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ON THE RANKIN–SELBERG ZETA FUNCTION

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Abstract

We obtain the approximate functional equation for the Rankin–Selberg zeta function in the critical strip and, in particular, on the critical line $\operatorname{Re} s = \frac{1}{2}$.

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1. Introduction

Let $\varphi(z)$ be a holomorphic cusp form of weight κ with respect to the full modular group $\mathrm{SL}(2, \mathbb{Z})$, so that

$$\varphi\left(\frac{az+b}{cz+d}\right) = (cz+d)^\kappa \varphi(z)$$

where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$, $\operatorname{Im} z > 0$ and $\lim_{\operatorname{Im} z \to \infty} \varphi(z) = 0$ (see, for instance, Rankin [12] for basic notions). We denote by $a(n)$ the nth Fourier coefficient of $\varphi(z)$ and suppose that $\varphi(z)$ is a normalized eigenfunction for the *Hecke operators* $T(n)$, that is, $a(1) = 1$ and $T(n)\varphi = a(n)\varphi$ for every $n \in \mathbb{N}$ (see Rankin [12] for the definition and properties of the Hecke operators). The classical example is $a(n) = \tau(n)$, when $\kappa = 12$. This is the well-known *Ramanujan tau function*, defined by

$$\sum_{n=1}^{\infty} \tau(n) x^n = x\{(1-x)(1-x^2)(1-x^3)\cdots\}^{24}$$

when $|x| < 1$.

Let c_n be the nonnegative convolution function defined by

$$c_n = n^{1-\kappa} \sum_{m^2 | n} m^{2(\kappa-1)} \left| a\left(\frac{n}{m^2}\right)\right|^2. \tag{1.1}$$

Note that c_n is a multiplicative arithmetic function, that is, $c_{mn} = c_m c_n$ when $(m, n) = 1$, since $a(n)$ is multiplicative.

The well-known *Rankin–Selberg problem* consists of the estimation of the error term function

$$\Delta(x) = \sum_{n \leqslant x} c_n - Cx. \tag{1.2}$$

The positive constant C in (1.2) may be written explicitly (see, for instance, [8]):

$$C = C(\varphi) = \frac{2\pi^2 (4\pi)^{\kappa-1}}{\Gamma(\kappa)} \iint_{\mathfrak{F}} y^{\kappa-2} |\varphi(x+iy)|^2 \, dx \, dy,$$

the integral being taken over a fundamental domain $\mathfrak{F}$ of the group $\mathrm{SL}(2, \mathbb{Z})$. The classical upper bound for $\Delta(x)$ (strictly speaking, $\Delta(x) = \Delta(x; \varphi)$) due to Rankin and Selberg, obtained independently in their important works [11, 14] published in 1939–1940, is

$$\Delta(x) = O(x^{3/5}). \tag{1.3}$$

This result is one of the longest-standing unimproved bounds of analytic number theory, but this paper is not concerned with this problem. Our object of study is the so-called *Rankin–Selberg zeta function*

$$Z(s) = \sum_{n=1}^{\infty} c_n n^{-s}, \tag{1.4}$$

which is the generating *Dirichlet series* for the sequence $\{c_n\}_{n \geq 1}$. One can define the Rankin–Selberg zeta function in various degrees of generality; see, for instance, Li and Wu [10] where the authors establish universality properties of such functions.

Note that the series in (1.4) converges absolutely if $\mathrm{Re}\, s > 1$. Indeed, from (1.2) and the estimate, due to Deligne [1], that $|a(n)| \leq n^{(\kappa-1)/2} d(n)$, where $d(n)$ is the number of positive divisors of n (note that $d(n) \ll_\varepsilon n^\varepsilon$),

$$c_n \ll_\varepsilon n^\varepsilon, \tag{1.5}$$

providing absolute convergence of $Z(s)$ when $\mathrm{Re}\, s > 1$. Here and later ε denotes an arbitrarily small constant, not necessarily the same at each occurrence, while $a = O_\varepsilon(b)$ and $a \ll_\varepsilon b$ mean that $a \leqslant Cb$, where C depends on ε.

