}{t^{2+\varepsilon}} \sup_{m \in I} \|\psi_m\| \qquad \text{if } |\tau| > |t|/2 \,. \tag{17.18}$$

This again decays for large t because τ is close to t and because $|\mathcal{B}(\tau)|_{C^0}$ decays for large τ.

Comparing (17.16) and (17.18), we find that the inequality in (17.16) even holds for all τ. Thus integrating this inequality over $\tau \in [0, t]$, we obtain the following estimate for the second summand in (17.12),

$$\left\| \int_I \mathrm{d}m \int_0^t U_m^{t,\tau} \left(\gamma^0 \mathcal{B}\, \psi_m \right) |_\tau \; \mathrm{d}\tau \right\|_t \leq \frac{C'}{t^{1+\varepsilon}} \sup_{m \in I} \sum_{b=0}^{2} \left\| \partial_m^b \psi|_{t=0} \right\|_{W^{2,2}}, \tag{17.19}$$

where $C' > 0$ is a new constant. Combining this inequality with the estimate (16.20) of the first summand in (17.12), we obtain the desired inequality (17.9). This concludes the proof of Proposition 17.2.3.

17.2.2 Proof of the Strong Mass Oscillation Property

In this section, we prove the following result.

Theorem 17.2.4 *Assume that the weak mass oscillation property holds and that the external potential $\mathcal{B}$ satisfies the condition*

$$\int_{-\infty}^{\infty} |\mathcal{B}(\tau)|_{C^0} \; \mathrm{d}\tau < \infty \,. \tag{17.20}$$

Then, the Dirac operator $\mathcal{D} = i\partial\!\!\!/ + \mathcal{B}$ has the strong mass oscillation property.

Combining this theorem with Theorem 17.2.2, one immediately obtains Theorem 17.2.1.

For the proof, we shall derive an explicit formula for the fermionic signature operator (Proposition 17.2.5). This formula is obtained by comparing the dynamics in the presence of the external potential with that in the Minkowski vacuum using the Lippmann–Schwinger equation and by employing distributional relations for products of fundamental solutions and Green's operators (Lemma 17.2.8).

In order to compare the dynamics in the presence of the external potential with that in the Minkowski vacuum, we work with the Hamiltonian formulation. We decompose the Dirac Hamiltonian (17.2) into the Hamiltonian in the Minkowski vacuum (17.4) plus a potential,

$$\tilde{H} = H + \mathcal{V} \qquad \text{with} \qquad \mathcal{V} := -\gamma^0 \mathcal{B} \,. \tag{17.21}$$

Proposition 17.2.5 *Assume that the potential $\mathcal{B}$ satisfies the condition* (17.20)*. Then, for every $\psi, \phi \in \mathcal{H}^\infty$,*

$$<\mathfrak{p}\psi|\mathfrak{p}\phi> = \int_I (\psi_m \,|\, \tilde{\mathcal{S}}_m \,\phi_m)_m \;\mathrm{d}m\,, \tag{17.22}$$

where $\tilde{\mathcal{S}}_m : \mathcal{H}_m \to \mathcal{H}_m$ are bounded linear operators that act on the wave functions at time t_0 by

$$\begin{aligned}\tilde{\mathcal{S}}_m &= \mathcal{S}_m \\ &\quad - \frac{\mathrm{i}}{2} \int_{-\infty}^{\infty} \epsilon(t-t_0) \left[\mathcal{S}_m\, U_m^{t_0,t}\, \mathcal{V}(t)\, \tilde{U}_m^{t,t_0} - \tilde{U}_m^{t_0,t}\, \mathcal{V}(t)\, \mathcal{S}_m\, U_m^{t,t_0}\right] \,\mathrm{d}t && (17.23)\\ &\quad + \frac{1}{2} \left(\int_{t_0}^{\infty}\!\int_{t_0}^{\infty} + \int_{-\infty}^{t_0}\!\int_{-\infty}^{t_0} \right) \tilde{U}_m^{t_0,t}\, \mathcal{V}(t)\, \mathcal{S}_m\, U_m^{t,t'}\, \mathcal{V}(t')\, \tilde{U}_m^{t',t_0} \;\mathrm{d}t\,\mathrm{d}t', && (17.24)\end{aligned}$$

and $\mathcal{S}_m$ is again the fermionic signature operator of the vacuum (16.58)*.*

Before entering the proof of this proposition, it is instructive to verify that the formula for $\tilde{\mathcal{S}}_m$ in (17.23) and (17.24) does not depend on the choice of t_0.

