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# A GENERAL ASYMPTOTIC RESULT FOR PARTITIONS

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§ **1.** In this paper we are concerned with partition functions $p_\gamma(n)$ that have generating functions of the form

$$\sum_{n=0}^{\infty} p_\gamma(n)x^n = \prod_{n=1}^{\infty} (1 - x^n)^{-\gamma(n)}$$

where $\gamma(n) \geqq 0$. We shall obtain an asymptotic relation for $p_\gamma(n)$ under suitable restrictions on $\gamma$ (see Theorem 1.1). These restrictions are weaker than those of Brigham [**2**] who considered this problem previously.

We have considered this problem when $\gamma(n)$ is 0 or 1 previously in [**7**] and [**8**]. Wright [**13**] has treated the case $\gamma(n) = n$ when $p_\gamma(n)$ is the number of plane partitions of $n$. Various other plane partition functions and their asymptotic expansions have been considered by Gordon and Houten [**5**] and Gordon [**4**].

In § 3 we apply our asymptotic result to a certain weighted partition function involving primes and powers of primes. We derive under no unproven hypothesis a result of Brigham [**3**] obtained previously under the assumption of the Riemann hypothesis. We also show that for $n$ sufficiently large, the $k$th difference of this particular weighted function is positive, thus proving the monotonicity of this function which was left undecided in [**3**]. Finally, we examine a connection between the zeros of the Riemann zeta function and the error term in the approximation of this partition function by elementary functions.

First of all, we require a series of definitions which are very similar to those of [**7**].

We define the function $f_\gamma$ for real $x > 0$ by

$$f_\gamma(x) = \sum_{m=1}^{\infty} \gamma(m)e^{-xm}.$$

We say that $\gamma$ has property (I) if with $\epsilon > 0$ an arbitrary constant and $\mu$ fixed

$$\sum_{m=1}^{\infty} \gamma(m)(xm)^{\mu}e^{-xm} = O\{f_\gamma^{\,1+\epsilon}(x)\}$$

and

$$f_\gamma(x)/f_\gamma(x(1 - f_\gamma^{\,-(1+\epsilon)/3}(x))) = O\{1\}$$

as $x \to 0$.

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Let

$$F_\gamma(x) = \sum_{m \leqq x} \gamma(m).$$

We say $\gamma$ has property (II) if there exists some constant $\eta$ with $\frac{1}{3} > \eta > 0$ such that

$$F_\gamma(x^{-1} f_\gamma^{-\delta}(x))/\log f_\gamma(x) \to \infty$$

and

$$F_\gamma(x^{-1}) > f_\gamma^{(2/3)+\eta}(x)$$

as $x \to 0$.

We shall see that $\gamma$ has properties (I) and (II) when either

$$\text{(i)}\ \left(\overline{\lim_{x\to\infty}} \frac{\log F_\gamma(x)}{\log x}\right) \Big/ \left(\underline{\lim_{x\to\infty}} \frac{\log F_\gamma(x)}{\log x}\right) < \frac{3}{2};$$

or with $c$ an arbitrary positive constant

(ii) $\log F_\gamma(x) \sim c \log \log x \quad (x \to \infty)$

or

(iii) $F_\gamma(2x) = O\{F_\gamma(x)\}$.

For each fixed positive integer $n$, we define $\alpha$ throughout this paper to be the unique solution of

$$(1.1)\quad n = \sum_{m=1}^{\infty} \gamma(m) m (e^{\alpha m} - 1)^{-1}.$$

Throughout this paper any equations or estimates involving $\alpha$ may only hold for $\alpha$ sufficiently small or equivalently $n$ sufficiently large.

We define $B_\mu = B_\mu(n)$ $(\mu = 2, 3, \ldots)$ by

$$B_\mu = \sum_{m=1}^{\infty} \gamma(m) g_\mu(e^{\alpha m}(e^{\alpha m} - 1)^{-\mu}$$

where $g_\mu(x)$ is a certain polynomial (the same as in [**7**] or the $g_\mu^*$ of Roth and Szekeres [**11**]) of degree $\leqq \mu - 1$ and in particular $g_1(x) = 1$ and $g_2(x) = x$. $D_\rho = D_\rho(n) (\rho = 1, 2, \ldots)$ is defined by

$$D_\rho = B_2^{-6\rho} \sum_{\mu_1=2}^{\infty} \cdots \sum_{\mu_{5\rho}=2}^{\infty} d_{\mu_1\mu_2\ldots\mu_{5\rho}} B_{\mu_1} B_{\mu_2} \ldots B_{\mu_{5\rho}}$$

the summation being subject to

$$\mu_1 + \mu_2 + \ldots + \mu_{5\rho} = 12\rho,$$

where the $d$'s are certain numerical constants.