When $\mathrm{Re}\, s \leq 1$, the function $Z(s)$ is defined by analytic continuation. It has a simple pole at $s = 1$ with residue C (compare with (1.1)), and is otherwise regular. For every $s \in \mathbb{C}$ it satisfies the functional equation

$$\Gamma(s+\kappa-1)\Gamma(s)Z(s) = (2\pi)^{4s-2}\Gamma(\kappa - s)\Gamma(1-s)Z(1-s), \tag{1.6}$$

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where $\Gamma(s)$ is the *gamma function*. One has the decomposition

$$Z(s) = \zeta(2s) \sum_{n=1}^{\infty} |a(n)|^2 n^{1-\kappa-s},$$

where $\zeta(s)$ is the familiar *Riemann zeta function* ($\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ when $\mathrm{Re}\, s > 1$). This formula is the analytic equivalent of the arithmetic relation (1.1). In our context, it is more important that one also has the decomposition

$$Z(s) = \sum_{n=1}^{\infty} c_n n^{-s} = \zeta(s) \sum_{n=1}^{\infty} b_n n^{-s} = \zeta(s) B(s), \tag{1.7}$$

say, where $B(s)$ belongs to the *Selberg class* of *Dirichlet series* of degree three. The coefficients b_n in (1.7) are multiplicative and satisfy

$$b_n \ll_\varepsilon n^\varepsilon. \tag{1.8}$$

This follows from the formula

$$b_n = \sum_{d|n} \mu(d) c_{n/d},$$

which is a consequence of (1.7), the *Möbius inversion formula* and (1.5). Actually the coefficients b_n are bounded by a log power (see [13]) in mean square, but this stronger property is not needed here. For the definition and basic properties of the Selberg class $\mathcal{S}$ of L-functions the reader is referred to Selberg's seminal paper [15] and the comprehensive survey paper of Kaczorowski and Perelli [9].

In view of (1.8), the series for $B(s)$ converges absolutely when $\mathrm{Re}\, s > 1$, but $B(s)$ has an analytic continuation that is holomorphic when $\mathrm{Re}\, s > 0$. This important fact follows from Shimura's work [16] (see also Sankaranarayanan [13]), and it implies that (1.7), that is, $Z(s) = \zeta(s)B(s)$, holds when $\mathrm{Re}\, s > 0$ and not only when $\mathrm{Re}\, s > 1$. The function $B(s)$ is of degree three in $\mathcal{S}$, as its functional equation (see, for instance, Sankaranarayanan [13]) is

$$B(s)\Delta_1(s) = B(1-s)\Delta_1(1-s),$$
$$\Delta_1(s) = \pi^{-3s/2}\Gamma(\tfrac{1}{2}(s+\kappa-1))\Gamma(\tfrac{1}{2}(s+\kappa))\Gamma(\tfrac{1}{2}(s+\kappa+1)).$$

It is very likely that $B(s)$ is primitive in $\mathcal{S}$, that is, it cannot be factored nontrivially as $F_1(s)F_2(s)$ with $F_1, F_2 \in \mathcal{S}$, but this seems hard to prove. Since $B(s)$ is holomorphic for $\mathrm{Re}\, s > 0$, it would follow that one of the factors, say $F_1(s)$, is $L(s + i\alpha, \chi)$ for some $\alpha \in \mathbb{R}$ and χ a primitive Dirichlet character. This follows from the fact that elements of degree one in $\mathcal{S}$ are $\zeta(s + i\alpha)$ and $L(s + i\alpha, \chi)$. However, then $F_2(s)$ would have degree two in $\mathcal{S}$, but the classification of functions in $\mathcal{S}$ of degree two is a difficult open problem.

2. The approximate functional equation for $Z(s)$

Approximate functional equations are an important tool in the study of Dirichlet series $F(s) = \sum_{n\ge1} f(n)n^{-s}$. Their purpose is to approximate $F(s)$ by *Dirichlet*

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polynomials of the type $\sum_{n\le x} f(n)n^{-s}$ in a certain region where the series defining $F(s)$ does not converge absolutely. In the case of the powers of $\zeta(s)$ they were studied, for instance, in [5, Ch. 4] and [6], and in a more general setting by the author [7].

Before we state our results, which involve approximations of $Z(s)$ by Dirichlet polynomials of the form $\sum_{n\le x} c_n n^{-s}$, we need some notation. Let (see (1.6))

$$X(s) = \frac{Z(s)}{Z(1-s)} = (2\pi)^{4s-2}\frac{\Gamma(\kappa-s)\Gamma(1-s)}{\Gamma(s+\kappa-1)\Gamma(s)}, \tag{2.1}$$

let $\tau = \tau(t)$ be defined by

$$\log \tau = -\frac{X'(\frac{1}{2}+it)}{X(\frac{1}{2}+it)} \tag{2.2}$$

where $t \ge 3$, and

$$\Phi(w) = \Phi(w;\, s, \tau) = \tau^{w-s}X(w) - X(s) \tag{2.3}$$

where $\frac{1}{2} \le \sigma = \mathrm{Re}\, s \le 1$. Then the following theorem holds.