Remark 17.2.6 (Independence of $\tilde{\mathcal{S}}_m$ on t_0) Our strategy is to differentiate the above formula for $\tilde{\mathcal{S}}_m$ with respect to t_0 and to verify that we obtain zero. We first observe that taking a solution $\phi_m \in \mathcal{H}_m$ of the Dirac equation in the presence of $\mathcal{B}$, evaluating at time t_0 and applying the time evolution operator $\tilde{U}_m^{t,t_0}$ gives ϕ_m at time t, that is, $\tilde{U}_m^{t,t_0}\phi_m|_{t_0} = \phi_m|_t$. Differentiating with respect to t_0 yields

$$\partial_{t_0} \tilde{U}_m^{t,t_0} \phi_m|_{t_0} = 0\,. \tag{17.25}$$

The situation is different when one considers the time evolution operator of the vacuum. Namely, in the expression $U_m^{t,t_0}\phi_m|_{t_0}$, the wave function ϕ_m satisfies the Dirac equation $(\mathrm{i}\partial_t - H)\phi_m = \mathcal{V}\phi_m$, whereas the time evolution operator solves the Dirac equation with $\mathcal{V} \equiv 0$. As a consequence,

$$\partial_{t_0} U_m^{t,t_0} \phi_m|_{t_0} = -\mathrm{i} U_m^{t,t_0} (\mathcal{V}\phi_m)|_{t_0}\,. \tag{17.26}$$

Using these formulas together with $U^{t_0,t_0} = \mathbb{1} = \tilde{U}^{t_0,t_0}$, a straightforward computation gives

$$\begin{aligned}&\partial_{t_0} \big(\psi_m \,|\, (17.23)\, \phi_m\big)\big|_{t_0} \\ &\quad = -\mathrm{i}(\psi_m \,|\, [\mathcal{S}_m, \mathcal{V}]\, \phi_m)|_{t_0} \\ &\qquad - \frac{\mathrm{i}}{2}\, (-2) \big(\psi_m \,\big|\, \big(\mathcal{S}_m\, \mathcal{V}(t_0) - \mathcal{V}(t_0)\, \mathcal{S}_m\big)\, \phi_m\big)\big|_{t_0} \\ &\qquad - \frac{\mathrm{i}}{2} \int_{-\infty}^{\infty} \epsilon(t-t_0)\, \big((-\mathrm{i}\mathcal{V}(t_0))\, \psi_m \,\big|\, \mathcal{S}_m\, U_m^{t_0,t}\, \mathcal{V}(t)\, \tilde{U}_m^{t,t_0}\, \phi_m\big)\big|_{t_0}\, \mathrm{d}t \\ &\qquad + \frac{\mathrm{i}}{2} \int_{-\infty}^{\infty} \epsilon(t-t_0)\, \big(\psi_m \,\big|\, \tilde{U}_m^{t_0,t}\, \mathcal{V}(t)\, \mathcal{S}_m\, U_m^{t,t_0}\, (-i\mathcal{V}(t_0))\, \phi_m\big)\big|_{t_0}\, \mathrm{d}t, && (17.27)\end{aligned}$$

