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ON COMPACT GROUP EXTENSION OF BERNOULLI SHIFTS

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Let $\rho : G \to \mathcal{U}(H)$ be an irreducible unitary representation of a compact group G where $\mathcal{U}(H)$ is a set of unitary operators of finite dimensional Hilbert space H. For the $(p_1, \cdots, p_L)$-Bernoulli shift, the solvability of $\rho(\phi(x))g(Tx) = g(x)$ is investigated, where $\phi(x)$ is a step function.

1. INTRODUCTION

Let $(X, \mathcal{B}, \mu)$ be a probability space and T a measure preserving transformation on X. A transformation T on X is called ergodic if the constant function is the only T-invariant function and it is called weakly mixing if the constant function is the only eigenfunction with respect to T. Let $\mathbf{1}_E$ be the characteristic function of a set E and consider the behaviour of the sequence $\sum\limits_{k=0}^{n-1} \mathbf{1}_E(T^k x)$ which equals the number of times that the points $T^k x$ visit E. The Birkhoff Ergodic Theorem applied to the ergodic transformation $T : x \mapsto \{Lx\}$ on $[0,1)$, where L is positive integer and $\{t\}$ is the fractional part of t, gives the classical Borel Theorem on normal numbers:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}_{[(j-1)/L, j/L)}(T^k x) = \frac{1}{L}$$

for $1 \leqslant j \leqslant L$. This implies that almost everywhere x is L-normal, that is, the relative frequency of the digit j in the L-adic expansion of x is $1/L$. See [**11**].

In this paper, we are interested in the uniform distribution of the sequence $d_n \in \{0, \cdots, M-1\}$ defined by

$$d_n(x) \equiv \sum_{k=0}^{n-1} \mathbf{1}_E(T^k x) \pmod M,$$

for $T : x \mapsto \{Lx\}$ and more general transformations.

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DEFINITION 1: Let T be a transformation on $[0,1)$ defined by

$$T(x) = \frac{x - \sum_{k=0}^{i-1} p_k}{p_i} \qquad \text{on } \left[\sum_{k=0}^{i-1} p_k, \sum_{k=0}^{i} p_k\right)$$

where $p_0 = 0$, $p_i > 0$ for $1 \leqslant i \leqslant L$ and $\sum_{k=0}^{L} p_k = 1$. We call this transformation the $(p_1, \cdots, p_L)$-transformation.

Let $\mathcal{P} = \{P_1, \cdots P_L\}$ be a partition on $[0,1)$ with $P_i = \left[\sum_{k=0}^{i-1} p_k, \sum_{k=0}^{i} p_k\right)$ for $1 \leqslant i \leqslant L$. Recall that the $(p_1, \cdots, p_L)$-transformation preserves Lebesgue measure μ on $[0,1)$ and that $\mathcal{P}$ is a generating partition on $[0,1)$ with respect to the $(p_1, \cdots, p_L)$-transformation. Hence almost every $x \in [0,1)$ has a symbolic representation $[a_1, a_2, \cdots]$ with respect to the $(p_1, \cdots, p_L)$-transformation and the partition $\mathcal{P}$ where $1 \leqslant a_i \leqslant L$. When x is represented by $[a_1, \cdots a_n]$ with a finite length, we call it a generalised *L-adic number*. Recall that a one-sided $(p_1, \cdots, p_L)$-Bernoulli shift, where $\sum_{i=1}^{L} p_i = 1$ and $p_i > 0$ is measure theoretically isomorphic to the $(p_1, \cdots, p_L)$-transformation on $X = [0,1)$ with Lebesgue measure μ and the partition $\mathcal{P} = \{P_1, \cdots P_L\}$.

This type of problem was first studied by Veech. He considered the case when the transformations are given by irrational rotations on the unit circle and $M = 2$, and obtained results which showed that the length of the interval E and the rotational angle θ are closely related. For example, he proved that when the irrational number θ has bounded partial quotients in its continued fraction expansion, the sequence d_n is evenly distributed if the length of the interval is not an integral multiple of θ modulo 1 [**10**].

In [**1**], Ahn, Choe and Lemánczyk consider the case of the $(1/L, \cdots, 1/L)$-transformation on $X = [0,1)$ and $M = 2$, and show that the sequence $\{d_n\}$ is evenly distributed if $\exp(\pi i \mathbf{1}_E(x))$ has finite L-adic discontinuity points $1/L \leqslant t_1 < \cdots < t_n \leqslant 1$. Recently, Choe, Hamachi and Nakada [**2**] show that $\{d_n\}$ is evenly distributed for more general sets and that the $\mathbb{Z}_2$-extension induced by $\phi(x) = \exp(\pi i \mathbf{1}_B(x))$ where $\mathbf{1}_B$ is the characteristic function of B, is ergodic. In this paper, we show that for all Bernoulli shifts the sequence $\{d_n\}$ is uniformly distributed and that the compact group extension by $\phi(x)$ is weakly mixing. When T is an irrational rotation, and $\phi(x)$ is a step function, the spectral type has been investigated by some mathematicians [**3, 4, 6**]. In connection with Veech's results, we also investigate the sequence $\{d_n\}$ induced by intervals.