Theorem 2.1. *If $\frac{1}{2} \le \sigma = \mathrm{Re}\, s \le 1$, $t \ge 3$, and $s = \sigma + it$, then*

$$\begin{aligned} Z(s) = {} & \sum_{n\le x} c_n n^{-s} + X(s)\sum_{n\le y} c_n n^{s-1} + C_1\frac{x^{1-s}}{1-s} + C_2X(s)\frac{y^s}{s} \\ & + O_\varepsilon\{t^\varepsilon(x^{-\sigma} + hx^{1-\sigma}) + t^{2+\varepsilon-4\sigma}(y^{\sigma-1} + hy^\sigma)\} \\ & - \frac{1}{2\pi i h^3}\int_{1/2-i\infty}^{1/2+i\infty} Z(1-z)\Phi(z;\, s, \tau)y^{s-z}(z-s)^{-4}(1-e^{-h(s-z)})^3\, dz, \end{aligned} \tag{2.4}$$

where $xy = \tau$, $1 \ll x \ll \tau$, $1 \ll y \ll \tau$, $0 < h \le 1$ is a parameter to be suitably chosen, and C_1 and C_2 are absolute constants.

The restriction $\frac{1}{2} \le \sigma = \mathrm{Re}\, s \le 1$ in Theorem 2.1 can be removed, and one can consider the whole range $0 \le \sigma \le 1$. For $0 \le \sigma \le \frac{1}{2}$ this is achieved by replacing s by $1-s$, interchanging x and y, and using $Z(1-s)X(s) = Z(s)$, together with (2.4) and (3.5) of Lemma 3.2.

The most important case of Theorem 2.1 is when s lies on the so-called *critical line* $\mathrm{Re}\, s = \frac{1}{2}$, that is, $s = \frac{1}{2} + it$. Then we obtain the following result from (2.4).

Theorem 2.2. *If $s = \frac{1}{2} + it$, $t \ge 3$, $xy = \tau$, $1 \ll x \ll \tau$ and $1 \ll y \ll \tau$, then*

$$\begin{aligned} Z(s) = {} & \sum_{n\le x} c_n n^{-s} + X(s)\sum_{n\le y} c_n n^{s-1} + C_1\frac{x^{1-s}}{1-s} + C_2X(s)\frac{y^s}{s} \\ & + O_\varepsilon(t^{\varepsilon-11/16}(x^{1/2} + t^2x^{-1/2})^{3/4}) + O_\varepsilon(t^{1/2+\mu(1/2)+\varepsilon}), \end{aligned} \tag{2.5}$$

where, for $\sigma \in \mathbb{R}$,

$$\mu(\sigma) = \limsup_{t\to\infty}\frac{\log|\zeta(\sigma+it)|}{\log t}.$$

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The best-known result, that $\mu(1/2) \le 32/205 = 0.15609\ldots$, is due to Huxley [4]. The famous *Lindelöf hypothesis* is that $\mu(1/2) = 0$ (this is equivalent to $\mu(\sigma) = 0$ for $\sigma \ge 1/2$), and it makes the second error term in (2.5) equal to $O_\varepsilon(t^{1/2+\varepsilon})$.

In general, if one introduces smooth weights in the sums in question, then the ensuing error terms are substantially improved. This was done, for instance, in [5, Ch. 4], in [6] and in [7]. From [7, equations (19) and (20)], with $\sigma = \frac{1}{2}$, $K = 4$, $t \ge 3$, $xy = \tau$, and $1 \ll x, y \ll \tau$,

$$Z(s) = \sum_{n\le x} \rho(n/x) c_n n^{-s} + X(s) \sum_{n\le y} \rho(n/y) c_n n^{s-1} + O_\varepsilon(t^\varepsilon), \tag{2.6}$$

where $s = \frac{1}{2} + it$. The smooth function $\rho(x)$ is defined as follows (see [6, Ch. 4] for an explicit construction). Let $b > 1$ be a fixed constant and $\rho(x) \in C^\infty(0, \infty)$. Then

$$\rho(x) + \rho(1/x) = 1 \quad \forall x > 0 \quad \text{and} \quad \rho(x) = 0 \quad \forall x \ge b.$$