Let us say that $\gamma$ is a $P$-function if the integers $l$ such that $\gamma(l) \neq 0$ do not have a common factor $> 1$ for all sufficiently large $n$.

Our first result can now be stated.

THEOREM 1.1. *Let $\gamma$ have properties (I) and (II). Suppose that $\gamma$ is a P-function and that*

$$\min_{\gamma(l)\neq 0} \gamma(l) > 0.$$

*Suppose furthermore that*

$$\varliminf_{x\to\infty} \frac{\log F_\gamma(x)}{\log\log x} > 0.$$

*Let $m$ be any fixed integer $\geqq 2$. Then*

$$p_\gamma(n) = (2\pi B_2)^{1/2} \exp\left\{\alpha n - \sum_{l=1}^{\infty} \gamma(l) \log (1 - e^{-\alpha l})\right\} \times \left[1 + \sum_{\rho=1}^{m-2} D_\rho + O\left\{f_\gamma^{\,1-(2m/3)}(\alpha)\right\}\right].$$

§**2.** The proof of Theorem 1.1 is similar to the proof of Theorem 1.1 of [**7**], hence we shall only consider those differences which are significant and sketch the proof.

First of all, let us show that when one of Equations (i), (ii) or (iii) holds, then $\gamma$ has properties (I) and (II).

That (iii) is sufficient follows from minor changes in the proof of Lemma 2.5 of [**7**] and in part *b*) of the proof of Lemma 3.3 of [**7**].

If (i) holds, it follows readily from

$$(2.1)\quad f_\gamma(x) = x\int_0^\infty F_\gamma(u)e^{-xu}du$$

with

$$l = \varliminf_{x\to\infty} \log F_\gamma(x)/\log x$$

$$L = \varlimsup_{x\to\infty} \log F_\gamma(x)/\log x$$

that for $\epsilon > 0$ an arbitrary constant $x \to 0$

$$(2.2)\quad x^{\epsilon-l} < f_\gamma(x) < x^{\epsilon-L}.$$

Clearly then, (i) is sufficient for property (II). Next we show that if (i) holds, then

$$(23)\quad x\int_0^\infty F_\gamma(u)(xu)^\mu e^{-xu}du = O\{f_\gamma^{\,1+\epsilon}(x)\}.$$

However, from (2.2) it follows that

$$(2.4)\quad x\int_{x^{-1}f_\gamma^{\,\epsilon/\mu}(\alpha)}^\infty F_\gamma(u)(ux)^\mu e^{-ux}du = O\{x^{1-\epsilon-L}\exp(-f_\gamma^{\,\epsilon/\mu}(\alpha)/2)\}.$$

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It follows from this and (2.2) that (2.3) holds. Let $\theta$ be the operator defined by

$$\theta = x\, d/dx.$$

We can differentiate under the integral sign in 2.1 and since

$$\theta^\mu f_\gamma(x) = \sum_{m=1}^{\infty} \gamma(m)(xm)^\mu e^{-\alpha m}$$

the first part of property (I) follows from (2.3). Now let

$$y = x(1 - f_\gamma^{-(1+\epsilon)/3}(x)).$$

Since $f_\gamma(x) \to \infty$ as $x \to 0$ we have $y \sim x$ as $x \to 0$ and it follows from (2.4) that with constant $\eta$, $0 < \eta < \frac{1}{3}$

$$f_\gamma(y) = y \int_0^{x^{-1} f_\gamma^{\eta}(x)} F_\gamma(u) e^{-\beta u} du + O\{1\};$$

however

$$e^{-yu}/e^{-xu} = \exp\,(ux\, f_\gamma^{-(1+\epsilon)/3}(x)) = O\{\exp\,(f_\gamma^{\eta-(1+\epsilon)/3}(x))\} = O\{1\}.$$

It readily follows that

$$f_\gamma(y) = O\{f_\gamma(x)\}$$

and the second part of property (II) holds.

If (ii) holds, then it follows as above that

$$\text{(2.5)} \quad \log f_\gamma(x) \sim c \log\log\,(1/x) \sim \log\, F_\gamma(1/x)$$

and property (II) follows. Furthermore we obtain that

$$\log\left[x \int_0^\infty F_\gamma(u)(xu)^\mu e^{-xu} du\right] \sim c \log\log\left(\frac{1}{x}\right) \sim \log F_\gamma\left(\frac{1}{x}\right)$$

and thus the first part of property (I) holds. The second part follows readily from (2.5).