To investigate the sequence $\{d_n(x)\}$, we consider the behaviour of the sequence $\exp((2\pi i/M)d_n(x))$ and check whether this sequence is uniformly distributed on the compact group G generated by $\exp(2\pi i/M)$. Weyl's criterion on uniform distribution says that the sequence $\exp\big((2\pi i/M)d_n(x)\big)$ is uniformly distributed if and only if

$$\lim_{N\to\infty}\sum_{n=1}^{N}\exp^k\left(\frac{2\pi i}{M}d_n(x)\right)=0$$

for all $1\leqslant k\leqslant L-1$.

We investigate the problem from the viewpoint of spectral theory. Let (X,μ) be a probability space and T an ergodic measure preserving transformation on X, which is not necessarily invertible. Let $\phi(x)$ be the G-valued function defined by $\phi(x)=\exp\big((2\pi i/M)\mathbf{1}_E(x)\big)$. Consider the skew product transformation T_ϕ on $X\times G$ defined by

$$T_\phi(x,g)=\big(Tx,\phi(x)g\big).$$

Then the problem is equivalent to checking whether T_ϕ is ergodic or not.

2. Compact group extension

Let G be a compact group with normalised right Haar measure ν, and (X,μ) a probability space and $T:X\to X$ an ergodic measure preserving transformation. Given a function $\phi:X\to G$, define a skew product transformation $T_\phi:X\times G\to X\times G$ by $(x,g)\mapsto\big(Tx,\phi(x)\cdot g\big)$. Then T_ϕ preserves the product measure $\mu\times\nu$. The ergodicity of T_ϕ can be checked by the decomposition of $L^2(X\times G)$. The Peter-Weyl Theorem says that the matrix coefficients of the irreducible unitary representation form an orthogonal basis for $L^2(G,\nu)$. Take any irreducible unitary representation ρ and let (ρ_{ij}) be its matrix representation. Then

$$\begin{aligned}U_{T_\phi}\big(\rho_{ij}(g)f(x)\big)&=\rho_{ij}\big(\phi(x)\cdot g\big)f(Tx)\\&=\sum_k\rho_{ik}(g)\rho_{kj}\big(\phi(x)\big)f(Tx).\end{aligned}$$

Hence we have the following U_{T_ϕ}-invariant orthogonal decomposition:

$$L^2(X\times G)=\oplus L^2_\rho(X\times G)$$

where the subspace $L^2_\rho(X\times G)$ is spanned by functions of the form $\rho_{ij}(g)f(x)$, $f\in L^2(X)$. For ρ is equal to the two Hilbert spaces $L^2_\rho(X\times G)$ and $L^2_\rho(X)$ are identical. The following is a well-known fact.

Lemma 1.

(i) *The skew product transformation $T_\phi:X\times G\to X\times G$ is not ergodic if and only if there exists an irreducible representation $\rho\neq 1$ satisfying $\rho\big(\phi(x)\big)h(Tx)=h(x)$ for some nonzero $h=(h_i)_{1\leqslant i\leqslant d}$, $h_i\in L^2(X)$ where d is the dimension of ρ.*

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(ii) *T_ϕ is not weakly mixing if and only if there exists an irreducible representation $\rho \neq 1$ and some constant $\lambda \in \mathbb{C}$, $|\lambda| = 1$, satisfying $\rho(\phi(x))h(Tx) = \lambda h(x)$. Here, $h = (h_i)_{1\leqslant i\leqslant d}$, $h_i \in L^2(X)$ is non zero and d is the dimension of ρ.*

From now on, let H be a finite dimensional Hilbert space and $\mathcal{U}(H)$ be a set of unitary operators on H.

Lemma 2. *Let $f(x)$ be a $\mathcal{U}(H)$-valued step function with finitely many points of discontinuity. For the $(p_1, \cdots, p_L)$-transformation T, if an H-valued function $h(x)$ satisfies the equation $f(x)h(Tx) = h(x)$, then $h(x)$ is also a step function with finitely many points of discontinuity.*

Proof: Since $f(x) \in \mathcal{U}(H)$ and T is an ergodic transformation, we may assume that $\|h(x)\|_H = 1$ where $\|\cdot\|_H$ is the Hilbert space norm.

For simplicity of proof we shall prove the theorem for the transformation defined by (p,q) where $p \geqslant q$. Let $\mathcal{P}$ be a partition and $\mathcal{P}_N = \bigvee_{k=0}^{N-1} T^{-k}\mathcal{P}$. Let m be the cardinality of the set of discontinuities Y and Y_ε be an ε-neighbourhood of Y. Then there exists ε_0 such that for all $0 < \varepsilon < \varepsilon_0$, $\mu(Y_\varepsilon) = 2m\varepsilon$. Now choose an integer N such that $p^N < \varepsilon_0$ and $(2m \cdot p^{N+1})/(1-p) < 1/2$.

