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RESEARCH ARTICLE

All Kronecker coefficients are reduced Kronecker coefficients

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Abstract
We settle the question of where exactly do the reduced Kronecker coefficients lie on the spectrum between the Littlewood-Richardson and Kronecker coefficients by showing that every Kronecker coefficient of the symmetric group is equal to a reduced Kronecker coefficient by an explicit construction. This implies the equivalence of an open problem by Stanley from 2000 and an open problem by Kirillov from 2004 about combinatorial interpretations of these two families of coefficients. Moreover, as a corollary, we deduce that deciding the positivity of reduced Kronecker coefficients is NP-hard, and computing them is #P-hard under parsimonious many-one reductions. Our proof also provides an explicit isomorphism of the corresponding highest weight vector spaces.

Contents

1. Introduction

The *Kronecker coefficients* $\mathtt{k}(\lambda, \mu, \nu)$ of the symmetric group S_n are some of the most classical, yet largely mysterious, quantities in algebraic combinatorics and representation theory. The Kronecker coefficient is the multiplicity of the irreducible S_n representation $\mathbb{S}_\nu$ in the tensor product $\mathbb{S}_\lambda \otimes \mathbb{S}_\mu$ of two other irreducible S_n representations. Murnaghan defined them in 1938 as an analogue of the Littlewood-Richardson coefficients $c^\lambda_{\mu\nu}$ of the general linear group GL_N, which are the multiplicity of the irreducible Weyl modules V_λ in the tensor products $V_\mu \otimes V_\nu$. Yet, the analogy has not translated far into their properties. The Littlewood-Richardson coefficients have a beautiful positive combinatorial interpretation, and their positivity is "easy" to decide, formally, it is in P. However, positive combinatorial formulas for the Kronecker coefficients have eluded us so far, see Section 1.2, and their positivity is hard to decide. The *reduced Kronecker coefficients* $\overline{\mathrm{k}}(\alpha, \beta, \gamma)$ are defined as the stable limit of the ordinary Kronecker coefficients

$$\overline{\mathrm{k}}(\alpha, \beta, \gamma) := \lim_{n \to \infty} \mathrm{k}(\,(n - |\alpha|, \alpha),\ (n - |\beta|, \beta),\ (n - |\gamma|, \gamma)\,). \tag{1.1}$$

These coefficients are called *extended Littlewood-Richardson numbers* in [Kir04], since in the special case when $|\alpha| = |\beta| + |\gamma|$, we have

$$\overline{\mathrm{k}}(\alpha, \beta, \gamma) = c^{\alpha}_{\beta,\gamma}, \tag{1.2}$$

the Littlewood-Richardson coefficient. As such, they have been considered as an intermediate, an interpolation, between the Littlewood-Richardson and Kronecker coefficients. Problem 2.32 in [Kir04] asks for a combinatorial interpretation of $\overline{\mathrm{k}}(\alpha, \beta, \gamma)$. They have been an object of independent interest, see [Mur38, Mur56, Bri93, Val99, Kir04, BOR11, BDVO15, CR15, Man15, SS16, IP17, PP20b, OZ19, OZ21], and considered better behaved than the ordinary Kronecker coefficients.

This is, however, not the case. As we show, every Kronecker coefficient is equal to an explicit reduced Kronecker coefficient of not much larger partitions, in particular:

Theorem 1. *For all partitions λ, μ, ν of equal sizes, we have*

$$\mathrm{k}(\lambda, \mu, \nu) \ = \ \overline{\mathrm{k}}\big(\, \nu_1^{\ell(\lambda)} + \lambda,\ \nu_1^{\ell(\mu)} + \mu,\ (\nu_1^{\ell(\lambda)+\ell(\mu)}, \nu)\,\big).$$

Here, $a^b := (\underbrace{a, \ldots, a}_{b \text{ many}})$ and $(\nu_1^b, \nu) := (\underbrace{\nu_1, \ldots, \nu_1}_{b \text{ many}}, \nu_1, \nu_2, \nu_3, \ldots)$.

Theorem 1 implies that in a very strong sense, on the spectrum between Littlewood-Richardson and Kronecker coefficients, the reduced Kronecker coefficients are at the same point as the ordinary Kronecker coefficients. In particular, Theorem 1 implies that Problem 2.32 in [Kir04] is equivalent to Problem 10 in [Sta00]: Finding a combinatorial interpretation for the Kronecker coefficient or for the reduced Kronecker coefficient are the same problem. Formally, Conjectures 9.1 and 9.4 in [Pak22] are the same. Our result can be interpreted in a positive or in a negative way. On the one hand, the reduced Kronecker coefficients cannot be easier to understand than the ordinary Kronecker coefficients. On the other hand, understanding the reduced Kronecker coefficients is sufficient to understand all ordinary Kronecker coefficients. As a corollary, we settle the conjecture from [PP20b, Section 4.4] on the hardness of deciding the positivity of $\overline{\mathrm{k}}(\alpha, \beta, \gamma)$.

Corollary 1 (settles conjecture in [PP20b, Section 4.4]). *Given α, β, γ in unary, deciding if $\overline{k}(\alpha, \beta, \gamma) > 0$ is NP-hard.*

Proof. This follows directly from Theorem 1 and the fact that deciding $\mathrm{k}(\lambda, \mu, \nu) > 0$ is NP-hard [IMW17]. □

Moreover, by the same immediate argument, it is now clear that computing the reduced Kronecker coefficient is strongly #P-hard under parsimonious many-one reductions (the argument in [PP20b] gives only the #P-hardness under Turing reductions).

We discovered the partition triple construction in Theorem 1 by analyzing the natural interpretation of $\mathrm{k}(\lambda, \mu, \nu)$ via the general linear group, see Section 3, and the relationship with 3-dimensional binary contingency arrays. Once the precise statement of Theorem 1 is found, one can give short and self-contained proofs (see Section 2). In particular, in Section 3, we prove Theorem 1 by giving an explicit isomorphism between the corresponding highest weight vector spaces. In Section 4, we give elementary purely combinatorial proofs via symmetric functions, which also bridge the two methodologies.

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Theorem 1 (but not the explicit isomorphism) can also be deduced from a formula from 2011 by E. Briand, R. Orellana and M. Rosas, [BOR11] (see the discussion at the end of Section 4.2).

1.1. Background and definitions

We refer to [JK84, Sta99, Sag13] for basic definitions and properties from algebraic combinatorics and representation theory, and employ the following notation. We write $[a, b] := \{a, a+1, \dots, b\}$, and $[n] := [1, n]$. A *composition* of n is a sequence of nonnegative integers that sum up to n. A *partition* $\lambda = (\lambda_1, \lambda_2, \dots)$ of n, denoted $\lambda \vdash n$, is a weakly decreasing composition. Its size is $|\lambda| := \sum_i \lambda_i$. Denote by $\ell(\lambda) = \max\{i \mid \lambda_i > 0\}$ the *length* of λ. We interpret λ as a vector of arbitrary length $\geq \ell(\lambda)$ by appending zeros. We denote by (n) the partition of n of length 1. To every partition, we associate its *Young diagram*, which is a list of left-justified rows of boxes, λ_i many boxes in row i. We write λ' do denote the *transpose* partition, that is, the partition that arises from reflecting the Young diagram at the main diagonal. Formally, $\lambda'_j := \max\{i \mid \lambda_i \geq j\}$. We add partitions row-wise: $(\lambda + \mu)_i = \lambda_i + \mu_i$. We define $\lambda \diamond \mu := (\lambda' + \mu')'$, adding partitions column-wise as Young diagrams. Note that $\diamond$ is commutative and associative, and that if $\lambda_{\ell(\lambda)} \geq \mu_1$, then $\lambda \diamond \mu = (\lambda_1, \dots, \mu_1, \dots)$ is just the concatenation of rows. The Specht modules $\mathbb{S}_\lambda$ for $\lambda \vdash n$ are the irreducible representation of the symmetric group S_n (see [JK84, Sta99, Sag13]).

The *Kronecker coefficient* $\mathrm{k}(\lambda, \mu, \nu)$ is the structure constant[1] defined via

$$\chi^\mu \cdot \chi^\nu = \sum_\lambda \mathrm{k}(\lambda, \mu, \nu)\chi^\lambda,$$

or, equivalently, via Specht modules as

$$\mathbb{S}_\nu \otimes \mathbb{S}_\mu = \sum_\lambda \mathbb{S}_\lambda^{\oplus \mathrm{k}(\lambda,\mu,\nu)}.$$

From this description, it is immediately clear that $\mathrm{k}(\lambda, \mu, \nu)$ is a nonnegative integer. Yet, the problem of finding a combinatorial interpretation of $\mathrm{k}(\lambda, \mu, \nu)$ is wide open [Sta00, IP22, Pan23].