There is another aspect of this subject worth mentioning. One can consider the function

$$\mathcal{Z}(t) = Z(\tfrac{1}{2} + it) X^{-1/2}(\tfrac{1}{2} + it) \tag{2.7}$$

where $t \in \mathbb{R}$. The functional equation for $Z(s)$ in the form $Z(s) = X(s)Z(1 - s)$ leads easily to $X(s)X(1 - s) = 1$, hence

$$\begin{aligned}\overline{\mathcal{Z}(t)} &= Z(\tfrac{1}{2} - it) X^{-1/2}(\tfrac{1}{2} - it) = Z(\tfrac{1}{2} + it) X(\tfrac{1}{2} - it) X^{-1/2}(\tfrac{1}{2} - it)\\ &= Z(\tfrac{1}{2} + it) X^{-1/2}(\tfrac{1}{2} + it) = \mathcal{Z}(t).\end{aligned}$$

Therefore $\mathcal{Z}(t) \in \mathbb{R}$ when $t \in \mathbb{R}$. The function $\mathcal{Z}(t)$ is the analogue of Hardy's classical function $\zeta(\frac{1}{2} + it)\chi^{-1/2}(\frac{1}{2} + it)$, where $\zeta(s) = \chi(s)\zeta(1 - s)$, which plays a fundamental role in the study of the zeros of $\zeta(s)$ on the *critical line* $\mathrm{Re}\, s = 1/2$. Taking $x = (t/2\pi)^2$ in Theorem 2.2, we then obtain, with the aid of Lemma 3.2, the following corollary.

Corollary 2.3. *For $t \in \mathbb{R}$ such that $|t| \ge 1$,*

$$\mathcal{Z}(t) = 2 \sum_{n\le (t/2\pi)^2} c_n n^{-1/2} \cos\Big(t \log\Big(\frac{(t/2\pi)^2}{n}\Big) - 2t + (\kappa - 1)\pi\Big) + O_\varepsilon(t^{1/2+\mu(1/2)+\varepsilon}). \tag{2.8}$$

One can compare (2.8) to the analogue for $Z^4(t) = |\zeta(\frac{1}{2} + it)|^4$, since [5, equation (4.29)] may be rewritten as

$$Z^4(t) = 2 \sum_{n\le (t/2\pi)^2} d_4(n) n^{-1/2} \cos\Big(t \log\Big(\frac{(t/2\pi)^2}{n}\Big) - 2t - \frac{1}{2}\pi\Big) + O_\varepsilon(t^{13/48+\varepsilon}), \tag{2.9}$$

where $d_4(n) = \sum_{abcd=n} 1$ is the divisor function generated by $\zeta^4(s)$. The reason why the error term in (2.9) is sharper than that in (2.8) is that we have much more information on $\zeta^4(s)$ than on $Z(s)$.

The rest of this paper is organized as follows. In Section 3 we shall formulate and prove the lemmas necessary for the proofs. In Section 4 we shall prove Theorem 2.1, and in Section 5 we shall prove Theorem 2.2.

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3. The necessary lemmas

Here is our first lemma.

LEMMA 3.1. *If $X > 1$, then*

$$\int_0^X \left|Z\left(\frac{1}{2}+it\right)\right| dt \ll_\varepsilon X^{5/4+\varepsilon}. \tag{3.1}$$

PROOF. From the decomposition (1.7) and the Cauchy–Schwarz inequality for integrals,

$$\int_{X/2}^X \left|Z\left(\frac{1}{2}+it\right)\right| dt \le \left(\int_{X/2}^X \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt \int_{X/2}^X \left|B\left(\frac{1}{2}+it\right)\right|^2 dt\right)^{1/2}. \tag{3.2}$$

Note that we have the elementary bound (see, for instance, [5, Ch. 1])

$$\int_0^X \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt \ll X \log X, \tag{3.3}$$

and that $B(s)$ belongs to the Selberg class of degree three. Therefore $B(s)$ is analogous to $\zeta^3(s)$, and by following the proof of [5, Theorem 4.4] (when $k = 3$) it may be seen that $B(s)$ satisfies an analogous approximate functional equation, where $M \ge (3X)^3/Y$ and $X^\varepsilon \le t \le X$. Taking $Y = X^{3/2}$ and applying the *mean value theorem for Dirichlet polynomials* (see [5, Theorem 5.2]), we obtain, in view of (1.8),

$$\int_{X/2}^X \left|B\left(\frac{1}{2}+it\right)\right|^2 dt \ll_\varepsilon X^{3/2+\varepsilon}. \tag{3.4}$$