If $I \in \mathcal{P}_N$ and if $I \cap Y \neq \emptyset$, then $I \subset Y_\varepsilon$ for $\varepsilon = p^N$. Hence the totality of $I \in \mathcal{P}_N$ with $I \cap Y \neq \emptyset$ has measure at most $2m \cdot p^N$. By a similar argument, the totality of $I \in \mathcal{P}_{N+j}$, $j \geqslant 0$ such that $I \cap Y \neq \emptyset$ has measure at most $2m \cdot p^{N+j}$.

Fix $L > 0$ and consider the collection of $I \in \mathcal{P}_{N+L}$ having the property that $T^j I \cap Y \neq \emptyset$ for some $0 \leqslant j \leqslant L-1$. Since $T^j \in \mathcal{P}_{N+L-j}$ for these j, and T is measure preserving, these intervals have total measure at most

$$2m \cdot p^{N+L-1} + 2m \cdot p^{N+L-2} \cdots 2m \cdot p^{N+1} \leqslant \frac{2m \cdot p^{N+1}}{1-p} \leqslant \frac{1}{2}.$$

Let $Q(N,L)$ be the sub collection of $\mathcal{P}_{N+L}$ such that $T^j I \cap Y = \emptyset$ for all $0 \leqslant j \leqslant L-1$. Then for each $I \in Q(N,L)$

$$f(x)f(Tx)\cdots f(T^{L-1}x)$$

is constant, say $\Lambda(I,L) \in \mathcal{U}(H)$. Since $h(x) = f(x)h(Tx)$,

$$h(x) = f(x)f(Tx)\cdots f(T^{L-1}x)h(T^L x).$$

Hence $h(x) = \Lambda(I,L)h(T^L x)$ holds almost everywhere on I. Letting $T^L I = J \in \mathcal{P}_N$, the map $T^L : I \to J$ is bijective and it is easily shown that

$$\text{(1)} \qquad \frac{1}{\mu(I)}\int_I h(x)\,d\mu(x) = \Lambda(I,L)\left(\frac{1}{\mu(J)}\int_J h(y)\,d\mu(y)\right).$$

Since $Q(N,L)$ measures at least $1/2$, the set of x which is interior to some $I \in Q(N,L)$ for an infinitely number of L must also measure at least $1/2$. Fixing such an

x, we have that (1) holds. We may assume that x is a Lebesgue point of h. Since $\mathcal{P}_N$ is finite, it can be assumed that J is always the same on the right side of (1). By the Lebesgue density theorem [8], we can assume that the left side of (1) tends to $h(x)$. By the compactness of $\mathcal{U}(H)$, we may assume that $\lim_{L\to\infty} \Lambda(I, L) = \Lambda \in \mathcal{U}(H)$. Hence

$$h(x) = \Lambda\left(\frac{1}{\mu(J)}\int_J h(y)\,d\mu(y)\right).$$

Since $\|h(x)\|_H = 1$ almost everywhere, we may assume that $\|h(x)\|_H = 1$. Since $\Lambda \in \mathcal{U}(H)$

$$\left\|\frac{1}{\mu(J)}\int_J h(y)\,d\mu(y)\right\|_H = 1.$$

$\|h(x)\|_H = 1$ almost everywhere implies h is constant on J.

Since $f(x)$ is a $\mathcal{U}(H)$-valued step function with finitely many discontinuities and $T^N J = X$, $h(x)$ is also step function with finitely many discontinuities. $\square$

LEMMA 3. *Let $\rho : G \to \mathcal{U}(H)$ be a unitary representation of the compact group G by unitary operators on a Hilbert space H, different from the zero representation. The following properties are equivalent:*

(i) *ρ is irreducible;*

(ii) *for every nonzero vector $h \in H$, the closed linear subspace generated by $\{\rho(g)h : g \in G\}$ is H;*

(iii) *the only bounded operators on H commuting with all $\rho(g)$ ($g \in G$) are of the form αI where $\alpha \in \mathbb{C}$ and I is the identity operator.*

PROOF: For the proof, see Hewitt and Ross's Book [5]. $\square$

THEOREM 1. *Let G be a compact group, H be a finite dimensional Hilbert space and $\mathcal{U}(H)$ be a set of unitary operators on H. Let $\rho : G \to \mathcal{U}(H)$ be a non trivial irreducible representation of G. Let T be the $(p_1, \cdots, p_L)$-transformation. Then $\rho(\phi(x))h(Tx) = h(x)$ has no solution if $\phi(x)$ is a step function with discontinuities at $p_1 \leqslant t_1 < \cdots < t_n = 1$ and the range of $\phi(x)$ is not contained in any closed proper subgroup of G.*

PROOF: Since $\rho \neq 1$ is an irreducible representation of G, it is sufficient to prove that $h(x)$ is constant by Lemma 3. Letting $\rho(\phi(x)) = f(x)$, $h(x)$ is a H-valued step function with finite discontinuity points by Lemma 2. Hence there exists $0 < r < p_1$ such that $h(x) = c$ on $[0, r)$. Hence $f(x)h(x) = h(x)$ on $[0, r)$. Since $f(x)$ is a unitary operator which is constant on $[0, p_1)$, the conclusion follows. $\square$

REMARK 1. Let G be a compact group. If $\phi(x)$ satisfies the condition of Theorem 1, then the skew product transformation is weakly mixing. Indeed if $\rho(\phi(x))h(Tx) = \lambda h(x)$ where $\lambda \in \mathbb{C}$ and $|\lambda| = 1$, then by a similar argument to that of Lemma 2, we can show that $h(x)$ is also step function with finitely many points of discontinuity. By the irreducible property of ρ and Lemma 3, the conclusion follows.