The Kronecker coefficients were defined by Murnaghan [Mur38] in 1938 as the analogues of the *Littlewood-Richardson coefficients* $c^\lambda_{\mu\nu}$, which are the structure constants in the ring of irreducible GL_N representations, the Weyl modules V_λ, given as $V_\mu \otimes V_\nu = \bigoplus_\lambda V_\lambda^{\oplus c^\lambda_{\mu\nu}}$. Some simple properties, see [JK84, Sag13], include the transposition invariance $\mathrm{k}(\lambda, \mu, \nu) = \mathrm{k}(\lambda', \mu', \nu)$, since $\mathbb{S}_{1^n} \otimes \mathbb{S}_\lambda = \mathbb{S}_{\lambda'}$ [JK84]. From their definition, and the fact that $\chi^\lambda(\pi) \in \mathbb{Z}$, see [JK84, Sag13], we have

$$\mathrm{k}(\lambda, \mu, \nu) = \tfrac{1}{n!} \sum_{\pi \in S_n} \chi^\lambda(\pi)\chi^\mu(\pi)\chi^\nu(\pi),$$

and thus we have the S_3 invariance $\mathrm{k}(\lambda, \mu, \nu) = \mathrm{k}(\lambda, \nu, \mu) = \mathrm{k}(\mu, \nu, \lambda) = \cdots$. Note that the Kronecker coefficient is not invariant under transposing an odd number of partitions, and we define

$$\mathrm{k}'(\lambda, \mu, \nu) := \mathrm{k}(\lambda', \mu', \nu') = \mathrm{k}(\lambda', \mu, \nu) = \mathrm{k}(\lambda, \mu', \nu) = \mathrm{k}(\lambda, \mu, \nu').$$

It is known that $\mathrm{k}(\lambda, \mu, \nu) = 0$ if $\ell(\lambda) > \ell(\mu) \cdot \ell(\nu)$ [Dvi93], which also follows by combining $\mathrm{k}(\lambda, \mu, \nu) = \mathrm{k}(\lambda, \mu', \nu')$ with Lemma 3. We define the *stable range* as the set of triples (λ, μ, ν) that satisfy

$$\forall i \geq 0 : \ \mathrm{k}(\lambda, \mu, \nu) \ = \ \mathrm{k}\big(\, \lambda + (i), \ \mu + (i), \ \nu + (i) \,\big).$$

[1]We remark that in the combinatorics literature, these coefficients have usually been denoted by g, for example, $g(\lambda, \mu, \nu)$, but here, we use k to avoid overlap with the notation used for the representation theory of GL_N.

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There are several proofs for the fact that for arbitrary (α, β, γ) with $|\alpha| = |\beta| = |\gamma|$, the triple $(\alpha + (i),\ \beta + (i), \gamma + (i))$ is in the stable range for i large enough (and hence for all i from then on), and upper bounds on the necessary i are known (see, e.g. [Bri93], [Dvi93], [Val99], [BOR11], [Ike12, Section 7.4], [PP14]). The reduced Kronecker coefficient is defined as the limit value in (1.1), namely, $\overline{\mathtt{k}}(\alpha, \beta, \gamma) := \lim_{n\to\infty} \mathtt{k}(\,(n - |\alpha|, \alpha),\ (n - |\beta|, \beta),\ (n - |\gamma|, \gamma)\,)$ for arbitrary partitions α, β, γ (in particular, we do *not* require $|\alpha| = |\beta| = |\gamma|$). When $|\alpha| = |\beta| + |\gamma|$, then it coincides with the Littlewood-Richardson coefficient $c^{\alpha}_{\beta\gamma}$ from (1.2) (see, e.g. [Mur56, Lit58, Dvi93], and [CŞW18, Section 6]).

1.2. Related work

The Littlewood-Richardson (LR) coefficients can be computed by the Littlewood-Richardson rule, stated in 1934 and proven formally about 40 years later. It says that $c^{\lambda}_{\mu\nu}$ is equal to the number of LR tableaux of shape λ/μ and content ν (see Section 4.1 and [Sta99, Sag13]). The apparent analogy in definitions motivates the community to search for such interpretations for the Kronecker coefficients. Interest in efficient ways to compute $\mathtt{k}(\lambda, \mu, \nu)$ and $\overline{\mathtt{k}}(\alpha, \beta\gamma)$ dates back at least to Murnaghan [Mur38]. Specific interest in nonnegative combinatorial interpretations of $\mathtt{k}(\lambda, \mu, \nu)$ can be found in [Las79, GR85], and was formulated clearly again by Stanley as Problem 10 in his list “Open Problems in Algebraic Combinatorics” [Sta00]: “Find a combinatorial interpretation of the Kronecker product coefficients $\mathtt{k}(\lambda, \mu, \nu)$, thereby combinatorially reproving that they are nonnegative” (see also [Pan23] for a detailed discussion on this topic). Despite its natural and fundamental nature and the variety of efforts, this question has seen relatively little progress. In 1989, Remmel found a combinatorial rule for $\mathtt{k}(\lambda, \mu, \nu)$ when two of the partitions are hooks [Rem89]. In 1994, Remmel and Whitehead [RW94] found $\mathtt{k}(\lambda, \mu, \nu)$ for $\ell(\lambda), \ell(\mu) \le 2$, which was subsequently studied also in [BMS15]. In 2006, Ballantine and Orellana [BO06] established a rule for $\mathtt{k}(\lambda, \mu, \nu)$ when $\mu = (n - k, k)$ and $\lambda_1 \ge 2k - 1$. In general, when the number of rows is fixed, $\mathtt{k}(\lambda, \mu, \nu)$ can be computed in polynomial time [CDW12] (see also [PP17b] for a different approach and related results). The most general rule for $\nu = (n - k, 1^k)$, a hook, and any other two partitions, was established by Blasiak in 2012 [Bla17], and later simplified in [Liu17, BL18]. Other special cases include multiplicity-free Kronecker products by Bessenrodt-Bowman [BB17], triples of partitions which are marginals of pyramids by Ikenmeyer-Mulmuley-Walter [IMW17], $\mathtt{k}(m^k, m^k, (mk - n, n))$ as counting labeled trees by Pak-Panova [Pan15, slide 9], near-rectangular partitions by Tewari in [Tew15], etc. As shown in [IMW17], computing the Kronecker coefficients is #P-hard, and deciding positivity is NP-hard, while in [BCG+23] it is shown that deciding positivity is in QMA.

It was shown by Murnaghan [Mur56] that the reduced Kronecker coefficients generalize the Littlewood-Richardson coefficients (see equation (1.2)). This motivated Kirillov to name $\overline{\mathtt{k}}$ as “extended Littlewood-Richardson numbers”. This relationship and other properties have motivated an independent interest in the reduced Kronecker coefficients as intermediates between Littlewood-Richardson and ordinary Kronecker coefficients. Some special cases of combinatorial interpretations can be derived from the existing ones for the ordinary Kronecker coefficients. A combinatorial interpretation of $\overline{\mathtt{k}}(\alpha, \beta, \gamma)$ in the subcase where $\ell(\alpha) = 1$ was obtained in [BO05, BO06] (see also [CR15]). Methods to compute them have been discussed in [Mur38, Mur56] and have been developed in a series of papers (see [BOR11, BDVO15, OZ19, OZ21]). As observed in [BDVO15], the reduced Kronecker coefficients are also the structure constants for the ring of so-called character polynomials [Mac98]. The reduced Kronecker coefficients are a special case of a more general stability phenomenon that, if $\mathtt{k}(i\alpha, i\beta, i\gamma) = 1$ for all i, then $\mathtt{k}(\lambda + N\alpha, \mu + N\beta, \nu + N\gamma)$ stabilizes as $N \to \infty$ (see [Ste, SS16, Val20]).

The Kronecker coefficients can be expressed as a small alternating sum of reduced Kronecker coefficients, and reduced Kronecker coefficients are certain sums of ordinary Kronecker coefficients for smaller partitions (see [BOR11]). These relationships showed that reduced Kronecker coefficients are also #P-hard to compute (see [PP20b]). However, these relations did not imply that deciding positivity of reduced Kronecker coefficients is NP-hard.

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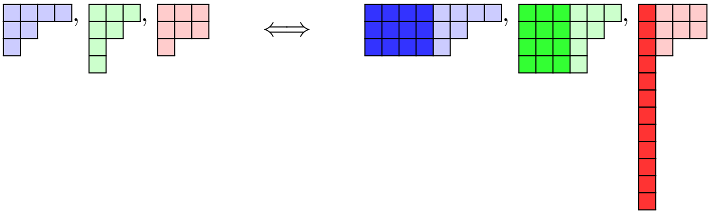

Figure 1. *An example of the situation in Lemma 1 with* $\lambda = (4, 2, 1)$*,* $\mu = (3, 2, 1, 1)$*,* $\nu = (3, 3, 1)$*,* $l = 3$ *and* $m = 4$*.*

It is important to note that deciding if $c^{\lambda}_{\mu\nu} > 0$ is in P, since they count integer points in a polytope that has an integral vertex whenever it is nonempty. This was shown in [MNS12, DLM06] and follows from Knutson-Tao's proof of the saturation theorem for Littlewood-Richardson coefficients [KT99], namely, that $c^{N\lambda}_{N\mu,N\nu} > 0 \iff c^{\lambda}_{\mu\nu} > 0$. The Kronecker coefficients do not satisfy the saturation property, because $\mathrm{k}(2^2, 2^2, 2^2) = 1$, but $\mathrm{k}(1^2, 1^2, 1^2) = 0$. Until recently, it was believed that the reduced Kronecker coefficients have the saturation property: It was conjectured in [Kir04, Kly04] that if $\overline{\mathrm{k}}(N\alpha, N\beta, N\gamma) > 0$ for some $N > 0$, then $\overline{\mathrm{k}}(\alpha, \beta, \gamma) > 0$. This was disproved in [PP20b] in 2020 and moved the reduced Kroneckers away from the Littlewood-Richardson coefficients on that spectrum.