The proof of Theorem 1.1 uses a saddle-point technique along the lines of Roth and Szekeres [**11**]. Using Cauchy's theorem, we express $p_\gamma(n)$ as a contour integral over the circle of radius $\exp\,(-\alpha n)$ centred at the origin. The point $\exp\,(-\alpha n)$ is a saddlepoint and the integration along the smaller arc of the circle from $\exp\,(-\alpha n - i\theta_0)$ to $\exp\,(-\alpha n + i\theta_0)$ where $\theta_0 = \alpha f_\gamma^{(1+\eta)/3}(\alpha)$ gives the dominant part of the integral. The asymptotic behaviour of this integral is obtained as in § 2 of [**7**]. The proof that the integral over the remaining arc is negligible is very similar to § 3 of [**7**]. Note however that it is necessary that $\log f_\gamma(x)/\log 1/x = O\{1\}$ for Theorem 3.1 of [**7**] to give a negligible order of magnitude and this is where the condition

$$\min_{\gamma(m)\neq 0} \gamma(m) > 0$$

is used.

If $\gamma$ is not a $P$-function, the asymptotic result is rather cumbersome to state and corresponds to Theorem 1.1 of [**8**]. For the sake of completeness, we give a result in this direction.

Let $A$ be the set of monotonically increasing integers $A = \{n|\gamma(n) \neq 0\}$. Let us say as do Bateman and Erdös [**1**] that $A$ has property $P_k$ if, when we remove an arbitrary subset of $k$ elements from $A$, the remaining elements have greatest common divisor unity. If $A$ has property $P_k$ but not $P_{k+1}$, let $\bar{A}$ denote a set of $k + 1$ elements of $A$ such that $A - \bar{A}$ has greatest common divisor greater than one.

The proof of Theorem 1.1 of [**8**] gives:

THEOREM 2.1. *Let $\gamma$ have properties (I) and (II). Let $\gamma$ have property $P_k$ but not $P_{k+1}$. Let $A$ and $\bar{A}$ be defined as above. Suppose that*

$$l = \min_{n \in A} \gamma(n) > 0.$$

*Suppose furthermore that*

$$\varliminf_{x \to \infty} \frac{\log F_\gamma(x)}{\log \log x} > 0.$$

*Let $m$ be any fixed integer $\geqq 2$. Then*

$$p_\gamma(n) = (2\pi B_2)^{-1/2} \exp\left\{\alpha n - \sum_{l=1}^{\infty} \gamma(l) \log\{1 - e^{-\alpha l}\}\right\} \times \left[1 + \sum_{\rho=1}^{m-2} D_\rho + O\{f_\gamma^{1-(2m/3)}(\alpha)\} + O\{\alpha^{\sum_{a \in \bar{A}} \gamma(a)} B_2^{1/2}\}\right].$$

§**3.** In this section we consider the weighted partition function $p_\Lambda(n)$ of Brigham obtained by choosing $\gamma(m) = \Lambda(m)$ where $\Lambda(p^b) = \log p$, $p$ prime, and $\Lambda(m) = 0$ for all other values of $n$. Asymptotic results of the type obtained by Brigham have been obtained for the number of partitions of $n$ into primes and distinct primes (see, for example [**11**]); however, Brigham's function can be approximated more accurately by elementary functions and is interesting for its own sake. We consider the function itself instead of the summatory function.

It is well-known that

$$\sum_{m=1}^{x} \Lambda(m) \sim x$$

and since $\Lambda$ is a $P$-function, the conditions of Theorem 1.1 are satisfied. We denote the Riemann zeta function and the gamma function as usual. In [**3**] it is shown that

$$g(\alpha) = \exp\left(- \sum_{m=1}^{\infty} \Lambda(m) \log(1 - e^{-\alpha m})\right) = K\, \alpha^{\log 2\pi} \exp\left(\frac{\pi^2}{6\alpha} + Z(\alpha) + O(\alpha)\right) \tag{3.1}$$

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where

$$K = \exp\left(-1 + \frac{\pi^2}{24} + \sum_\rho \frac{1}{\rho^2}\right)$$

and

$$Z(\alpha) = \sum_\rho \frac{\pi(2\pi)^\rho \zeta(-\rho)}{\rho\alpha^\rho \sin\left(\pi\frac{\rho}{2}\right)} = -\sum_\rho \frac{\zeta(1+\rho)\Gamma(\rho)}{\alpha^\rho},$$

the summations being taken over the complex zeros of $\zeta(s)$ counted according to their multiplicities.