Let $(Y, \mathcal{C}, \mu)$ be a probability space, $f \in L^1(Y, \mathcal{C}, \mu)$ and $\mathcal{B} \subset \mathcal{C}$ a sub σ-algebra. Put $\nu(B) = \int_B f\, d\mu$ for $B \in \mathcal{B}$. The Radon-Nikodym Theorem implies that there is a function $h \in L^1(Y, \mathcal{B}, \mu)$ such that $\nu(B) = \int_B h\, d\mu$ for $B \in \mathcal{B}$. We use the notation $E(f \mid \mathcal{B})$ for h, and call it the *conditional expectation* of f with respect to $\mathcal{B}$. Let S be a transformation defined on Y and $\mathcal{B}$ be *exhaustive* that is, $S^{-1}\mathcal{B} \subset \mathcal{B}$ and $S^n\mathcal{B} \uparrow \mathcal{C}$. The Martingale Theorem says that $E(f \mid S^n\mathcal{B})$ converges to f almost everywhere and in $L^1(Y, \mathcal{C}, \mu)$ for $f \in L^1(Y, \mathcal{C}, \mu)$

LEMMA 4. *Let S be a transformation on $(Y, \mathcal{C}, \mu)$, and $\mathcal{B} \subset \mathcal{C}$ be an exhaustive sub σ-algebra, and let $\phi : Y \to \mathcal{U}(H)$ be a $\mathcal{B}$-measurable. If $q : Y \to H$ is a $\mathcal{C}$-measurable solution to the equation $\phi \cdot q = q \circ S$, then q is $\mathcal{B}$-measurable.*

PROOF: We follow an idea of Parry [7]. Applying the conditional expectation operator $E(\cdot \mid \mathcal{B})$ to the equation

$$\phi \cdot q = q \circ S \tag{1}$$

we have

$$\phi \cdot E(q \mid \mathcal{B}) = E(q \circ S \mid \mathcal{B})$$

or

$$\phi \cdot E(q \mid \mathcal{B}) = E(q \mid S\mathcal{B}) \circ S. \tag{2}$$

Multiplying (2) by the Hermitian conjugate of (1) we obtain

$$q^*(y) \cdot E(q \mid \mathcal{B})(y) = q^*(Sy) \cdot E(q \mid S\mathcal{B}) \circ S(y) \qquad \text{almost everywhere}$$

where q^* is the conjugate of q.
Hence

$$\int_Y q^* \cdot E(q \mid \mathcal{B})\, d\mu = \int_Y q^* \cdot E(q \mid S\mathcal{B})\, d\mu.$$

By exactly the same argument, using $S^n\mathcal{B}$ in place of $\mathcal{B}$, we have

$$\int_Y q^* \cdot E(q \mid S^n\mathcal{B})\, d\mu = \int_Y q^* \cdot E(q \mid S^{n+1}\mathcal{B})\, d\mu,$$

so that

$$\int_Y q^* \cdot E(q \mid \mathcal{B})\, d\mu = \int_Y q^* \cdot E(q \mid S^n\mathcal{B})\, d\mu.$$

Taking limits, and using the Martingale Theorem, we get

$$\int_Y q^* \cdot E(q \mid \mathcal{B})\, d\mu = \int_Y \|q\|_H^2\, d\mu,$$

where $\|\cdot\|_H$ is the Hilbert space norm. Thus $E(q \mid \mathcal{B}) = q$ almost everywhere, and q is $\mathcal{B}$-measurable. ∎

REMARK 2. For the $(p_1, \cdots, p_L)$-transformation and $\phi(x)$ which satisfies the condition of Theorem 1, consider the corresponding two-sided $(p_1, \cdots, p_L)$-Bernoulli transformation and the skew product transformation. Then by Lemma 4, and Remark 1, this skew-product is weakly mixing. Hence if G is metrisable, it is also Bernoulli by Rudolph's Theorem [**9**].

3. MOD M NORMALITY OF BERNOULLI SHIFTS

To investigate the mod M normality of the $(p_1, \cdots, p_L)$-transformation, we consider the function $\phi(x) = \exp\big((2\pi i/M)\mathbf{1}_E(x)\big)$. Recall that a function $f(x)$ is called a *coboundary* if $f(x)q(Tx) = q(x)$, $|q(x)| = 1$ almost everywhere on X. In the following two Lemmas, we consider more general functions $\phi(x)$ with finitely many discontinuity points. In the following, the unit circle in the complex plane is denoted by $\mathbb{T}$.