It is known that the Kronecker coefficients (and hence also the reduced Kronecker coefficients) satisfy the so-called *semigroup property* [Chr06, Theorem 2.7], which implies that if $\mathrm{k}(\lambda, \mu, \nu) > 0$, then $\forall N > 0 : \mathrm{k}(N\lambda, N\mu, N\nu) > 0$, and if $\overline{\mathrm{k}}(\alpha, \beta, \gamma) > 0$, then $\forall N > 0 : \overline{\mathrm{k}}(N\alpha, N\beta, N\gamma) > 0$. Deciding whether or not $\exists N : \mathrm{k}(N\lambda, N\mu, N\nu) > 0$ is in NP ∩ coNP, and analogously for $\overline{\mathrm{k}}(N\alpha, N\beta, N\gamma)$ [BCMW17].

2. Setting up the proof of Theorem 1

We set up the proof in this section, reducing to a more general Theorem 2, which has a short proof via GL_N in Section 3. We also give two short, self-contained proofs using basic symmetric function techniques in Section 4.

We prove a slightly stronger statement than Theorem 1: For $l \geq \ell(\lambda)$, $m \geq \ell(\mu)$, $c \geq \nu_1$, we have

$$\mathrm{k}(\lambda, \mu, \nu) = \overline{\mathrm{k}}\big(c^l + \lambda,\ c^m + \mu,\ c^{l+m} \diamond \nu\big). \tag{2.1}$$

We start with Lemma 1, a classical identity that can be proved in several ways (see, e.g. [Dvi93, Theorem 2.4'], [BOR09, Proof of Lemma 2.1], [Val09, Theorem 3.1], [Ike12, Corollary 4.4.14]).

Lemma 1. *Let* λ, μ, ν *be partitions with* $\ell(\lambda) \leq l$*,* $\ell(\mu) \leq m$*. Then*

$$k(\lambda, \mu, \nu) \;=\; k(\, m^l + \lambda,\ l^m + \mu,\ 1^{lm} + \nu\,).$$

The situation is depicted in Figure 1.

In terms of k', instead of k, we can alternatively phrase Lemma 1 as

$$\mathrm{k}'(\lambda, \mu, \nu') \;=\; \mathrm{k}'(\, m^l + \lambda,\ l^m + \mu,\ (lm) \diamond \nu'\,), \tag{2.2}$$

which has a direct proof via an isomorphism of highest weight vector spaces (see (3.3)).

Note that if $\ell(\nu) > lm$, then $\ell(\nu) > lm \geq \ell(\lambda) \cdot \ell(\mu)$, and hence $\mathrm{k}(\lambda, \mu, \nu) = 0$. Moreover, $\ell(1^{lm} + \nu) = \ell(\nu) > lm \geq \ell(\lambda) \cdot \ell(\mu) = \ell(m^l + \lambda) \cdot \ell(l^m + \mu)$, and hence $\mathrm{k}(\, m^l + \lambda,\ l^m + \mu,\ 1^{lm} + \nu\,) = 0$. So we can assume that $\ell(\nu) \leq lm$. We give two proofs in this case, one in Section 3 and one in Section 4.

The following Lemma 2 is proved by applying Lemma 1 twice, in different directions. An illustration of the situation is given in Figure 2.

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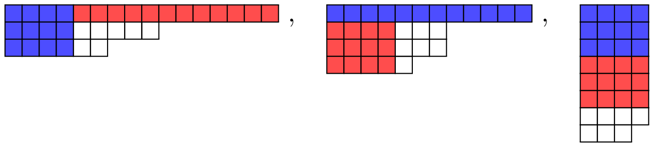

Figure 2. *An example of the proof of Lemma 2 with $\lambda = (5, 2)$, $\mu = (3, 3, 1)$ and $\nu = (4, 3)$, with $l = 2$, $m = 3$ and $c = 4$. The red boxes are the addition from the first application of Lemma 1, and the blue boxes are the second application.*

Lemma 2. *Let λ, μ, ν be partitions of the same size, and let $l \geq \ell(\lambda)$, $m \geq \ell(\mu)$ and $c \geq \nu_1$. Let $d = (m + 1)c$, $e = (l + 1)c$. Then*

$$\mathbf{k}(\lambda, \mu, \nu) \;=\; \mathbf{k}\big((d) \diamond (c^l + \lambda),\; (e) \diamond (c^m + \mu),\; c^{l+m+1} \diamond \nu \big).$$

Proof. We apply Lemma 1 twice as follows.

$$\begin{aligned} \mathrm{k}'(\lambda, \mu, \nu') &\overset{(2.2)}{=} \mathrm{k}'(\, (mc) \diamond \lambda,\; c^m + \mu,\; m^c + \nu' \,) \\ &\overset{(2.2)}{=} \mathrm{k}'(\, c^{l+1} + ((mc) \diamond \lambda),\; (e) \diamond (m^c + \mu),\; (l + m + 1)^c + \nu' \,). \end{aligned}$$ □

Theorem 2. *Let λ, μ, ν be partitions of the same size, such that $\lambda_1 \geq \ell(\mu) \cdot \nu_1$ and $\mu_1 \geq \ell(\lambda) \cdot \nu_1$. Then, for every $h \geq 0$, we have*

$$\mathbf{k}(\lambda, \mu, \nu) = \mathbf{k}(\, \lambda + h,\; \mu + h,\; \nu + h \,).$$

We provide three proofs of this fact, one in Section 3, and two in Section 4. Those sections can be read independently of each other. The proofs make use of an observation on 3-dimensional contingency arrays with zeros and ones as entries (Lemma 4), but they use it in different ways.

We identify subsets $Q \subseteq \mathbb{N}^3$ with their characteristic functions $Q : \mathbb{N}^3 \to \{0, 1\}$, and we call Q a binary or $\{0, 1\}$-contingency array. This means, we interpret Q as a function to $\{0, 1\}$, and as the point set of its support. The interpretation will always be clear from the context. The 2-dimensional marginals of Q are defined as $Q_{i**} := \sum_{j,k} Q_{i,j,k} = |Q \cap (\{i\} \times \mathbb{N} \times \mathbb{N})|$, $Q_{*i*} := \sum_{j,k} Q_{j,i,k} = |Q \cap (\mathbb{N} \times \{i\} \times \mathbb{N})|$, $Q_{**i} := \sum_{j,k} Q_{j,k,i} = |Q \cap (\mathbb{N} \times \mathbb{N} \times \{i\})|$. For $\alpha \in \mathbb{N}^{\mathbb{N}}$, $\beta \in \mathbb{N}^{\mathbb{N}}$, $\gamma \in \mathbb{N}^{\mathbb{N}}$, $|\alpha| = |\beta| = |\gamma| < \infty$, we denote by

$$\mathcal{C}(\alpha, \beta, \gamma) := \{Q \subseteq \mathbb{N}^3 \mid Q_{i**} = \alpha_i,\; Q_{*i*} = \beta_i,\; Q_{**i} = \gamma_i \text{ for every } i\}.$$

There is a close connection to the Kronecker coefficients via the following lemma.

Lemma 3. *For partitions α, β, γ of equal size, we have $\mathbf{k}'(\alpha, \beta, \gamma) \leq |\mathcal{C}(\alpha, \beta, \gamma)|$.*

Proof. There are different proofs of this fact, for example [IMW17, Lemma 2.6] and [PP20a, Theorem 5.3] (see also Sections 3.1 and 4.1). □

The following lemma shows how restrictions on the marginals can result in strong restrictions on the sets Q, a technique that was also applied in [IMW17].

Lemma 4. *Let α, β, γ be compositions with $|\alpha| = |\beta| = |\gamma|$. Let $a \geq \ell(\alpha)$, $b \geq \ell(\beta)$, and let the integers c, h be such that $c + h \geq \ell(\gamma)$ and $\sum_{i>c} \gamma_i \leq h$. Furthermore, let $\alpha_1 \geq bc + h$, $\beta_1 \geq ac + h$.*

Then, for every $Q \in \mathcal{C}(\alpha, \beta, \gamma)$, we have

$$\begin{aligned} &\{1\} \times [b] \times [c] \subseteq Q, \quad [a] \times \{1\} \times [c] \subseteq Q, \quad \{1\} \times \{1\} \times [c + h] \subseteq Q, \text{ and} \\ &Q \cap (\mathbb{N} \times \mathbb{N} \times [c + 1, c + h]) = \{1\} \times \{1\} \times [c + 1, c + h]. \end{aligned}$$

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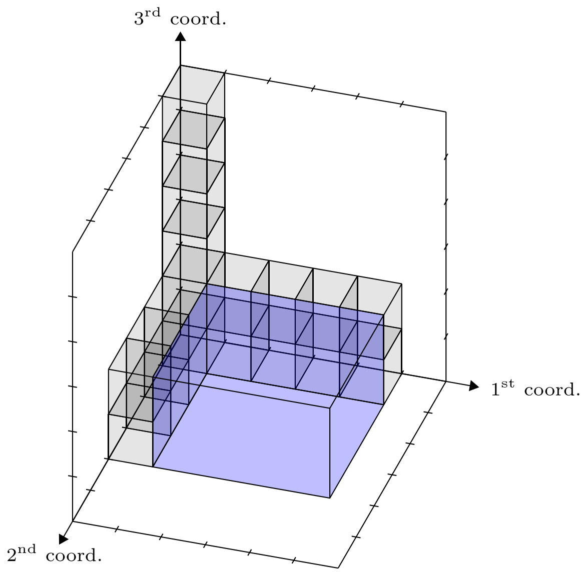

Figure 3. *Lemma 4 for $a = 5$, $b = 4$, $c = 2$, $h = 4$. A gray cube represents a forced 1 in the contingency array. Absence of color represents a forced 0 in the contingency array. The blue box shows the area where both zeros and ones are possible.*

In particular, if $\mathcal{C}(\alpha, \beta, \gamma)$ is nonempty, then $a = \ell(\alpha)$, $b = \ell(\beta)$, $\gamma_i = 1$ for all $c + 1 \leq i \leq c + h$, and $\alpha_1 = bc + h$, $\beta_1 = ac + h$, $\alpha_2 \leq bc$ and $\beta_2 \leq ac$.