$$\begin{aligned}
&\partial_{t_0} \big(\psi_m \,|\, (17.24)\, \phi_m \big)\big|_{t_0} \\
&\quad = -\frac{1}{2} \int_{-\infty}^{\infty} \epsilon(t'-t_0)\, \big(\psi_m \,\big|\, \mathcal{V}(t_0)\, \mathcal{S}_m\, U_m^{t_0,t'}\, \mathcal{V}(t')\, \tilde{U}_m^{t',t_0}\, \phi_m \big)\big|_{t_0}\, \mathrm{d}t' \\
&\qquad - \frac{1}{2} \int_{-\infty}^{\infty} \epsilon(t-t_0)\, \big(\psi_m \,\big|\, \tilde{U}_m^{t_0,t}\, \mathcal{V}(t)\, \mathcal{S}_m\, U_m^{t,t_0}\, \mathcal{V}(t_0)\, \phi_m \big)\big|_{t_0}\, \mathrm{d}t \,, \qquad (17.28)
\end{aligned}$$

where for notational simplicity we here omitted the restrictions $|_{t_0}$ for the solutions ψ_m and ϕ_m. Adding the terms gives zero. $\Diamond$

The remainder of this section is devoted to the proof of Proposition 17.2.5. Our strategy is to combine the Lippmann–Schwinger equation with estimates in momentum space. We begin with two technical lemmas.

Lemma 17.2.7 *Assume that the external potential $\mathcal{B}$ satisfies condition* (17.20)*. For any $t_0 \in \mathbb{R}$, we denote the characteristic functions in the future and past, respectively, of this hypersurface $t = t_0$ by $\chi_{t_0}^{\pm}(x)$ (i.e., $\chi_{t_0}^{\pm}(x) = \Theta(\pm(x^0 - t_0))$, where Θ is the Heaviside function). Then, for any $\psi_m \in C^\infty_{\mathrm{sc}}(\mathscr{M}, S\mathscr{M}) \cap \mathcal{H}_m$, the wave function $k_m(\chi_{t_0}^{\pm} \mathcal{B} \psi_m)$ is a well-defined vector in $\mathcal{H}_{t_0}$ and*

$$\| k_m(\chi_{t_0}^{\pm} \mathcal{B}\psi_m) \|_{t_0} \leq \frac{1}{2\pi}\, \|\psi_m\|_m \int_{-\infty}^{\infty} \chi_{t_0}^{\pm}(\tau)\, |\mathcal{B}(\tau)|_{C^0}\, \mathrm{d}\tau \,. \qquad (17.29)$$

Proof Using the integral kernel representation (16.13) and (16.14) together with the fact that the time evolution in the vacuum is unitary, we obtain

$$\begin{aligned}
&2\pi \left\| \int_{\mathbb{R}^3} k_m\big((t_0,.), (\tau, \vec{y})\big)\, \big(\chi_{t_0}^{\pm} \mathcal{B}\psi_m\big)(\tau,\vec{y})\, \mathrm{d}^3y \right\|_{t_0} = \big\| U_m^{t_0,\tau} \gamma^0 (\chi_{t_0}^{\pm} \mathcal{B}\psi_m)|_\tau \big\|_{t_0} \\
&\quad = \big\| \gamma^0 (\chi_{t_0}^{\pm} \mathcal{B}\psi_m)|_\tau \big\|_\tau \leq |\mathcal{B}(\tau)|_{C^0}\, \|\psi_m\|_m \,. \qquad (17.30)
\end{aligned}$$

Integrating over τ and using (17.20) gives the result. $\square$

The following lemma is proved in [54, Eqs. (2.13)–(2.17)] (see Exercises 16.9 and 16.10).