The bound in (3.1) follows immediately from equations (3.2)–(3.4) if we replace X by $X/2^j$ (where $j = 1, 2, \ldots$) and add the resulting expressions. The best bound for the integral in (3.1) is $X^{1+\varepsilon}$, up to ε. This follows, for instance, by obvious modifications of the arguments used in the proof of [5, Theorem 9.5]. It would improve the bound in (1.3) to $O_\varepsilon(x^{1/2+\varepsilon})$. □

LEMMA 3.2. *For $0 \le \sigma \le 1$ fixed and $t \ge 3$,*

$$X(\sigma+it) = \left(\frac{t}{2\pi}\right)^{2-4\sigma} \exp\left(4it - 4it\log\left(\frac{t}{2\pi}\right) + (1-\kappa)\pi i\right) \times \left(1 + O\left(\frac{1}{t}\right)\right), \tag{3.5}$$

where the O-term admits an asymptotic expansion in negative powers of t.

PROOF. This follows from (2.1) and the full form of Stirling's formula, that is,

$$\log\Gamma(s+b) = \left(s+b-\frac{1}{2}\right)\log s - s + \frac{1}{2}\log 2\pi + \sum_{j=1}^{K} \frac{(-1)^j B_{j+1}(b)}{j(j+1)s^j} + O_\delta\left(\frac{1}{|s|^{K+1}}\right),$$

which holds for a constant b, any fixed integer $K \ge 1$, and $|\arg s| \le \pi - \delta$ for $\delta > 0$, where the points $s = 0$ and the neighbourhoods of the poles of $\Gamma(s+b)$ are excluded, and the $B_j(b)$ are *Bernoulli polynomials*; see, for instance, Erdélyi *et al.* [2]. □

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Lemma 3.3. *Let $\tau = \tau(t)$ be defined by (2.2). Then*

$$\tau = \Big(\frac{t}{2\pi}\Big)^4\Big(1 + O\Big(\frac{1}{t^2}\Big)\Big), \tag{3.6}$$

where $t \ge 3$; the O-term admits an asymptotic expansion in negative powers of t. If $\Phi(w)$ is defined by (2.3), then $\Phi(w)(s-w)^{-2}$ is regular for $\mathrm{Re}\, w \le \frac{1}{2}$ and also for $\mathrm{Re}\, w < \sigma$ if $\frac{1}{2} < \sigma \le 1$. Moreover, uniformly in s for $\mathrm{Re}\, w = \frac{1}{2}$ and $t \ge 3$,

$$\Phi(w) \ll t^{2-4\sigma} \min\{1, (t^{-1}|w-s|^2)\}. \tag{3.7}$$

Proof. The functions τ and Φ were introduced, in the case of $\zeta^2(s)$, by Hardy and Littlewood [3] in their classical proof of the approximate functional equation for $\zeta^2(s)$. To prove (3.6), recall from (2.1) that

$$X(s) = \frac{Z(s)}{Z(1-s)} = (2\pi)^{4s-2}\frac{\Gamma(\kappa - s)\Gamma(1-s)}{\Gamma(s+\kappa-1)\Gamma(s)}.$$

Logarithmic differentiation then gives

$$-\frac{X'(\frac{1}{2}+it)}{X(\frac{1}{2}+it)}$$
$$= -4\log(2\pi) + \frac{\Gamma'(\kappa - \frac{1}{2} - it)}{\Gamma(\kappa - \frac{1}{2} - it)} + \frac{\Gamma'(\frac{1}{2} - it)}{\Gamma(\frac{1}{2} - it)} + \frac{\Gamma'(\kappa - \frac{1}{2} + it)}{\Gamma(\kappa - \frac{1}{2} + it)} + \frac{\Gamma'(\frac{1}{2} + it)}{\Gamma(\frac{1}{2} + it)}.$$

If we use (see [5, equation (A.35)])

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s - \frac{1}{2s} + O\Big(\frac{1}{|s|^2}\Big)$$

(when $|\arg s| \le \pi - \delta$ and $|s| \ge \delta$), where the O-term has an asymptotic expansion in term of negative powers of s,

$$\log\tau = -\frac{X'(\frac{1}{2}+it)}{X(\frac{1}{2}+it)} = 4\log t - 4\log(2\pi) + O\Big(\frac{1}{t^2}\Big)$$

when $t \ge 3$, which is equivalent to (3.6).