In other words, if we have 3-dimensional point configurations with such marginals, then the walls consist of two rectangles and a long column as depicted in Figure 3.

Proof. Assume that there exists a binary contingency array $Q \in \mathcal{C}(\alpha, \beta, \gamma)$. Let $B_\cup := \{1\} \times [b] \times [c + h] \cup [a] \times \{1\} \times [c+h]$ be the set of points in the planes $x = 1$ and $y = 1$, and let $B_\cap := \{1\} \times \{1\} \times [c+h]$ be the set of points on the line $x = y = 1$. Let $H_i := Q \cap (\mathbb{N} \times \mathbb{N} \times \{i\}) \cap B_\cup$ be the entries of Q in $B_\cup$ at the section with the plane $z = i$. In particular,

$$\sum_{i=1}^{c+h} |H_i| = |Q \cap B_\cup|.$$

We have $\sum_{i=c+1}^{c+h} |H_i| \leq \sum_{i=c+1}^{c+h} \gamma_i \leq h$, $|H_i| \leq a + b - 1$ for all $0 < i \leq c$ and $|Q \cap B_\cap| \leq c + h$. All these inequalities must be met with equality, because

$$\begin{aligned} \alpha_1 + \beta_1 &= |Q \cap B_\cap| + |Q \cap B_\cup| \\ &= |Q \cap B_\cap| + \textstyle\sum_{i=1}^{c+h} |H_i| \\ &= |Q \cap B_\cap| + \textstyle\sum_{i=1}^{c} |H_i| + \sum_{i=c+1}^{c+h} |H_i| \\ &\leq (c + h) + (a + b - 1)c + h \\ &= (a + b)c + 2h \\ &\leq \alpha_1 + \beta_1. \end{aligned}$$

We thus have the following equalities: $|Q \cap B_\cap| = c + h = |B_\cap|$ and $\forall i \in [c]$, we have $|H_i| = a + b - 1 = |(\mathbb{N} \times \mathbb{N} \times \{i\}) \cap B_\cup|$. Thus, we have $B_\cap \subseteq Q$ and $\{1\} \times [b] \times [c] \subseteq Q$ and $[a] \times \{1\} \times [c] \subseteq Q$ and

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$Q \cap (\mathbb{N} \times \mathbb{N} \times [c+1, c+h]) = \{1\} \times \{1\} \times [c+1, c+h]$. This gives the desired marginals, and the claim follows. $\square$

We now prove (2.1), which implies Theorem 1.

Proof of (2.1). Let $d = mc + c$ and $e = lc + c$. Suppose first that $\lambda_1 \le mc$ and $\mu_1 \le lc$. We apply Lemma 2, and obtain

$$\mathrm{k}(\lambda, \mu, \nu) = \mathrm{k}\big(\underbrace{(d) \diamond (c^l + \lambda)}_{=:\hat{\lambda}}, \underbrace{(e) \diamond (c^m + \mu)}_{=:\hat{\mu}}, \underbrace{c^{l+m+1} \diamond \nu}_{=:\hat{\nu}} \big).$$

The top rows of $\hat{\lambda}, \hat{\mu}, \hat{\nu}$ are d, e, c, respectively, and thus Theorem 2 gives that for all $h \in \mathbb{N}$ we have

$$\begin{aligned} \mathrm{k}(\hat{\lambda},\ \hat{\mu},\ \hat{\nu}) &= \mathrm{k}(\hat{\lambda} + h,\ \hat{\mu} + h,\ \hat{\nu} + h) \\ &= \mathrm{k}(\ (d+h) \diamond (c^l + \lambda),\ (e+h) \diamond (c^m + \mu),\ (c+h) \diamond c^{l+m} \diamond \nu\ ) \\ &= \overline{\mathrm{k}}(c^l + \lambda,\ c^m + \mu,\ c^{l+m} \diamond \nu), \end{aligned}$$

where the last identity follows by letting $h \to \infty$. This proves (2.1) in the first case.

Suppose now that $\lambda_1 > mc$, the case $\mu_1 > lc$ is completely analogous. Set $b := m + 1$. Then we have $\mathrm{k}(\lambda, \mu, \nu) = \mathrm{k}(\lambda', \mu, \nu') = 0$ since $\ell(\lambda') = \lambda_1 > mc \ge \ell(\mu)\ell(\nu')$. On the other hand, the reduced Kronecker coefficient is obtained by adding long first rows, $cm + c + h$, $cl + c + h$, $c + h$, respectively, so

$$\begin{aligned} &\overline{\mathrm{k}}(c^l + \lambda,\ c^m + \mu,\ c^{l+m} \diamond \nu) \\ &= \mathrm{k}\big((cm + c + h) \diamond (c^l + \lambda),\ (lc + c + h) \diamond (c^m + \mu),\ (c+h) \diamond c^{l+m} \diamond \nu)\big) \\ &= \mathrm{k}'\big(\underbrace{(cm + c + h) \diamond (c^l + \lambda)}_{=:\alpha},\ \underbrace{(lc + c + h) \diamond (c^m + \mu)}_{=:\beta},\ \underbrace{((l+b)^c + \nu') \diamond (1^h)}_{=:\gamma} \big) \end{aligned}$$

for sufficiently large h. Let $\hat{\gamma} = (l+b)^c + \nu'$ be γ without the h many trailing 1s. We observe that $\alpha_2 = \lambda_1 + c$, $\ell(\beta) = b$ and $\ell(\hat{\gamma}) = c$. From $\lambda_1 > mc$, we conclude $\alpha_2 > bc$. Lemma 4 shows that $\mathcal{C}(\alpha, \beta, \gamma) = \emptyset$. Hence, $\mathrm{k}'(\alpha, \beta, \gamma) = 0$ by Lemma 3. $\square$

3. Proofs via the general linear group

3.1. Tools from the general linear group viewpoint

We refer to [Ful97, Section 8] for the basic properties of the irreducible representations of the general linear group. The irreducible representations $V_{a^b}(\mathbb{C}^b)$ of the general linear group GL_b are 1-dimensional: For $g \in \mathrm{GL}_b$, $v \in V_{a^b}(\mathbb{C}^b)$, we have $gv := \det(g)^a v$. Hence, if we decompose $V_{1^{ab}}(\mathbb{C}^{ab})$ as a $\mathrm{GL}_a \times \mathrm{GL}_b$ representation via the group homomorphism $\mathrm{GL}_a \times \mathrm{GL}_b \to \mathrm{GL}_{ab}$, $(g_1, g_2) \mapsto g_1 \otimes g_2$, then we obtain $V_{1^{ab}}(\mathbb{C}^{ab}) \simeq V_{b^a}(\mathbb{C}^a) \otimes V_{a^b}(\mathbb{C}^b)$. Tensoring with such a 1-dimensional representation preserves irreducibility: $V_\lambda(\mathbb{C}^a) \otimes V_{b^a}(\mathbb{C}^a) \simeq V_{b^a + \lambda}(\mathbb{C}^a)$.

The Kronecker coefficients have an interpretation as the structure coefficients arising when decomposing irreducible GL_{ab} representations as $\mathrm{GL}_a \times \mathrm{GL}_b$ representations:

$$V_\nu(\mathbb{C}^{ab}) \simeq \bigoplus_{\substack{\lambda \vdash_a |\nu| \\ \mu \vdash_b |\nu|}} \Big(V_\lambda(\mathbb{C}^a) \otimes V_\mu(\mathbb{C}^b) \Big)^{\oplus \mathrm{k}(\lambda, \mu, \nu)}.$$

This can be seen directly from Schur-Weyl duality (see, e.g. [Chr06, (2.2)] or [Ike12, Proposition 4.4.11]). Another formulation is via the multiplicity of the irreducible $G := \mathrm{GL}_a \times \mathrm{GL}_b \times \mathrm{GL}_c$ representation

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$V_\alpha(\mathbb{C}^a) \otimes V_\beta(\mathbb{C}^b) \otimes V_\gamma(\mathbb{C}^c)$ in the D-th wedge power of $\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ (see [IMW17]). Formally for partitions $\alpha, \beta, \gamma \vdash D$, we have

$$\mathrm{k}'(\alpha,\beta,\gamma) \;=\; \mathrm{mult}_{\alpha,\beta,\gamma}\left(\bigwedge^D(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c)\right).$$

A vector v for which $\big(\mathrm{diag}(r_1,\dots,r_a), \mathrm{diag}(s_1,\dots,s_b), \mathrm{diag}(t_1,\dots,t_c)\big)v = r_1^{\lambda_1}\cdots r_a^{\lambda_a}\cdot s_1^{\mu_1}\cdots s_b^{\mu_b}\cdot t_1^{\nu_1}\cdots t_c^{\nu_c} v$is called a *weight vector* of weight (λ,μ,ν).