Lemma 17.2.8 *In the Minkowski vacuum, the fundamental solution k_m and the Green's operator s_m defined by*

$$s_m := \frac{1}{2} \left(s_m^\vee + s_m^\wedge \right) \qquad (17.31)$$

satisfy the distributional relations in the mass parameters m and m',

$$k_m\, k_{m'} = \delta(m - m')\, p_m, \qquad (17.32)$$

$$k_m\, s_{m'} = s_{m'}\, k_m = \frac{\mathrm{PP}}{m - m'}\, k_m, \qquad (17.33)$$

$$s_m\, s_{m'} = \frac{\mathrm{PP}}{m - m'}\, (s_m - s_{m'}) + \pi^2\, \delta(m - m')\, p_m \,, \qquad (17.34)$$

where PP *denotes the principal part, and p_m is the distribution*

$$p_m(k) = (\not{k} + m)\, \delta(k^2 - m^2) \,. \qquad (17.35)$$

Proof of Proposition 17.2.5. Let $\psi \in \mathcal{H}^\infty$ be a family of solutions of the Dirac equation for varying mass. We denote the boundary values at time t_0 by $\psi_m^0 := \psi_m|_{t_0}$. Then, we can write the Lippmann–Schwinger equation (17.3) as

$$\psi_m|_t = U_m^{t,t_0} \psi_m^0 + \mathrm{i} \int_{t_0}^t U_m^{t,\tau} \big(\gamma^0 \mathcal{B}\, \psi_m \big)\big|_\tau \;\mathrm{d}\tau \,. \tag{17.36}$$

We now bring this equation into a more useful form. Expressing the time evolution operator with the help of (16.14) in terms of the fundamental solution, we obtain

$$\begin{aligned}\psi_m(x) &= 2\pi \int_{\mathbb{R}^3} k_m\big(x, (t_0, \vec{y})\big)\, \gamma^0 \psi_m^0(t_0, \vec{y})\; \mathrm{d}^3y \\ &\qquad + 2\pi\mathrm{i} \int_{t_0}^{x^0} \mathrm{d}y^0 \int_{\mathbb{R}^3} \mathrm{d}^3y\, k_m(x,y) (\mathcal{B}\, \psi_m)(y) \,. \end{aligned} \tag{17.37}$$

Applying (16.9) and using that the advanced and retarded Green's operators are supported in the future and past light cones, respectively, we can rewrite the last integral in terms of the advanced and retarded Green's operators,

$$\psi_m = 2\pi\, k_m\big(\gamma^0 \delta_{t_0} \psi_m^0\big) - s_m^\wedge\big(\chi_{t_0}^+ \mathcal{B}\psi_m\big) - s_m^\vee\big(\chi_{t_0}^- \mathcal{B}\psi_m\big) \,, \tag{17.38}$$

where $\delta_{t_0}(x) := \delta(t_0 - x^0)$ is the δ distribution supported on the hypersurface $x^0 = t_0$. Next, we express the advanced and retarded Green's operators in terms of the Green's operator (17.31): According to (16.9), we have the relations

$$s_m = s_m^\vee - \mathrm{i}\pi k_m = s_m^\wedge + \mathrm{i}\pi k_m, \tag{17.39}$$

and thus

$$\psi_m = k_m g_m - s_m \mathcal{B}\psi_m \quad \text{with} \quad g_m := 2\pi\, \gamma^0 \delta_{t_0} \psi_m^0 + \mathrm{i}\pi\, \epsilon_{t_0} \mathcal{B}\psi_m \,, \tag{17.40}$$

where ϵ_{t_0} is the step function

$$\epsilon_{t_0}(x) := \epsilon(x^0 - t_0), \tag{17.41}$$

and we omitted the brackets in expressions like $k_m g_m \equiv k_m(g_m)$. Note that the expression $k_m g_m$ is well defined according to Lemma 17.2.7. We also remark that by applying the operator $(i\partial\!\!\!/ - m)$ to the distribution g_m in (17.40), one immediately verifies that ψ_m indeed satisfies the Dirac equation $(\mathrm{i}\partial\!\!\!/ - m)\psi_m = -\mathcal{B}\psi_m$.