The only nontrivial case concerning the regularity of $\Phi(w)(s-w)^{-2}$ is when $w = \frac{1}{2} + iv$ and $s = \frac{1}{2} + it$, and this follows from (3.7). If $w = \frac{1}{2} + iv$, then

$$|\Phi(w)| \le \tau^{1/2-\sigma}|X(\tfrac{1}{2}+iv)| + |X(\sigma+it)| \ll t^{2-4\sigma},$$

in view of (3.6) and (3.5).

To obtain the other bound in (3.7) suppose that $|w-s| \ll \sqrt{t}$, which is the relevant range of its validity. Then $v \asymp t$ (that is, $v \ll t$ and $t \ll v$) for $w = \frac{1}{2} + iv$, and

$$\frac{d^2}{dw^2}X(w) \asymp \frac{1}{t}$$

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when $w = \frac{1}{2} + iv$ and $v \asymp t$. Write (2.3) as

$$\Phi(w) = \tau^{w-s} X(w)\Big(1 - \frac{X(s)}{X(w)}\tau^{s-w}\Big) \tag{3.8}$$

and note that, by Taylor's formula,

$$\begin{aligned}
\frac{X(s)}{X(w)}\tau^{s-w} &= \exp(\log X(s) - \log X(w) + (s-w)\log\tau) \\
&= \exp\Big((s-w)\frac{X'(w)}{X(w)} + O(|s-w|^2 t^{-1}) + (s-w)\log\tau\Big) \\
&= \exp\Big((s-w)\frac{X'(\frac{1}{2}+it)}{X(\frac{1}{2}+it)} + O(|s-w|^2 t^{-1}) + (s-w)\log\tau\Big) \\
&= 1 + O(|s-w|^2 t^{-1}),
\end{aligned}$$

in view of (2.2) and (2.6). If we insert this in (3.8), then we obtain the second estimate in (3.7) from (3.5) and (3.6). □

4. Proof of Theorem 2.1

The idea of the proof of Theorem 2.1 goes back to Hardy and Littlewood [3], who considered the approximate functional equation for $\zeta^2(s)$. Wiebelitz [17] generalized their method to deal with $\zeta^k(s)$ when $k \in \mathbb{N}$ and $k > 2$, and this was refined in [5, Theorem 4.3]. In what follows we shall make the modifications which are necessary in the case of $Z(s)$. Let the hypotheses of Theorem 2.1 hold and set

$$\begin{aligned}
I = I(s, x) &= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} Z(s+w)x^w w^{-4}\,dw \\
&= \sum_{n=1}^{\infty} c_n n^{-s}\Big\{\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Big(\frac{x}{n}\Big)^w w^{-4}\,dw\Big\} \\
&= \frac{1}{3!}\sum_{n\le x} c_n n^{-s}\log^3(x/n) = S_x,
\end{aligned}$$

say, where we used the absolute convergence of $Z(s)$ for $\sigma > 1$ and [5, equation (A.12)] with $k = 4$, reflecting the fact that $Z(s)$ belongs to the Selberg class of degree $k = 4$. The basic idea is to use a differencing argument to recover $\sum_{n\le x} c_n n^{-s}$ from the same sum weighted by $\log^3(x/n)$. To achieve this, first we move the line of integration in I to $\operatorname{Re} w = -1/4$. In doing this we pass over the poles $w = 0$ and $w = 1 - s$ of the integrand, with the residues

$$F_x = \sum_{m=0}^{3}\frac{Z^{(m)}(s)}{m!\,(3-m)!}(\log x)^{3-m}$$

and

$$Q_x := \frac{Cx^{1-s}}{(1-s)^4},$$

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respectively. Hence by the residue theorem,

$$J_0 = \frac{1}{2\pi i}\int_{-1/4-i\infty}^{-1/4+i\infty} Z(s+w)x^w w^{-4}\,dw = I - F_x - Q_x = S_x - F_x - Q_x. \tag{4.1}$$