For $(A,B,C) \in \mathbb{C}^{a\times a}\times\mathbb{C}^{b\times b}\times\mathbb{C}^{c\times c}$, the Lie algebra action on $\bigwedge^D(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c)$ is defined as $(A,B,C).v := \lim_{\varepsilon\to 0}\varepsilon^{-1}((\varepsilon(A,B,C) + (\mathrm{id}_a,\mathrm{id}_b,\mathrm{id}_c))v - v)$. A raising operator is the Lie algebra action of $(E_{i-1,i},0,0)$, where $E_{i,j}$ is the matrix with a 1 at position (i,j) and zeros everywhere else. The other raising operators are $(0,E_{i-1,i},0)$ and $(0,0,E_{i-1,i})$. Let $e_i := (0,\dots,0,1,0,\dots,0)^T$, and let $e_{i,j,k} := e_i \otimes e_j \otimes e_k$. Then, for example, $(E_{i,j},0,0)e_{r,1,1} = e_{i,1,1}$ iff $r = j$ and 0 otherwise. A highest weight vector (HWV) of weight (α,β,γ) is a nonzero weight vector of weight (α,β,γ) that is mapped to zero by all raising operators. The irreducible $\mathrm{GL}_a\times\mathrm{GL}_b\times\mathrm{GL}_c$ representation $V_\alpha\otimes V_\beta\otimes V_\gamma$ contains exactly one HWV (up to scale), and that is of weight (α,β,γ). Hence, ([IMW17, Lemma 2.1]),

$$\mathrm{k}'(\alpha,\beta,\gamma) = \dim\left(\mathrm{HWV}_{\alpha,\beta,\gamma}\bigwedge^D(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c)\right), \tag{3.1}$$

where $\mathrm{HWV}_{\alpha,\beta,\gamma}$ denotes the space of HWVs of weight (α,β,γ). Note that each standard basis vector in $\bigwedge^D(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c)$ is a weight vector, and hence for each weight vector space of weight w, we have a basis given by the set of standard basis vectors of weight w. Let $e_{i,j,k} := e_i\otimes e_j\otimes e_k$, and for a list of points $Q \in (\mathbb{N}^3)^D$, we define $\psi_Q := e_{Q_1}\wedge e_{Q_2}\wedge\cdots\wedge e_{Q_D}$. If Q has marginals (α,β,γ), then ψ_Q has weight (α,β,γ). This immediately implies the result of Lemma 3.

We illustrate the concept with some examples. The HWVs in $\bigwedge^2(\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^1)$ are $e_{1,1,1}\wedge e_{2,1,1}$ and $e_{1,1,1}\wedge e_{1,2,1}$. A nontrivial example is the HWV

$$t := e_{1,1,1}\wedge e_{2,1,1}\wedge e_{1,2,2} + e_{1,1,1}\wedge e_{1,2,1}\wedge e_{2,1,2} + e_{1,1,1}\wedge e_{1,1,2}\wedge e_{2,2,1}$$

of weight $((2,1),(2,1),(2,1))$ in $\bigwedge^3(\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2)$, which can be seen as follows:

$$\begin{aligned}
(E_{1,2},0,0)t &= e_{1,1,1}\wedge e_{1,2,1}\wedge e_{1,1,2} + e_{1,1,1}\wedge e_{1,1,2}\wedge e_{1,2,1} = 0,\\
(0,E_{1,2},0)t &= e_{1,1,1}\wedge e_{2,1,1}\wedge e_{1,1,2} + e_{1,1,1}\wedge e_{1,1,2}\wedge e_{2,1,1} = 0,\\
(0,0,E_{1,2})t &= e_{1,1,1}\wedge e_{2,1,1}\wedge e_{1,2,1} + e_{1,1,1}\wedge e_{1,2,1}\wedge e_{2,1,1} = 0.
\end{aligned}$$

3.2. Proofs from the general linear group viewpoint

For the sake of completeness, we present the short proof of Lemma 1 from [BOR09, Proof of Lemma 2.1] and [Ike12, Corollary 4.4.14].

Proof of Lemma 1 via the general linear group. We have that $V_{m^l+\lambda}(\mathbb{C}^l)\otimes V_{l^m+\mu}(\mathbb{C}^m)$ occurs in $V_{1^{lm}+\nu}(\mathbb{C}^{lm})$ with multiplicity $\mathrm{k}(m^l+\lambda,\ l^m+\mu,\ 1^{lm}+\nu)$. But we also have

$$\begin{aligned}
V_{1^{lm}+\nu}(\mathbb{C}^{lm}) &\simeq V_{1^{lm}}(\mathbb{C}^{lm})\otimes V_\nu(\mathbb{C}^{lm})\\
&\simeq (V_{m^l}(\mathbb{C}^l)\otimes V_{l^m}(\mathbb{C}^m)) \;\otimes \bigoplus_{\substack{\lambda\vdash_l|\nu|\\ \mu\vdash_m|\nu|}}\Big(V_\lambda(\mathbb{C}^l)\otimes V_\mu(\mathbb{C}^m)\Big)^{\oplus\mathrm{k}(\lambda,\mu,\nu)}\\
&\simeq \bigoplus_{\substack{\lambda\vdash_a|\nu|\\ \mu\vdash_b|\nu|}}\Big(V_{m^l+\lambda}(\mathbb{C}^l)\otimes V_{l^m+\mu}(\mathbb{C}^m)\Big)^{\oplus\mathrm{k}(\lambda,\mu,\nu)}.
\end{aligned}$$

□

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Proof of Theorem 2 via contingency arrays and highest weight vectors. Let $a := \ell(\lambda)$, $b := \ell(\mu)$, $c := \nu_1$. Let $\gamma := \nu'$, so $\ell(\gamma) = c$. We have $\lambda_1 \geq bc$ and $\mu_1 \geq ac$. Observe that $\mathtt{k}(\lambda, \mu, \nu) = \mathtt{k}'(\lambda, \mu, \gamma)$. Let $\widetilde{\lambda} = \lambda + (h)$, $\widetilde{\mu} = \mu + (h)$, $\widetilde{\gamma} = \gamma \diamond (1^h)$. We define an injective linear map φ as follows

$$\begin{aligned}\varphi : \bigwedge^D (\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c) &\to \bigwedge^{D+h} (\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^{c+h}) \\ v &\mapsto v \wedge e_{1,1,c+1} \wedge e_{1,1,c+2} \wedge \cdots \wedge e_{1,1,c+h}.\end{aligned} \tag{3.2}$$

Note that φ maps vectors of weight (λ, μ, γ) to vectors of weight $(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\gamma})$. It remains to show that φ maps HWVs to HWVs, and that every HWV of weight $(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\gamma})$ has a preimage under φ.

We first prove that φ sends HWVs to HWVs. By construction of φ, we observe that for $1 \leq i < i' \leq a$, we have

$$(E_{i,i'}, 0, 0)\varphi(u) = \varphi((E_{i,i'}, 0, 0)u) = \varphi(0) = 0.$$

Analogously, $(0, E_{j,j'}, 0)\varphi(u) = 0$ for $1 \leq j < j' \leq b$, and $(0, 0, E_{k,k'})\varphi(u) = 0$ for $1 \leq k < k' \leq c$. The remaining raising operators also vanish by construction of φ, because

$$\begin{aligned}&(0, 0, E_{c+k,c+k'})(v \wedge e_{1,1,c+1} \wedge \cdots \wedge e_{1,1,c+h}) \\ &\quad = v \wedge e_{1,1,c+1} \wedge \cdots \wedge e_{1,1,c+k} \wedge e_{1,1,c+k} \wedge \widehat{e_{1,1,c+k'}} \wedge \cdots \wedge e_{1,1,c+h} \\ &\quad = 0,\end{aligned}$$

because of the repeated factor $e_{1,1,c+k}$. Here, the $\widehat{e_{1,1,c+k'}}$ means omission of that factor. For every basis vector ψ_Q of weight $(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\gamma})$, by Lemma 4, we have $(1, 1, c) \in Q$ and $Q \cap (\mathbb{N} \times \mathbb{N} \times \{c+1\}) = \{(1, 1, c+1)\}$, hence, $(0, 0, E_{c,c+1})\psi_Q = 0$ as well.