Now we can compute the inner product $<\mathfrak{p}\psi|\mathfrak{p}\psi>$ with the help of Lemma 17.2.8. Namely, using (17.40),

$$\begin{aligned} <\mathfrak{p}\psi|\mathfrak{p}\psi> &= \iint_{I\times I} <k_m g_m - s_m \mathcal{B}\psi_m \,|\, k_{m'} g_{m'} - s_{m'} \mathcal{B}\psi_{m'}>\; \mathrm{d}m\; \mathrm{d}m' \\ &= \int_I \Big(<g_m \,|\, p_m g_m> + \pi^2 <\mathcal{B}\psi_m \,|\, p_m \mathcal{B}\psi_m> \Big)\, \mathrm{d}m \\ &\quad + \iint_{I\times I} \frac{\mathrm{PP}}{m-m'} \Big(<\mathcal{B}\psi_m \,|\, k_{m'} g_{m'}> - <k_m g_m \,|\, \mathcal{B}\psi_{m'}> \\ &\quad + <\mathcal{B}\psi_m \,|\, (s_m - s_{m'}) \mathcal{B}\psi_{m'}> \Big)\; \mathrm{d}m\; \mathrm{d}m' \,. \end{aligned} \tag{17.42}$$

Note that this computation is mathematically well defined in the distributional sense because ψ_m and g_m are smooth and compactly supported in the mass parameter m. Employing the explicit formula for g_m in (17.40), we obtain

$$<\mathfrak{p}\psi|\mathfrak{p}\psi> = \int_I \Big(<g_m \,|\, p_m g_m> + \pi^2 <\mathcal{B}\psi_m \,|\, p_m \mathcal{B}\psi_m> \Big)\, \mathrm{d}m\,. \tag{17.43}$$

Comparing (16.29) with (17.35) and taking into account that the operator $\mathcal{S}_m$ defined by (16.58) gives a minus sign for the states of negative frequency, we get

$$p_m = \mathcal{S}_m\, k_m\,. \tag{17.44}$$

Using this identity together with Proposition 13.4.4 in the vacuum yields the relations

$$<g_m \,|\, p_m g_m> = (k_m g_m \,|\, \mathcal{S}_m\, k_m g_m)|_{t_0}\,, \tag{17.45}$$

$$<\mathcal{B}\psi_m \,|\, p_m \mathcal{B}\psi_m> = (k_m \mathcal{B}\psi_m \,|\, \mathcal{S}_m\, k_m \mathcal{B}\psi_m)|_{t_0}\,. \tag{17.46}$$

We finally apply Proposition 13.6.1 to obtain the representation

$$<\mathfrak{p}\psi|\mathfrak{p}\psi> = \int_I \Big((h_m \,|\, \mathcal{S}_m\, h_m)|_{t_0} + \pi^2\, (k_m \mathcal{B}\psi_m \,|\, \mathcal{S}_m\, k_m \mathcal{B}\psi_m)|_{t_0} \Big)\, \mathrm{d}m\,, \tag{17.47}$$

where

$$h_m := \psi_m + \mathrm{i}\pi\, k_m(\epsilon_{t_0} \mathcal{B}\psi_m)\,. \tag{17.48}$$

Comparing (17.22) with (17.47), we get

$$(\psi_m \,|\, \tilde{\mathcal{S}}_m\, \psi_m)_m = (h_m \,|\, \mathcal{S}_m\, h_m)|_{t_0} + \pi^2\, (k_m \mathcal{B}\psi_m \,|\, \mathcal{S}_m\, k_m \mathcal{B}\psi_m)|_{t_0}\,. \tag{17.49}$$

Expressing the operators k_m according to (16.14) by the time evolution operator and writing ψ_m in terms of the initial data as