In the integral in (4.1), set $z = s + w$, replace x by τ/y, and use the functional equation for $Z(s)$ and (2.3) in the form

$$\tau^{u-s}X(u) = X(s) + \Phi(u; s, \tau)$$

to obtain

$$\begin{aligned} J_0 &= \frac{1}{2\pi i}\int_{-1/4-i\infty}^{-1/4+i\infty} Z(1-z)X(s)y^{s-z}(z-s)^{-4}\,dz \\ &\quad + \frac{1}{2\pi i}\int_{-1/4-i\infty}^{-1/4+i\infty} Z(1-z)\Phi(z; s, \tau)y^{s-z}(z-s)^{-4}\,dz \\ &= X(s)J_1 + J_2, \end{aligned}$$

say. This is the point that explains the definition of the function Φ in (2.3). We use [5, equation (A.12)] again to deduce that

$$J_1 = \frac{1}{3!}\sum_{n\le y} c_n n^{s-1}\log^3(x/n) = S_y,$$

with notation similar to when we evaluated I. The line of integration in J_2 is moved to $\operatorname{Re} z = 1/4$. We pass over the pole $z = 0$ of the integrand, picking up the residue $-X(s)Q_y$, where

$$Q_y = -\frac{Cy^s}{s^4}.$$

Therefore from (4.1),

$$F_x - S_x + Q_x = -X(s)(S_y - Q_y) - J_y \tag{4.2}$$

with

$$J_y = \frac{1}{2\pi i}\int_{1/4-i\infty}^{1/4+i\infty} Z(1-z)\Phi(z; s, \tau)y^{s-z}(z-s)^{-4}\,dz.$$

In (4.2) we replace x and y by $xe^{\nu h}$ and $ye^{-\nu h}$ (where $0 \le \nu \le 3$), respectively, so that the condition $xe^{\nu h}\cdot ye^{-\nu h} = \tau$ is preserved. We use (see [5, equations (4.39) and (4.40)])

$$\sum_{\nu=0}^{m}(-1)^\nu\binom{m}{\nu}\nu^p = m! \quad \forall p \in \mathbb{N} \tag{4.3}$$

when $p = m$, and the result that the sum is equal to 0 when $p < m$, and the estimate

$$e^z = \sum_{n=0}^{M}\frac{z^n}{n!} + O(|z|^{M+1}),$$

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(when $M \geq 1$ and $a \leq \operatorname{Re} z \leq b$), where a and b are fixed. To better distinguish the sums which will arise in this process, we introduce left indices to obtain, from (4.2),

$$\sum_{\nu=0}^{3}(-1)^{\nu}\binom{3}{\nu}({}_{\nu}F_{x} - {}_{\nu}S_{x} + {}_{\nu}Q_{x} + X(s)({}_{\nu}S_{y} - {}_{\nu}Q_{y}) + {}_{\nu}J_{y}) = 0,$$

or abbreviating,

$$\bar{F}_x - \bar{S}_x + \bar{Q}_x + X(s)\bar{S}_y - X(s)\bar{Q}_y + \bar{J}_y = 0. \tag{4.4}$$

Each term in (4.4) will be evaluated or estimated separately. First,

$$\bar{F}_x = \sum_{m=0}^{3} \frac{Z^{(m)}(s)}{3!\,(3-m)!} A_m(x),$$

where

$$A_m(x) = \sum_{\nu=0}^{3}(-1)^{\nu}\binom{3}{\nu}(\log x + \nu h)^{3-m}$$
$$= \sum_{r=0}^{3-m}\binom{3-m}{r} h^r \log^{3-m-r} x \sum_{\nu=0}^{3}(-1)^{\nu}\binom{3}{\nu}\nu^r = 3!h^3$$

for $m = 0$, and otherwise $A_m(x) = 0$, where we used (4.3). Therefore

$$\bar{F}_x = h^3 Z(s),$$

and this is exactly what is needed for the approximate functional equation that will follow on dividing (4.4) by h^3. Consider next

$$\bar{S}_x = \frac{1}{3!}\sum_{n\leq x} c_n n^{-s} \sum_{\nu=0}^{3}\binom{3}{\nu}(-1)^{\nu}(\nu h + \log(x/n))^3$$
$$+ \frac{1}{3!}\sum_{\nu=0}^{3}\binom{3}{\nu}(-1)^{\nu} \sum_{x<n\leq xe^{\nu h}} c_n n^{-s}(\nu h + \log(x/n))^3$$
$$= \Sigma_1 + \Sigma_2,$$

say. Analogously to the evaluation of $\bar{F}_x$ it follows that

$$\Sigma_1 = h^3 \sum_{n\leq x} c_n n^{-s}.$$

We estimate Σ_2 trivially, using (1.5), to obtain

$$|\Sigma_2| \leq \frac{1}{3!}\sum_{\nu=0}^{3}\binom{3}{\nu}(2\nu h)^3 x^{-\sigma} \sum_{x<n\leq xe^{3h}} c_n$$
$$\ll_{\varepsilon} h^3 x^{-\sigma} t^{\varepsilon}(1 + x(e^{3h} - 1)) \ll_{\varepsilon} t^{\varepsilon}(h^3 x^{-\sigma} + h^4 x^{1-\sigma}).$$