Recall that

$$\begin{aligned}(3.2)\quad n &= \sum_{m=1}^{\infty} m\Lambda(m)(e^{\alpha m}-1)^{-1} \\ &= \sum_{m=1}^{\infty} m\Lambda(m)\sum_{l=1}^{\infty} e^{-\alpha m l} \\ &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \alpha^{-s}\Gamma(s)\zeta(s)\sum_{m=1}^{\infty}\frac{\Lambda(m)}{m^{s-1}}\,d_s, \quad \sigma > 2 \\ &= \frac{-1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \alpha^{-s}\Gamma(s)\zeta(s)\frac{\zeta'(s-1)}{\zeta(s-1)}\,ds, \quad \sigma > 2.\end{aligned}$$

It is shown in the same way in [**3**] that

$$(3.3)\quad \log g(\alpha) = \frac{-1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \alpha^{-s}\Gamma(s)\zeta(1+s)\frac{\zeta'(s)}{\zeta(s)}\,ds, \quad \sigma > 1.$$

Note that

$$\frac{d^2}{d\alpha^2}\left\{\sum_{m=1}^{\infty} \Lambda(m)\log(1-e^{-\alpha m})\right\} = B_2(\alpha),$$

hence the first term of the asymptotic expansion of $p_\Lambda(n)$ obtained from Theorem 1.1 is $\alpha$ times the asymptotic formula obtained for the summatory function in [**3**]. More generally, if we consider the $k$th difference $p_\gamma^{(k)}(n)$ defined by

$$\sum_{n=1}^{\infty} p_\gamma^{(k)}(n)x^n = (1-x)^k \sum_{n=1}^{\infty} p_\gamma(n)x^n,$$

then the proof of Theorem 5.1 of [**9**] is easily modified to show that under the assumptions of Theorem 1.1

$$p_\gamma^{(k)}(n) \sim [\alpha_{(k)}]^{(k)} p_\gamma(n)$$

where $\alpha_{(k)}$ is defined by an equation corresponding to (1.1). However, the proof of Theorem 5.2 of [**10**] shows that the same $\alpha$, in particular the $\alpha$ of

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equation (1.1), may be used for all $k$. In our case, we have

$$p_\Lambda^{(k)}(n) \sim \alpha^k p_\Lambda(n)$$

which shows that the $k$th differences are positive for $n$ sufficiently large.

We have verified Brigham's result and furthermore shown that all the $k$th differences of $p_\Lambda(n)$ are positive for $n$ sufficiently large.

Let us determine as far as seems possible the asymptotic estimation of $p_\Lambda(n)$ in terms of elementary functions. From (3.2)

$$(3.4) \quad n = \frac{\alpha^{-1}}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \alpha^{-s}\Gamma(1+s)\zeta(1+s)\left(-\frac{\zeta'(s)}{\zeta s}\right) ds, \quad \sigma > 1.$$

Furthermore

$$(3.5) \quad \Gamma(1+s)\zeta(1+s)\left(-\frac{\zeta'(s)}{\zeta(s)}\right) = \frac{\zeta(2)}{s-1} + b_0 + b_1(s-1) + 1\,.\,.$$

in a neighbourhood of 1. From [**12**, p. 114] there exists a constant $c > 0$ such that $\zeta'(s)/\zeta(s)$ has no singularities in the region $s = \sigma + it$

$$\sigma \geqq 1 - c(\log|s|)^{-3/4}(\log\log|s|)^{-3/4}.$$

We may estimate the integral in (3.4) much as in [**6**, pp. 77–81] to obtain

$$n = \frac{\pi^2}{6\alpha^2} + O\left\{\alpha^{-2}\exp\left(-c\log\frac{1}{\alpha}\right)^{4/7}\left(\log\log\frac{1}{\alpha}\right)^{-3/7}\right\}$$

for some constant $c > 0$.