LEMMA 5. *For the $(p_1, \cdots, p_L)$-transformation, if a $\mathbb{T}$-valued function $\phi(x)$ is a step function with finitely many discontinuity points $p_1 \leqslant t_1 < \cdots < t_n < 1$, then $\phi(x)$ is not a coboundary.*

PROOF: Assume that $\phi(x)h(Tx) = h(x)$. Since $\phi(x)$ is step function with finitely many discontinuity points, $h(x)$ is also a step function with finitely many discontinuity points. Hence there exists $0 < r \leqslant p_1$ such that $h(x)$ is constant on $[0, r)$. Thus $\phi(x)h(x) = h(x)$ on $[0, r)$. So $h(x)$ is constant on $[0, 1)$. Hence the conclusion follows. ∎

EXAMPLE 1. For the $(1/2, 1/2)$-transformation, let $I = [3/4, 1]$, $F = \bigcup_{k=0}^{\infty}(1/2^k)I$ and $E = F \vartriangle T^{-1}F$. Then $\phi(x) = \exp\big(\pi i\mathbf{1}_E(x)\big)$ is a coboundary even if the discontinuity points of $\phi(x)$ are contained in $[1/2, 1)$ where the cobounding function is $h(x) = \exp\big(\pi i\mathbf{1}_F(x)\big)$.

Now let $F = \bigcup_{k=1}^{\infty}(1/2^k)I$ and $E = F \vartriangle T^{-1}F$. Then $\phi(x) = \exp\big(\pi i\mathbf{1}_E(x)\big)$ is a coboundary even if there exists $r > 0$ such that $\phi(x) \neq 1$ on $[r, 1)$. But this phenomenon disappears when $\phi(x)$ has finitely many discontinuity points. Hence we have the following Lemma.

LEMMA 6. *Let $\phi(x)$ be a $\mathbb{T}$-valued step function on $X = [0, 1)$ with finitely many discontinuity points. If there exists $r > 0$ such that $\phi(x) \neq 1$ on $[0, r)$ or $[r, 1)$, then $\phi(x)$ is not a coboundary for the $(p_1, \cdots, p_L)$-transformation.*

PROOF: Assume that $\phi(x)h(Tx) = h(x)$. As in the proof of Lemma 5, there exists $0 < r < p_1$ such that $h(x)$ is constant on $[0, r)$. Hence there exists $t > 0$ such that $\phi(x) = 1$ on $[0, t)$. ∎

PROPOSITION 1. *For the $(p_1, \cdots, p_L)$-transformation, a complex-valued function $\phi(x) = \exp\big((2\pi i/M)\mathbf{1}_{(a,b)}(x)\big)$ is a coboundary if and only if $L = 2, M = 2$ and $(a, b) = (p_1^2, p_2p_1 + p_1)$ or $(a, b) = \big(p_1^3/(1 - p_1 + p_1^2), (p_1^3 - 2p_1^2 + 2p_1)/(1 - p_1 + p_1^2)\big)$.*

PROOF: We may assume that $0 < a < b < 1$ by Lemma 6. Assume that $\phi(x)h(Tx) = h(x)$. Since $\phi^L(x) = 1$, $\phi^L(x)h^L(Tx) = h^L(x)$ is equivalent to $h^L(Tx) = h^L(x)$. Since T is ergodic, $h^L(x)$ is constant. Hence we may assume that $h^L(x) = 1$. By this fact and by Lemma 3, $h(x)$ can be expressed as

$$h(x) = \exp\left(\frac{2\pi i}{M}\sum_{k=1}^{n-1} b_k \mathbf{1}_{[a_k,a_{k+1}]}(x)\right)$$

where b_k is an integer and $0 = a_1 < a_2 < \cdots < a_n = 1$. We already know that if $f(x) = \lambda h(x)$, then $\phi(x)f(Tx) = f(x)$ also holds. Hence we may also assume that $b_1 = 1$ and $b_2 = 0$.

Since $h(x)$ has $n-2$ discontinuity points and $h(Tx)$ has at least $L(n-2)$ discontinuity points, $h(x)\overline{h(Tx)}$ has at most $(L-1)(n-2)$ discontinuity points. Since $\phi(x)$ has two discontinuity points, we have

$$0 \leqslant n-2 \leqslant \frac{2}{L-1}.$$

Hence if $L \geqslant 4$, then $\phi(x)$ can not be a coboundary. Thus the remaining case is $L = 2, 3$. If $L = 2$, then $n = 3, 4$ and if $L = 3$, then $n = 3$.

In the following, we write by $\beta = \exp(2\pi i/M)$ for convenience.

CASE I. Assume that $L = 2$ and $n = 3$. In this case, we may assume that $h(x) = \beta$ on $[0,c)$ and $h(x) = 1$ on $[c,1)$.

If $c \leqslant p_1$, then $\phi(x) = 1$ on $[0,p_1c)$, $\phi(x) = \beta$ on $[p_1c,c)$, $\phi(x) = 1$ on $[c,p_1)$, $\phi(x) = \overline{\beta}$ on $[p_1,(1-p_1)c+p_1)$ and $\phi(x) = 1$ on $[0,p_1c)$. Hence $\phi(x)h(Tx) \neq h(x)$.