We now show that every weight vector of weight $(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\gamma})$ has a preimage under φ, which finishes the proof. It is sufficient to show this for basis vectors. Let ψ_Q be a basis weight vector of weight $(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\gamma})$, that is, $Q \subseteq \mathbb{N}^3$ with marginals $(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\gamma})$. We apply Lemma 4 to see that $\{1\} \times \{1\} \times [c + 1, c + h] \subset Q$ and $Q \cap (\mathbb{N} \times \mathbb{N} \times \{i\}) = \{(1, 1, i)\}$ for all $c + 1 \leq i \leq c + h$. Therefore, ψ_Q has a preimage under φ, namely, ψ_P, where P arises from Q by deleting all points with 3rd coordinate $> c$. $\square$

Note that (2.2) (and hence also Lemma 2) can alternatively be proved by a simple explicit linear map between highest weight vector spaces

$$\begin{aligned}\mathrm{HWV}_{\lambda,\mu,\nu^t}\Big(\bigwedge^D (\mathbb{C}^l \otimes \mathbb{C}^m \otimes \mathbb{C}^c)\Big) &\to \mathrm{HWV}_{m^l+\lambda,\ l^m+\mu,\ (lm)\diamond\nu^t}\Big(\bigwedge^{D+lm} (\mathbb{C}^l \otimes \mathbb{C}^m \otimes \mathbb{C}^{c+1})\Big) \\ e_{i_1,j_1,k_1} \wedge \cdots \wedge e_{i_D,j_D,k_D} &\mapsto e_{i_1,j_1,k_1+1} \wedge \cdots \wedge e_{i_D,j_D,k_D+1} \wedge e_{1,1,1} \wedge \cdots \wedge e_{l,m,1}\end{aligned} \tag{3.3}$$

for $D = |\lambda|$. Combining this with the construction in Theorem 2, Theorem 1 is now fully proved by an explicit isomorphism of highest weight vector spaces.

4. Proofs via symmetric functions

4.1. Tools from symmetric functions

Here, we recall basic definitions and facts from symmetric function theory (see [Sta99, Sag13, Mac98]).

The standard Young tableaux (SYT) of shape $\lambda \vdash n$ are assignments of $1, 2, \ldots, n$ to the Young diagram of λ, so that the numbers are decreasing along rows and down columns, and each number appears exactly once. A semi-standard Young tableaux (SSYT) of skew shape λ/μ is an assignment of integers $1, 2, \ldots, N$ to the boxes of the skew Young diagram λ/μ, such that the values weakly increase along rows and strictly down columns. We say that an SSYT T has type (weight) $type(T) = \alpha$ if there are α_i entries equal to i for each i.

As an example, $\begin{array}{|c|c|c|c|c|}\cline{3-5} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 & 1 & 3 \\ \cline{2-5} \multicolumn{1}{c|}{} & 2 & 3 & 3 & \multicolumn{1}{c}{} \\ \cline{1-4} 1 & 3 & 4 & 4 & \multicolumn{1}{c}{} \\ \cline{1-4} \end{array}$ is an SSYT of shape $(5,4,4)/(2,1)$ and type $(3,1,4,2)$.

The ring of symmetric functions Λ has several fundamental bases. Here, we will use the monomial basis $\{m_\lambda\}$ given by $m_\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \ldots + \cdots$, summing over all distinct monomials with exponents $\lambda_1, \lambda_2, \ldots$. We will also use the homogeneous symmetric functions $\{h_\lambda\}$ given by

$$h_m := \sum_{i_1 \le i_2 \le \cdots \le i_m} x_{i_1} x_{i_2} \cdots x_{i_m} \text{ for } m > 0;$$

$h_0 = 1$; $h_m = 0$ for $m < 0$ and $h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots$. The *Schur functions* s_λ are one of the fundamental bases of the ring Λ of symmetric functions. Moreover, $s_\lambda(x_1, \ldots, x_N)$ is the value of the character of V_λ at a matrix with eigenvalues $x_1, \ldots, x_N$. We have the following formulas for them, where $\ell = \ell(\lambda)$,

$$\text{Jacobi-Trudi identity:} \quad s_\lambda = \det[h_{\lambda_i - i + j}]_{i,j=1}^{\ell} \tag{4.1}$$

$$\text{Weyl determinantal formula:} \quad s_\lambda(x_1, \ldots, x_N) = \frac{\det[x_i^{\lambda_j + N - j}]_{i,j=1}^N}{\det[x_i^{N-j}]} \tag{4.2}$$

$$\text{via SSYTs:} \quad s_\lambda = \sum_{T \in SSYT(\lambda)} x^{type(T)}. \tag{4.3}$$

The Littlewood-Richardson coefficients are the structure constants in the ring of symmetric functions as

$$s_\mu(x) s_\nu(x) = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda(x).$$

They can be computed combinatorially via the Littlewood-Richardson rule: $c^\lambda_{\mu\nu}$ is equal to the number of SSYTs T of shape λ/μ, type ν and whose reading words are a ballot sequence. The reading word is obtained by reading the tableaux right to left along rows, top to bottom, and a word is a ballot sequence if, in every prefix, the number of i's is not less than the number of $i+1$'s for every i. The multi-LR coefficients $c^\lambda_{\alpha^1 \cdots \alpha^k}$ are defined as

$$c^\lambda_{\alpha^1 \cdots \alpha^k} := \langle s_\lambda, s_{\alpha_1} s_{\alpha^2} \cdots s_{\alpha^k} \rangle = \sum_{\beta^1, \beta^2, \ldots} c^\lambda_{\alpha^1 \beta^1} c^{\beta^1}_{\alpha^2 \beta^2} \cdots c^{\beta^{k-1}}_{\alpha^{k-1} \alpha^k}, \tag{4.4}$$

where the sum is over partitions $\beta^i \vdash |\beta^{i-1}| - |\alpha^i|$ with $\beta^0 := \lambda$. It is then easy to see that they count SSYTs T of shape λ and type $(\alpha^1 \diamond \alpha^2 \diamond \cdots)$, such that the reading word of each skew subtableau corresponding to the entries with values between $1 + \sum_{i=1}^r \ell(\alpha^i)$ and $\sum_{i=1}^{r+1} \ell(\alpha^i)$ is a lattice permutation for every $r = 1, \ldots, k-1$. For example,

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&1&4&4&6\\ \hline 2&2&2&4&5&7&\multicolumn{1}{c}{}\\ \cline{1-6} 3&5&5&6&6&\multicolumn{1}{c}{}&\multicolumn{1}{c}{}\\ \cline{1-5}\end{array} \quad \text{and} \quad \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&1&4&4&6\\ \hline 2&2&2&4&6&6&\multicolumn{1}{c}{}\\ \cline{1-6} 3&5&5&5&7&\multicolumn{1}{c}{}&\multicolumn{1}{c}{}\\ \cline{1-5}\end{array}$$

are two multi-LR tableaux of shape $\lambda = (7,6,5)$ and types $\alpha^1 = (4,3,1)$, $\alpha^2 = (3,3)$, $\alpha^3 = (3,1)$.

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The Kronecker coefficient can be studied via the following expansions

$$s_\lambda[x\cdot y] = \sum_{\mu,\nu} \mathrm{k}(\lambda,\mu,\nu) s_\mu(x) s_\nu(y), \tag{4.5}$$

where $x\cdot y = (x_1y_1, x_1y_2, \dots, x_2y_1, \dots)$ consists of the pairwise products of the two sets of variables. In particular, this gives that

$$h_m[x\cdot y] = \sum_\lambda s_\lambda(x)s_\lambda(y).$$

From the Jacobi-Trudy identity, we thus obtain

$$\begin{aligned} s_\lambda[x\cdot y] &= \det[h_{\lambda_i-i+j}[x\cdot y]] \\ &= \sum_{\sigma\in S_\ell} \mathrm{sgn}(\sigma) \sum_{\alpha^i\vdash\lambda_i-i+\sigma_i} s_{\alpha^1}(x)\cdots s_{\alpha^\ell}(x) s_{\alpha^1}(y)\cdots s_{\alpha^\ell}(y), \end{aligned}$$

so

$$\mathrm{k}(\lambda,\mu,\nu) = \sum_{\sigma\in S_\ell} \mathrm{sgn}(\sigma) \sum_{\alpha^i\vdash\lambda_i-i+\sigma_i} c^{\mu}_{\alpha^1\cdots\alpha^k} c^{\nu}_{\alpha^1\cdots\alpha^k}. \tag{4.6}$$

Note that this identity appears many times in the literature, including [Val09, PP17b, PP17a].

The following are referred to as the “triple Cauchy identities” (see, e.g. [Sta99, Exercise 7.78]):

$$\sum_{\lambda,\mu,\nu} \mathrm{k}(\lambda,\mu,\nu) s_\lambda(x)s_\mu(y)s_\nu(z) = \prod_{i,j,k} \frac{1}{1-x_iy_jz_k},$$

$$\sum_{\lambda,\mu,\nu} \mathrm{k}(\lambda,\mu,\nu) s_\lambda(x)s_\mu(y)s_{\nu'}(z) = \prod_{i,j,k} (1+x_iy_jz_k),$$

where the second identity follows from the first via the involution ω on the symmetric functions in the variables z. Denote by $C(\alpha,\beta,\gamma) := |\mathcal{C}(\alpha,\beta,\gamma)|$. Then the second identity becomes

$$\sum_{\lambda,\mu,\nu} \mathrm{k}(\lambda,\mu,\nu) s_\lambda(x)s_\mu(y)s_{\nu'}(z) = \sum_{\alpha,\beta,\gamma} C(\alpha,\beta,\gamma) x^\alpha y^\beta z^\gamma. \tag{4.7}$$

Note that this identity immediately gives the upper bound in Lemma 3 by comparing coefficients at $x^\lambda y^\mu z^{\nu'}$ on both sides.