Let us temporarily assume $n$ to be a continuous variable in Equation (1.1) in order to derive relations which shall hold for all $n$ (in particular integral $n$)

$$\alpha = \frac{\pi}{\sqrt{6}} n^{-1/2} g(n)$$

Since $\alpha \to 0$ as $n \to \infty$ it follows that

$$n = n\,g^{-2}(n) + o\{n\,g^{-2}(n)\}$$
$$g^2(n) = 1 + o(1)$$

and since $\alpha > 0$ we conclude that $g(n) \sim 1$ as $n \to \infty$. Now let $g(n) = 1 + f(n)$ where $f(n) = o(1)$,

$$\log\left(\frac{1}{\alpha}\right) = \log\left[\frac{\sqrt{6}}{\pi} n^{1/2}(1 - f(n) + O\{f^2(n)\}\right]$$
$$= (1/2)\log n + O\{1\}.$$

Thus

$$\left(\log\frac{1}{\alpha}\right)^{4/7} \sim ((1/2)\log n)^{4/7} \text{ as } n \to \infty$$

$$\left(\log\log\frac{1}{\alpha}\right)^{-3/7} \sim (\log\log n)^{-3/7} \text{ as } n \to \infty$$

and we conclude that there is a constant $c_1 > 0$ such that

$$\exp\left(-c\left(\log\frac{1}{\alpha}\right)^{4/7}\left(\log\log\frac{1}{\alpha}\right)\right)^{-3/7}$$
$$= O\{\exp(-c_1(\log n)^{4/7}(\log\log n)^{-3/7})\}$$

Now we write

$$\alpha = \frac{\pi}{\sqrt{6}}n^{-1/2}(1+f(n))$$

and obtain

$$n = n(1 - 2f(n) + O\{f^2(n)\}) + O\{n\exp(-c_1(\log n)^{4/7}(\log\log n)^{-3/7})\}$$

from which it follows that the last $O$-term is an estimate for $f(n)$. Thus for some constant $c > 0$,

$$\alpha = \frac{\pi}{\sqrt{6}}n^{-1/2}[1 + O\{\exp(-c(\log n)^{4/7}(\log\log n)^{-3/7})\}] \tag{3.6}$$

We now revert to assuming $n$ integral.

Finally, we conclude from (3.1), (3.6) and Theorem 1.1 that:

THEOREM 3.1. *There exists a constant $c > 0$ such that*

$$\log p_\Lambda(n) = \pi\sqrt{\frac{2}{3}}n^{1/2}[1 + O\{\exp(-c(\log n)^{4/7}(\log\log n)^{-3/7})\}].$$

Define $\theta$ to be the least upper bound of the real parts of the imaginary zeros of the Riemann zeta function.

THEOREM 3.2. *With $\theta$ define as above*

$$\log p_\Lambda(n) = \pi\sqrt{\frac{2}{3}}n^{1/2} + O\{n^{\theta/2}\}.$$

*Proof.* Applying the argument of [**3**, pp. 194–197] to (3–4) one obtains

$$n = \alpha^{-2}\frac{\pi^2}{6} - \alpha^{-1}\sum_\rho \frac{\zeta(1+\rho)\Gamma(1+\rho)}{\alpha^\rho} - \log 2\pi + O\{\alpha\}.$$

Now assuming $0 < \alpha < 1$,

$$\left|\left(\frac{1}{\alpha}\right)^\rho\right| \leqq \alpha^{-\theta}.$$

Furthermore since $N(T)$, the number of zeros of $\zeta(s)$ in the region $0 \leqq \sigma \leqq 1$, $0 < t \leqq T$, satisfies the relation [**12**, p. 178]

$$N(T+1) - N(T) = O(\log T)$$

it follows from Stirling's formula and the fact that $\zeta(s)$ is a function of finite order that

$$\sum_{\rho} |\zeta(1+\rho)\Gamma(1+\rho)| = O\{1\}$$

and we have that

$$n = \alpha^{-2}\frac{\pi^2}{6} + O\{\alpha^{-1-\theta}\}.$$

Now with $c = \pi/\sqrt{6}$

$$|\alpha n - c\sqrt{n}| = \frac{|\alpha^2 n^2 - c^2 n|}{\alpha n + c\sqrt{n}} < \frac{D\alpha^{1-\theta} n}{C\sqrt{n}} = D_1(n\alpha^2)^{(1-\theta)/2} \leqq D_2 n^{\theta/2}$$

and

$$\alpha n = \frac{\pi}{\sqrt{6}} n^{1/2} + O\{n^{\theta/2}\}.$$

Similarly from (3.1)

$$\log g(\alpha) = \frac{\pi}{\sqrt{6}} n^{1/2} + O\{n^{\theta/2}\}$$

thus the theorem follows from Theorem 1.1.

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