If $c > p_1$, then $\phi(x) = 1$ on $[0,p_1c)$, $\phi(x) = \beta$ on $[p_1c,p_1)$, $\phi(x) = 1$ on $[p_1,c)$, $\phi(x) = \overline{\beta}$ on $[c,(1-p_1)c+p_1)$, and $\phi(x) = 1$ on $[(1-p_1)c+p_1,1)$. Hence $\phi(x)h(Tx) \neq h(x)$.

If $c = p_1$, then $\phi(x) = 1$ on $[0,p_1^2)$, $\phi(x) = \beta$ on $[p_1^2,p_1)$, $\phi(x) = \overline{\beta}$ on $[p_1,(1-p_1)p_1 + p_1)$, and $\phi(x) = 1$ on $[(1-p_1)p_1+p_1,1)$.

Therefore

$$\beta^2 = 1$$

and

$$(a,b) = (p_1^2, (1-p_1)p_1 + p_1).$$

CASE II. Assume that $L = 2$ and $n = 4$. In this case, we may assume that $h(x) = \beta$ on $[0,c)$, $h(x) = 1$ on $[c,d)$ and $h(x) = \gamma$ on $[d,1)$ where $\gamma \neq 1$. Indeed, there exists $s > p_1c$ and $t < (1-p_1)d + p_1$ such that $\phi(x) = 1$ on $[0,p_1c)$, $\phi(x) = \beta$ on $[p_1c,s)$, $\phi(x) = \gamma$ on $[t,(1-p_1)d+p_1)$, and $\phi(x) = 1$ on $[(1-p_1)d+p_1,1)$. Hence $\beta = \gamma$.

If $p_1d > c$, then there exists $t < (1-p_1)d + p_1$ such that $\phi(x) = 1$ on $[0,p_1c)$, $\phi(x) = \beta$ on $[p_1c,c)$, $\phi(x) = 1$ on $[cp_1d)$, $\phi(x) = \beta$ on $[t,(1-p_1)d+p_1)$ and $\phi(x) = 1$ on $[(1-p_1)d+p_1,1)$. Hence $p_1d \leqslant c$.

If $p_1 d < c$, then there exists $t < (1-p_1)d + p_1$ such that $\phi(x) = 1$ on $[0, p_1 c)$, $\phi(x) = \beta$ on $[p_1 c, p_1 d)$, $\phi(x) = 1$ on $[p_1 d, c)$, $\phi(x) = \beta$ on $\big[t, (1-p_1)d + p_1\big)$ and $\phi(x) = 1$ on $\big[(1-p_1)d + p_1, 1\big)$.

Thus $p_1 d \leqslant c$ and by a similar argument, we can show that $(1-p_1)c + p = d$. Therefore $c = p_1^2/(1-p_1+p_1^2)$ and $d = p_1/(1-p_1+p_1^2)$. In this case, $\phi(x) = 1$ on $\big[0, p_1^3/(1-p_1+p_1^2)\big)$, $\phi(x) = \beta$ on $\big[p_1^3/(1-p_1+p_1^2), p_1^2/(1-p_1+p_1^2)\big)$, $\phi(x) = \overline{\beta}$ on $\big[p_1^2/(1-p_1+p_1^2), p_1/(1-p_1+p_1^2)\big)$, $\phi(x) = \beta$ on $\big[p_1/(1-p_1+p_1^2), (p_1^3-2p_1^2+2p_1)/(1-p_1+p_1^2)\big)$ and $\phi(x) = 1$ on $\big[(p_1^3-2p_1^2+2p_1)/(1-p_1+p_1^2), 1\big)$.

Hence

$$\beta^2 = 1$$

and

$$(a,b) = \left(\frac{p_1^3}{1-p_1+p_1^2}, \frac{p_1^3-2p_1^2+2p_1}{1-p_1+p_1^2}\right).$$

CASE III. Assume that $L = 3$ and $n = 3$. In this case, we may assume that $h(x) = \beta$ on $[0, c)$ and $h(x) = 1$ on $[c, 1)$.

If $c < p_1$, then $\phi(x) = 1$ on $[0, p_1 c)$, $\phi(x) = \beta$ on $[p_1 c, c)$, $\phi(x) = 1$ on $[c, p_1)$, $\phi(x) = \overline{\beta}$ on $\big[p_1 + p_2, (1-p_1-p_2)c + p_1 + p_2\big)$ and $\phi(x) = 1$ on $\big[(1-p_1-p_2)c + p_1 + p_2, 1\big)$. Hence $\phi(x)h(Tx) \neq h(x)$. The other case is also similarly verified. ∎

REMARK 3. By a similar argument to that of the above proof, we can show that for the $(p_1, \cdots, p_L)$-transformation, $\phi(x) = \exp\big((2k\pi i)/M \mathbf{1}_{(a,b)}(x)\big)$ is a coboundary if and only if $L = 2$, $(k/M) = 1/2$ and $(a,b) = (p_1^2, p_2 p_1 + p_1)$ or $(a,b) = \big(p_1^3/(1-p_1+p_1^2), (p_1^3 - 2p_1^2 + 2p_1)/(1-p_1+p_1^2)\big)$.