$$\psi_m|_t = \tilde{U}^{t,t_0} \psi|_{t_0}\,, \tag{17.50}$$

we obtain

$$\begin{aligned}
&(\psi_m \,|\, \tilde{\mathcal{S}}_m\, \psi_m)_m = (\psi|\mathcal{S}_m\psi)|_{t_0} \\
&\quad - \frac{\mathrm{i}}{2} \int_{-\infty}^{\infty} \epsilon(t-t_0)\, \big(\psi \,\big|\, \mathcal{S}_m\, U^{t_0,t}\, \mathcal{V}(t)\, \tilde{U}^{t,t_0}\, \psi\big)\big|_{t_0}\, \mathrm{d}t \\
&\quad + \frac{\mathrm{i}}{2} \int_{-\infty}^{\infty} \epsilon(t-t_0)\, \big(U^{t_0,t}\, \mathcal{V}(t)\, \tilde{U}^{t,t_0}\, \psi \,\big|\, \mathcal{S}_m\, \psi\big)\big|_{t_0}\, \mathrm{d}t \\
&\quad + \frac{1}{4} \iint_{\mathbb{R}\times\mathbb{R}} \epsilon(t-t_0)\, \epsilon(t'-t_0) \\
&\qquad \times \big(U^{t_0,t}\, \mathcal{V}(t)\, \tilde{U}^{t,t_0}\, \psi \,\big|\, \mathcal{S}_m\, U^{t_0,t'}\, \mathcal{V}(t')\, \tilde{U}^{t',t_0}\, \psi\big)\big|_{t_0}\, \mathrm{d}t\, \mathrm{d}t' \\
&\quad + \frac{1}{4} \iint_{\mathbb{R}\times\mathbb{R}} \big(U^{t_0,t}\, \mathcal{V}(t)\, \tilde{U}^{t,t_0}\, \psi \,\big|\, \mathcal{S}_m\, U^{t_0,t'}\, \mathcal{V}(t')\, \tilde{U}^{t',t_0}\, \psi\big)\big|_{t_0}\, \mathrm{d}t\, \mathrm{d}t'\,.
\end{aligned} \tag{17.51}$$

Rearranging the terms and polarizing gives the result. □

Proof of Theorem 17.2.4. Since the time evolution operators are unitary and the operators $\mathcal{S}_m$ have norm one (see (16.58)), the representation (17.23) and (17.24) gives rise to the following estimate for the sup-norm of $\tilde{S}_m$,

$$\big\|\tilde{\mathcal{S}}_m\big\| \leq 1 + \int_{\mathbb{R}} |\mathcal{V}(t)|_{C^0}\, \mathrm{d}t + \iint_{\mathbb{R}\times\mathbb{R}} |\mathcal{V}(t)|_{C^0}\, |\mathcal{V}(t')|_{C^0}\, \mathrm{d}t\, \mathrm{d}t' \,. \tag{17.52}$$

The decay assumption (17.20) implies that the sup-norm of $\tilde{S}_m$ is bounded uniformly in m. Using this fact in (17.22) gives the inequality (15.23), thereby establishing the strong mass oscillation property. □

We finally remark that the uniqueness statement in Proposition 15.3.3 implies that the relations (17.23) and (17.24) yield an explicit representation of the fermionic signature operator in the presence of a time-dependent external potential.

17.3 Exercises

Exercise 17.1 For a smooth one-parameter family of matrices $F(\alpha)$, $\alpha \in \mathbb{R}$, the **ordered exponential** $\mathrm{Pexp}(\int F(\alpha)\, d\alpha)$

$$\begin{aligned}\mathrm{Pexp}\left(\int_a^b F(\alpha)\, d\alpha\right) &= \mathbb{1} + \int_a^b F(t_0)\, \mathrm{d}t_0 + \int_a^b \mathrm{d}t_0\, F(t_0) \int_{t_0}^b \mathrm{d}t_1\, F(t_1) \\ &\quad + \int_a^b \mathrm{d}t_0\, F(t_0) \int_{t_0}^b \mathrm{d}t_1\, F(t_1) \int_{t_1}^b \mathrm{d}t_2\, F(t_2) + \cdots \,.\end{aligned} \tag{17.53}$$

In this exercise, we collect a few elementary properties of the ordered exponential.