We now express $\mathrm{k}(\lambda,\mu,\nu)$ as an alternating sum over contingency arrays. Denote by

$$\Delta(x) = \det[x_i^{a-j}] = \prod_{i<j}(x_i-x_j) = \sum_{\sigma\in S_a} \mathrm{sgn}(\sigma) x_1^{a-\sigma_1}\cdots x_a^{a-\sigma_a},$$

and multiply by $\Delta(x)\Delta(y)\Delta(z)$ both sides of equation (4.7). Expressing the Schur functions via the ratio of determinants in 4.2, we obtain

$$\begin{aligned} &\sum_{\lambda,\mu,\nu} \mathrm{k}(\lambda,\mu,\nu) \det[x_i^{\lambda_j+a-j}] \det[y_i^{\mu_j+b-j}] \det[z_i^{\nu'_j+c-j}] \\ &\qquad = \Delta(x)\Delta(y)\Delta(z) \sum_{\alpha,\beta,\gamma} C(\alpha,\beta,\gamma) x^\alpha y^\beta z^\gamma. \end{aligned}$$

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Comparing coefficients at $x_1^{\lambda_1+a-1}\dots y_1^{\mu_1+b-1}\dots z_1^{\nu_1'+c-1}\dots$ on both sides, we obtain $\mathrm{k}(\lambda,\mu,\nu)=$

$$[x_1^{\lambda_1+a-1}\dots y_1^{\mu_1+b-1}\dots z_1^{\nu_1'+c-1}\dots]\Delta(x)\Delta(y)\Delta(z)\sum_{\alpha,\beta,\gamma}C(\alpha,\beta,\gamma)x^\alpha y^\beta z^\gamma,$$

where the $[\cdots]$ denotes the coefficient extraction. Expanding the Δs into monomials, whose marginals we incorporate, we get that we must have $\lambda_i+a-i=\alpha_i+a-\sigma_i$ etc., so $\alpha_i=\lambda_i+\sigma_i-i$, and we obtain, see also [PP20a], $\mathrm{k}(\lambda,\mu,\nu)=$

$$\sum_{\sigma\in S_a,\,\pi\in S_b,\,\rho\in S_c}\operatorname{sgn}(\sigma)\operatorname{sgn}(\pi)\operatorname{sgn}(\rho)C(\lambda+\sigma-\mathrm{id},\mu+\pi-\mathrm{id},\nu'+\rho-\mathrm{id}),\tag{4.8}$$

where a permutation σ is interpreted as the vector $(\sigma(1),\dots,\sigma(a))$ and $\mathrm{id}=(1,2,\dots)$ is the identity permutation of the corresponding size.

4.2. Proofs via symmetric functions

Proof of Lemma 1 via symmetric functions. Let $\hat\nu=1^{lm}+\nu$. We use Schur functions as follows. We apply equation (4.5) with variables $x_1,\dots,x_\ell$ and $y_1,\dots,y_m$, so $x_i=0$ for $i>l$ and $y_j=0$ for $j>m$, and we obtain

$$s_{\hat\nu}[x\cdot y]=\sum_{\theta,\tau}\mathrm{k}(\hat\nu,\theta,\tau)s_\theta(x)s_\tau(y).\tag{4.9}$$

If $\mathrm{k}(\hat\nu,\theta,\tau)>0$, we must have $\ell(\theta)\ell(\tau)\ge\ell(\hat\nu)=lm$. Since $s_\theta(x_1,\dots,x_l)=0$ if $\ell(\theta)>l$ and $s_\tau(y_1,\dots,y_m)=0$ if $\ell(\tau)>m$, we then must have only the partitions with $\ell(\theta)=l$, $\ell(\tau)=m$ appearing.

Since $s_{\hat\nu}$ is the generating function over SSYTs with entries $x_1y_1,\dots,x_ly_m$, and its first column has length exactly lc, we must have all the entries x_iy_j appearing exactly once in that column. As this is the minimal possible column, the rest of the SSYT can be any of the SSYTs of the remaining shape and entries $x_1y_1,\dots,x_ly_m$. Thus

$$s_{\hat\nu}[x\cdot y]=s_\nu[x\cdot y]\prod_{i,j}x_iy_j=(x_1\dots x_l)^m(y_1,\dots,y_m)^l\sum_{\rho,\eta}\mathrm{k}(\nu,\rho,\eta)s_\rho(x)s_\eta(y).\tag{4.10}$$

We also note that $s_{l^m+\mu}(y_1,\dots,y_m)=(y_1\dots y_m)^l s_\mu(y)$, since the first l columns of length m are forced to be filled with $1,\dots,m$, and for the remaining tableaux, there are no restrictions other than being an SSYT. Similarly, $s_{m^l+\lambda}(x_1,\dots,x_l)=(x_1\dots x_l)^m s_\lambda(x)$. Comparing coefficients of $s_\lambda(x)s_\mu(y)$ at equations (4.9) and (4.10), we thus see that

$$\mathrm{k}(\hat\nu,l^m+\mu,m^l+\lambda)=\mathrm{k}(\nu,\lambda,\mu).$$

□

Proof of Theorem 2 via contingency arrays and symmetric functions. From now on, we will use formula (4.8) and Lemma 4 to show that the only possible contingency arrays are the ones in Figure 2.

Consider now $\mathrm{k}(\lambda+h,\mu+h,\nu+h)$ as in the problem, and let $\alpha=(\lambda+h),\beta=(\mu+h),\gamma=(\nu+h)'$, so that $\mathrm{k}(\alpha,\beta,\gamma')=\mathrm{k}(\lambda+h,\mu+h,\nu+h)$. Let $\nu_1=c,\ell(\lambda)=a$ and $\ell(\mu)=b$, so we have $\alpha_1\ge bc+h,\beta_1\ge ac+h$, $\gamma_i=1$ for $i=c+1,\dots,c+h$ and $\mathrm{k}(\alpha,\beta,\gamma')=$

$$\sum_{\sigma\in S_a,\,\pi\in S_b,\,\rho\in S_{c+h}}\operatorname{sgn}(\sigma)\operatorname{sgn}(\pi)\operatorname{sgn}(\rho)C(\alpha+\sigma-\mathrm{id},\beta+\pi-\mathrm{id},\gamma+\rho-\mathrm{id}).\tag{4.11}$$

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In formula (4.11), we then consider $\{0,1\}$-contingency arrays Q with marginals

$$Q_{1**} := \sum_{j,k} Q_{1,j,k} = \lambda_1 + \sigma_1 - 1 \geq bc + h,$$
$$Q_{*1*} := \sum_{i,k} Q_{i,1,k} = \mu_1 + \pi_1 - 1 \geq ac + h,$$
$$Q_{**k} := \sum_{i,j} Q_{i,j,k} = 1 + \rho_k - k, \text{ for } k = c+1, \dots, c+h.$$

Note that then we have

$$\sum_{k>c} Q_{**k} = h + \sum_{k=c+1}^{c+h} \rho_k - \sum_{k=c+1}^{c+h} k \leq h, \tag{4.12}$$

and the support of the array is in $[1,a] \times [1,b] \times [1,c+h]$, so we can apply Lemma 4 and conclude that $Q_{1,j,k} = 0$ iff $(j,k) \in [2,b] \times [c+1,c+h]$ and $Q_{i,1,k} = 0$ iff $(i,k) \in [2,a] \times [c+1,c+h]$.

Thus, we must have $Q_{1**} = bc + h$, $Q_{*1*} = ac + h$, and so $\sigma_1 = \pi_1 = 1$, $\{\rho_{c+1}, \dots, \rho_{c+h}\} = \{c+1, \dots, c+h\}$, and for $k \in [c+1, c+h]$, we must have $Q_{i,j,k} = 0$ unless $i = j = 1$. This also forces us to have $Q_{1,1,k} = 1$ for all these k, and so $\rho_k = k$ for $k = c+1, \dots, c+h$.

This completely determines $Q_{i,j,k}$ for $k > c$, as well as ρ_k for $k > c$, and $\rho = \bar{\rho}, (c+1), \dots, (c+h)$ for $\bar{\rho} \in S_c$. We can thus write formula (4.11) as

$$\begin{aligned}
&\mathsf{k}(\lambda + h, \mu + h, \nu + h) \\
&= \sum_{\sigma \in S_a,\, \pi \in S_b,\, \rho \in S_{c+h}} \mathrm{sgn}(\sigma)\mathrm{sgn}(\pi)\mathrm{sgn}(\rho)\, C(\alpha + \sigma - id, \beta + \pi - id, \gamma + \rho - id) \\
&= \sum_{\sigma \in S_a,\, \pi \in S_b,\, \eta \in S_c} \mathrm{sgn}(\sigma)\mathrm{sgn}(\pi)\mathrm{sgn}(\eta)\, C(\bar{\alpha} + \sigma - id, \bar{\beta} + \pi - id, \bar{\gamma} + \eta - id),
\end{aligned}$$

where $\bar{\alpha} = \alpha - (h) = \lambda$, $\bar{\beta} = \beta - (h) = \mu$ and $\bar{\gamma} = (\gamma_1 \dots, \gamma_c) = \nu'$. As the last part coincides with the expression for $\mathsf{k}(\lambda, \mu, \nu)$ in (4.8), we get the desired identity. □

Proof of Theorem 2 via Littlewood-Richardson coefficients. Let again $\ell(\lambda) = a$, $\ell(\mu) = b$ and $\nu_1 = c$.