REMARK 4. Let G be the subgroup of $\mathbb{T}$ generated by $\exp(2\pi i/M)$, $\phi(x) = \exp\big((2\pi i)/M \mathbf{1}_E(x)\big)$ be a G-valued function on $X = [0,1)$ and T_ϕ be the skew product transformation on $X \times G$ defined by $T_\phi(x, g) = \big(Tx, \phi(x) \cdot g\big)$. For the $(p_1, \cdots, p_L)$ transformation, T_ϕ is weakly mixing if $\phi(x)$ has discontinuities $p_1 \leqslant t_1 < \cdots < t_n < 1$ or E is an interval and $L \geqslant 3$. Hence T_ϕ is Bernoulli and mod M normality holds almost everywhere.

PROOF: Let U_{T_ϕ} be an unitary operator on $L^2(X \times G)$. Recall that the dual group of G consists of the trivial homomorphism 1 and γ_k defined by $\gamma_k(z) = z^k$ for $1 \leqslant k \leqslant M-1$. Hence

$$L^2(X \times G) = \bigoplus_{k=0}^{L-1} L^2(X) \cdot z^k$$

and each $L^2(X) \cdot z^k$ is an invariant subspace of U_{T_ϕ}. If $f(x,z)$ is an eigen-function with eigenvalue λ then $f(x,z) = \sum_{k=0}^{L-1} f_k(x) \cdot z^k$ and

$$U_{T_\phi} f(x,z) = \sum_{k=0}^{L-1} \phi^k(x) f_k(Tx) \cdot z^k.$$

Since T is weakly mixing, $f_0(x)$ is a constant function, $\phi^k(x)f_k(Tx) = \lambda f_k(x)$ and $\lambda^L = 1$ by the property of $\phi(x)$. Since $\overline{\lambda}\phi^k(x)$ satisfies the conditions of Proposition 1 and Lemma 5, the conclusion follows. ∎

Now we consider the case of the (p_1, p_2)-transformation, $\phi(x) = \exp(\pi i \mathbf{1}_E(x))$ and E being an interval. To check whether $\lim_{N\to\infty} \sum_{n=1}^{N} \exp(\pi i d_n(x)) = 0$ or not, consider the skew product transformation T_ϕ on $[0,1) \times \{-1,1\}$ defined by $T_\phi(x,z) = (Tx, \phi(x)\cdot z)$. Then

$$\lim_{N\to\infty} \sum_{1}^{N} \exp(\pi i d_n(x)) \cdot z = \lim_{N\to\infty} \sum_{1}^{N} U_{T_\phi} f(x,z)$$

where U_{T_ϕ} is an isometry on $L^2(X \times \{-1,1\})$ induced by T_ϕ and $f(x,z) = z$. Hence if T_ϕ is ergodic, then $\lim_{N\to\infty} \sum_{1}^{N} \exp(\pi i d_n(x)) = 0$ by an application of the Birkhoff Ergodic theorem to $f(x,z) = z$. If T_ϕ is not ergodic, then there exists $q(x)$ such that $q(x) = \exp(\pi i \mathbf{1}_F(x))$ for some measurable set F and $\exp(\pi i \mathbf{1}_E(x)) = q(x)q(Tx)$. Furthermore,

$$\lim_{N\to\infty} \sum_{1}^{N} \exp(\pi i d_n(x)) = q(x) \int_{[0,1)} q(t)\, d\mu(t).$$

Hence

(i) if $(a,b) = (p_1^2, (1-p_1)p_1 + p_1)$, then

$$\lim_{N\to\infty} \sum_{1}^{N} \exp(\pi i d_n(x)) = (2p_1 - 1)\exp(\pi i \mathbf{1}_{(c,d)}(x))$$

where $(c,d) = (p_1, 1)$.

(ii) If $(a,b) = (p_1^3/(1-p_1+p_1^2), (p_1^3 - 2p_1^2 + 2p_1)/(1-p_1+p_1^2))$, then

$$\lim_{N\to\infty} \sum_{1}^{N} \exp(\pi i d_n(x)) = \left(\frac{1 - 3p_1 + 3p_1^2}{1 - p_1 + p_1^2}\right) \exp(\pi i \mathbf{1}_{(c,d)}(x))$$

where $(c,d) = (p_1^2/(1-p_1+p_1^2), p_1/(1-p_1+p_1^2))$.

Now we consider some spectral properties of the skew product $T_\phi(x)$.