(a) Assume that the matrix-valued function F is commutative in the sense that

$$\big[F(\alpha), F(\beta)\big] = 0 \qquad \text{for all } \alpha, \beta \in [a,b] \,. \tag{17.54}$$

Show that the ordered exponential reduces to the ordinary exponential,

$$\mathrm{Pexp}\left(\int_a^b F(\alpha)\, \mathrm{d}\alpha\right) = \exp\left(\int_a^b F(\alpha)\, \mathrm{d}\alpha\right). \tag{17.55}$$

Hint: Show inductively that

$$\begin{aligned}&\int_a^b \mathrm{d}t_0\, F(t_0) \int_{t_0}^b \mathrm{d}t_1\, F(t_1) \cdots \int_{t_{n-1}}^b \mathrm{d}t_n\, F(t_n) \\ &\quad = \frac{1}{(n+1)!}\left(\int_a^b F(t)\, \mathrm{d}t\right)^{n+1} .\end{aligned} \tag{17.56}$$

(b) Assume that F is continuous on $[a,b]$. Show that the Dyson series converges absolutely and that

$$\left\| \mathrm{Pexp}\left(\int_a^b F(\alpha)\, \mathrm{d}\alpha\right)\right\| \leq \exp\left(\int_a^b \big\|F(\alpha)\big\|\, \mathrm{d}\alpha\right). \tag{17.57}$$

Hint: Estimate the integrals and apply (a).

(c) Show by direct computation that the ordered exponential satisfies the equations

$$\frac{\mathrm{d}}{\mathrm{d}a} \operatorname{Pexp}\left(\int_a^b F(\alpha)\ \mathrm{d}\alpha \right) = -F(a)\ \operatorname{Pexp}\left(\int_a^b F(\alpha)\ \mathrm{d}\alpha \right), \tag{17.58}$$

$$\operatorname{Pexp}\left(\int_a^a F(\alpha)\ \mathrm{d}\alpha \right) = \mathbb{1}\,. \tag{17.59}$$

Use the uniqueness theorem for solutions of ordinary differential equations to give an alternative definition in terms of the solution of an initial-value problem. Use this reformulation to show the group property

$$\operatorname{Pexp}\left(\int_a^b F(\alpha)\,\mathrm{d}\alpha \right) \operatorname{Pexp}\left(\int_b^c F(\alpha)\,\mathrm{d}\alpha \right) = \operatorname{Pexp}\left(\int_a^c F(\alpha)\,\mathrm{d}\alpha \right). \tag{17.60}$$

(d) Show that

$$\frac{\mathrm{d}}{\mathrm{d}b} \operatorname{Pexp}\left(\int_a^b F(\alpha)\ \mathrm{d}\alpha \right) = \operatorname{Pexp}\left(\int_a^b F(\alpha)\ \mathrm{d}\alpha \right) F(b)\,. \tag{17.61}$$

Hint: Differentiate the identity (17.60) in the case $c = a$ and use the group properties (17.59) and (17.60).

(e) Show that

$$\operatorname{Pexp}\left(\int_a^b F(\alpha)\ \mathrm{d}\alpha \right)^* = \operatorname{Pexp}\left(\int_b^a \big(-F(\alpha)^* \big)\ \mathrm{d}\alpha \right). \tag{17.62}$$

Deduce that if $F(\alpha)$ is an anti-Hermitian matrix, then the ordered exponential is a unitary matrix. *Hint:* There are two alternative methods. One method is to argue using the differential equations (17.58) and (17.61) or with the group property. A more computational approach is to take the adjoint of the Dyson series and reparametrize the integrals.

Exercise 17.2 Given $\omega \in \mathbb{R}$ and a smooth function $V(t)$, we consider the ordinary differential equation

$$\big(\mathrm{i}\partial_t + \omega\big)\phi(t) = V(t)\,\phi(t)\,. \tag{17.63}$$

(a) Write down the Lippmann–Schwinger equation, taking the right-hand side of the equation as the perturbation. *Hint:* The free time evolution operator $U^{t,t'}$ was computed in Exercise 16.5.

(b) Express the Lippmann–Schwinger equation in the case $\omega = 0$ explicitly as an integral equation. How is it related to the integral equation used in the Picard iteration (in the proof of the Picard–Lindelöf theorem)?