We have that $\mathsf{k}(\lambda + h, \mu + h, \nu + h) = \mathsf{k}(\nu' \diamond (1^h), \lambda' \diamond (1^h), \mu + h)$, and we are going to apply formula (4.6) with that triple of partitions. Set $\hat{\mu} = \mu + h$, $\hat{\lambda} = \lambda' \diamond (1^h) = (\lambda + h)'$ and $\hat{\nu} = \nu' \diamond (1^h)(\nu + h)'$. Here, $\ell(\nu' \diamond (1^h)) = c + h$, so

$$\mathsf{k}(\lambda + h, \mu + h, \nu + h) = \sum_{\sigma \in S_{c+h}} \mathrm{sgn}(\sigma) \sum_{\alpha^i \vdash \hat{\nu}_i - i + \sigma_i} c^{\hat{\lambda}}_{\alpha^1 \alpha^2 \dots} c^{\hat{\mu}}_{\alpha^1 \alpha^2 \dots}.$$

We will now characterize the possible partitions α^i involved in this sum. From the iterated definition of the multi-LR coefficients (4.4), we see that in order for the coefficients to be nonzero, we must have $\alpha^i \subset \hat{\mu}$ and $\alpha^i \subset \hat{\lambda}$. In particular, then $\ell(\alpha^i) \leq \ell(\mu) = b$ and $\alpha^i_1 \leq \hat{\lambda}_1 = a$. Note that multi-LR coefficients count certain SSYTs of type $(\alpha^1 \diamond \alpha^2 \diamond \dots \diamond \alpha^c \diamond \dots)$, and thus in the shape $\hat{\lambda}$, the first column will have at most $\ell(\alpha^1) + \dots + \ell(\alpha^c) \leq bc$ many entries from the first c partitions. So there are at least h boxes in the first column which need to be covered by the partitions $\alpha^{c+1}, \dots, \alpha^{c+h}$. We then have

$$h \leq \ell(\alpha^{c+1}) + \dots + \ell(\alpha^{c+h}) \leq |\alpha^{c+1}| + \dots + |\alpha^{c+h}| = \sum_{i=c+1}^{c+h} 1 - i + \sigma_i \leq h,$$

as $\sigma_{c+1} + \cdots + \sigma_{c+h} \leq c + 1 + \cdots c + h$. Thus, we need to have equalities, and so

$$|\alpha^{c+1}| + \cdots + |\alpha^{c+h}| = h, \ell(\alpha^i) = |\alpha^i|,$$

so α^i are single column partitions, possibly empty.

Further, we have $\alpha^i \leq a$, $\alpha^i \subset \hat{\mu}$. As there is a multi-LR of type $(\alpha^1 \diamond \alpha^2 \cdots)$, the first row of that tableaux can only be occupied by the smallest entries of each type. So we must have

$$ac + h = \hat{\mu}_1 \leq \sum_i \alpha_1^i \leq \sum_{i=1}^{c} a + \sum_{i=c+1}^{c+h} \alpha_1^i.$$

Thus, $\alpha_1^{c+1} + \cdots + \alpha_1^{c+h} \geq h$. Since $\alpha_1^i \leq 1$ by the above consideration, we must have $\alpha^i = (1)$ for all $i > c$. So $\sigma_i = i$ for $i = c + 1, \ldots, c + h$.

Then

$$c^{\hat{\lambda}}_{\alpha^1 \alpha^2 \cdots \alpha^{c+h}} = c^{\lambda'}_{\alpha^1 \cdots \alpha^c} \quad \text{and} \quad c^{\hat{\mu}}_{\alpha^1 \alpha^2 \cdots \alpha^{c+h}} = c^{\mu}_{\alpha^1 \cdots \alpha^c}.$$

We thus get that

$$\begin{aligned} \mathrm{k}(\lambda + h, \mu + h, \nu + h) &= \sum_{\sigma \in S_{c+h}} \mathrm{sgn}(\sigma) \sum_{\alpha^i \vdash \hat{\nu}_i - i + \sigma_i} c^{\hat{\lambda}}_{\alpha^1 \alpha^2 \ldots} c^{\hat{\mu}}_{\alpha^1 \alpha^2 \ldots} \\ = \sum_{\sigma \in S_c} \mathrm{sgn}(\sigma) \sum_{\alpha^i \vdash \nu'_i - i + \sigma_i} c^{\lambda'}_{\alpha^1 \alpha^2 \ldots} c^{\mu}_{\alpha^1 \alpha^2 \ldots} &= \mathrm{k}(\nu', \lambda', \mu) = \mathrm{k}(\lambda, \mu, \nu), \end{aligned}$$

which completes the proof. □

We now discuss how Theorem 1 can be seen from the following identity, which first appeared in [BOR11], where it was proven using symmetric function operators. It was then reformulated in [BDVO15, Theorem 4.3], which studies the partition algebra, as follows.

Set $m = r + s$, and let $\nu \vdash m - l$, $\lambda \vdash r$, $\mu \vdash s$ for some nonnegative integer l. Then

$$\overline{\mathrm{k}}(\lambda, \mu, \nu) = \sum_{\substack{l_1, l_2 \\ l = l_1 + 2l_2}} \sum_{\substack{\alpha \vdash r - l_1 - l_2 \\ \beta \vdash s - l_1 - l_2}} \sum_{\substack{\pi, \rho, \sigma \vdash l_1 \\ \gamma \vdash l_2}} c^{\nu}_{\alpha,\beta,\pi} c^{\lambda}_{\alpha,\rho,\gamma} c^{\mu}_{\gamma,\sigma,\beta} \mathrm{k}(\pi, \rho, \sigma).$$

We now apply it to compute the reduced Kronecker coefficient in Theorem 1. Let $\hat{\lambda} = \nu_1^{\ell(\lambda)} + \lambda$, $\hat{\mu} = \nu_1^{\ell(\mu)} + \mu$ and $\hat{\nu} = (\nu_1^{\ell(\lambda)+\ell(\mu)}, \nu)$. Then $m - l = (\ell(\lambda) + \ell(\mu))\nu_1 + n$, $r = \ell(\lambda)\nu_1 + n$, $s = \ell(\mu)\nu_1 + n$, so $l = n$ and the above identity translates to

$$\overline{\mathrm{k}}(\hat{\lambda}, \hat{\mu}, \hat{\nu}) = \sum_{\substack{l_1, l_2 \\ n = l_1 + 2l_2}} \sum_{\substack{\alpha \vdash r - l_1 - l_2 \\ \beta \vdash s - l_1 - l_2}} \sum_{\substack{\pi, \rho, \sigma \vdash l_1 \\ \gamma \vdash l_2}} c^{\hat{\nu}}_{\alpha,\beta,\pi} c^{\hat{\lambda}}_{\alpha,\rho,\gamma} c^{\hat{\mu}}_{\gamma,\sigma,\beta} \mathrm{k}(\pi, \rho, \sigma).$$

We now observe that $|\alpha| \geq \ell(\lambda)\nu_1$, and if $c^{\hat{\lambda}}_{\alpha,\rho,\gamma} > 0$, $c^{\hat{\nu}}_{\alpha,\beta,\pi} > 0$, then $\alpha \subset \hat{\nu} \cap \hat{\lambda} = (\nu_1^{\ell(\lambda)})$. Then we must have $\alpha = \nu_1^{\ell(\lambda)}$, and so $l_1 + l_2 = n$. Since $l_1 + 2l_2 = n$, we must have $l_2 = 0$ and $l_1 = n$, so $\gamma = \emptyset$. Similarly, we obtain $\beta = (\nu_1^{\ell(\mu)})$, leaving us with

$$\overline{\mathrm{k}}(\hat{\lambda}, \hat{\mu}, \hat{\nu}) = \sum_{\pi, \rho, \sigma \vdash n} c^{\hat{\nu}}_{\alpha,\beta,\pi} c^{\hat{\lambda}}_{\alpha,\rho} c^{\hat{\mu}}_{\sigma,\beta} \mathrm{k}(\pi, \rho, \sigma).$$

Observe that the Littlewood-Richardson rule gives, since α is the rectangle in the beginning of $\hat{\lambda}$, that $c^{\hat{\lambda}}_{\alpha,\rho} = 0$ if $\rho \neq \hat{\lambda}/\alpha = \lambda$, and is 1 otherwise. Similarly, the other Littlewood-Richardson coefficients

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are zero unless $\sigma = \mu$ and $\pi = \nu$, we are left with only one partition triple $(\pi, \rho, \sigma) = (\nu, \lambda, \mu)$, whose coefficient is 1 and the identity in Theorem 1 follows.

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