PROPOSITION 2. *Let T be an weakly mixing transformation on a probability space (X,μ) and $H_\lambda^k = \{h(x) \mid \phi^k(x)h(Tx) = \lambda h(x)\}$ where $\phi(x)$ is a $\mathbb{T}$-valued function. Then the dimension of H_λ^k is 0 or 1. For each k, there exists at most one λ such that the dimension of H_λ^k is 1.*

PROOF: Assume that $f(x)$, $g(x) \in H_\lambda^k$. Then $\phi^k(x)f(Tx) = \lambda f(x)$ and $\phi^k(x)g(Tx) = \lambda g(x)$. Hence $f(Tx)\overline{g(Tx)} = f(x)\overline{g(x)}$. By the ergodicity of T, $f(x)\overline{g(x)} = C$ where C is constant. Thus the first assertion is proved.

Now we shall prove the second assertion. Assume that $\phi^k(x)f(Tx) = \lambda f(x)$ and $\phi^k(x)g(Tx) = \lambda' g(x)$. Hence $f(Tx)\overline{g(Tx)} = \lambda \cdot \lambda' f(x)\overline{g(x)}$. By the mixing property of T, $f(x)\overline{g(x)} = C$ where C is constant and $\lambda \cdot \lambda' = 1$. ∎

PROPOSITION 3. *Let T be an ergodic transformation on X, G the finite subgroup of $\mathbb{T}$ generated by $\exp(2\pi i/M)$ and $\phi(x)$ be a G-valued function. Let T_ϕ be the skew product transformation defined by $T_\phi(x,g) = \big(Tx, \phi(x)\cdot g\big)$ on $X \times G$. If $\phi^k(x)h(Tx) = h(x)$, then there exists $q(x)$ such that the following diagram commutes*

$$\begin{array}{ccc} X \times G & \xrightarrow{T_\phi} & X \times G \\ {\scriptstyle Q}\big\downarrow & & \big\downarrow{\scriptstyle Q} \\ X \times G^k & \xrightarrow{S} & X \times G^k \end{array}$$

where $Q(x,g) = \big(x, q(x)g^k\big)$ and $S(x,g) = (Tx, g)$. Hence T_ϕ has at least r ergodic components where r is the cardinality of G^k.

PROOF: Since $\big(\phi^k(x)\big)^M\big(h(Tx)\big)^M = \big(h(x)\big)^M$ is equivalent to $\big(h(Tx)\big)^M = \big(h(x)\big)^M$ and T is ergodic, we may assume that $\big(h(x)\big)^M = 1$. Hence there exists a G-valued function $q(x)$ such that $\phi^k(x)q(Tx) = q(x)$. For this $q(x)$, it is easy to see that the diagram commutes. ∎

EXAMPLE 2. Consider the $(1/2,1/2)$-transformation and $\phi(x) = \exp\big(\pi i \mathbf{1}_{[1/4,3/4]}(x)\big)$. Let $q(x) = \exp\big(\pi i \mathbf{1}_{[1/2,1]}(x)\big)$. Since $[1/4,3/4] = [1/2,1] \,\triangle\, T^{-1}[1/2,1]$, $\phi(x) = q(x)q(Tx)$. Hence T_ϕ has two ergodic components. Indeed, we can give many examples in which T_ϕ has two ergodic components: For a given F, let $E = F \,\triangle\, T^{-1}F$, $\phi(x) = \exp\big(\pi i \mathbf{1}_E(x)\big)$ and $q(x) = \exp\big(\pi i \mathbf{1}_F(x)\big)$. Then T_ϕ has two ergodic components, $\big\{\big(x, q(x)\big) : x \in X\big\}$ and $\big\{\big(x, -q(x)\big) : x \in X\big\}$.

REFERENCES

[1] Y. Ahn and G.H. Choe, 'Spectral types of skewed Bernoulli shift', *Proc. Amer. Math. Soc.* **128** (2000), 503–510.

[2] G.H. Choe, T. Hamachi and H. Nakada, 'Skew product and mod 2 normal numbers', (preprint).

[3] J.P. Conze, 'Remarques sur les transformations cylindriques et les equations fonctionnelles', *Séminaire de Probabilité I* (Univ. Rennes, Rennes, France 1976).

[4] J.-M. Derrien, 'Critéres d'ergodicité de cocycles en escalier. Exemples', *C.R. Acad. Sci. Paris Ser. I Math.* **316** (1993), 73–76.

[5] E. Hewitt and K. Ross, *Abstract Harmonic analysis I, II* (Springer-Verlag, Berlin, Heidelberg, New York, 1963, 1970).

[6] K.D. Merrill, 'Cohomology of step functions under irrational rotations', *Israel J. Math.* **52** (1985), 320–340.

[7] W. Parry, 'A Cocycle equation for shift', *Contemp. Math.* **135** (1992), 327–333.

[8] W. Rudin, *Real and complex analysis* (McGraw-Hill, New York, 1987).

[9] D.J. Rudolph, 'Classifying the isometric extensions of a Bernoulli shift', *J. Analyse Math.* **34** (1978), 36–60.

[10] W.A. Veech, 'Strict ergodicity of uniform distribution and Kronecker-Weyl theorem mod 2', *Tran. Amer. Math. Soc.* **140** (1969), 1–33.

[11] P. Walters, *An introduction to Ergodic theory* (Springer-Verlag, Berlin, Heidelberg, New York, 1982).

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