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Similarly, it follows that

$$-X(s)\bar{S}_y = h^3X(s)\sum_{n\le y} c_n n^{s-1} + O_\varepsilon\Big(h^3|X(\sigma+it)|\sum_{\nu=0}^{3}\sum_{ye^{-3h}<n\le y} c_n n^{\sigma-1}\Big)$$
$$= h^3X(s)\sum_{n\le y} c_n n^{s-1} + O_\varepsilon(h^3t^{2+\varepsilon-4\sigma}(y^{\sigma-1}+hy^\sigma)).$$

Also

$$\bar{Q}_x = 3!h^3C\frac{x^{1-s}}{1-s} + O(h^4x^{1-\sigma})$$

and

$$X(s)\bar{Q}_y = C_2X(s)h^3\frac{y^s}{s} + O_\varepsilon(t^{2+\varepsilon-4\sigma}h^4y^\sigma).$$

Therefore we are left with the evaluation of

$$\bar{J}_y = \frac{1}{2\pi i}\int_{1/4-i\infty}^{1/4+i\infty} Z(1-z)\Phi(z;s,\tau)y^{s-z}(z-s)^{-4}\sum_{\nu=0}^{3}(-1)^\nu\binom{3}{\nu}e^{-\nu h(s-z)}\,dz.$$

Observing that (3.7) holds and that the function

$$\sum_{\nu=0}^{3}(-1)^\nu\binom{3}{\nu}e^{-\nu h(s-z)} = (1-e^{-h(s-z)})^3$$

has a zero of order three at $z = s$, we can move the line of integration in $\bar{J}_y$ to $\operatorname{Re} z = \frac{1}{2}$. Hence

$$\bar{J}_y = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} Z(1-z)\Phi(z;s,\tau)y^{s-z}(z-s)^{-4}(1-e^{-h(s-z)})^3\,dz.$$

Therefore we obtain the assertion of Theorem 2.1 from (4.4) by dividing the whole expression by h^3 and collecting the above estimates for the error terms.

5. Proof of Theorem 2.2

We set $s = \frac{1}{2} + it$ and $z = \frac{1}{2} + iv$ in (2.4), and write the right-hand-side integral as

$$i\int_{-\infty}^{\infty}\cdots\,dv = i\Big(\int_{-\infty}^{t/2} + \int_{t/2}^{2t} + \int_{2t}^{\infty}\Big)\cdots\,dv = i(I_1+I_2+I_3), \tag{5.1}$$

say. The integrals I_1 and I_3 are estimated similarly. The latter is, by trivial estimation and the first bound in (3.7),

$$\begin{aligned}&\int_{2t}^{\infty} Z\Big(\frac{1}{2}-iv\Big)\Phi\Big(\frac{1}{2}+iv;s,\tau\Big)y^{i(t-v)}(t-v)^{-4}(1-e^{-hi(t-v)})^3\,dv\\ &\quad\ll \int_{2t}^{\infty}\Big|Z\Big(\frac{1}{2}+iv\Big)\Big|v^{-4}\,dv \ll_\varepsilon t^{\varepsilon-11/4},\end{aligned} \tag{5.2}$$

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where we used (3.1). From (2.4), (5.1) and (5.2), it follows that

$$
\begin{aligned}
Z(s) = \sum_{n\le x} c_n n^{-s} + X(s)\sum_{n\le y} c_n n^{s-1} + C_1\frac{x^{1-s}}{1-s} + C_2 X(s)\frac{y^s}{s} \\
+ O_\varepsilon(1 + t^{\varepsilon-11/16}(x^{1/2} + t^2x^{-1/2})^{3/4}) - \frac{1}{2\pi i h^3} I_2,
\end{aligned}
\tag{5.3}
$$

with the choice

$$h = t^{-11/16}(x^{1/2} + t^2x^{-1/2})^{-1/4},$$

so that $0 < h \le 1$. To estimate I_2, we use

$$(1 - e^{-hi(t-v)})^3 \ll h^3|t - v|^3$$

and the second bound in (3.7) ($\sigma = \frac{1}{2}$). This gives, on using the Cauchy–Schwarz inequality for integrals,

$$
\begin{aligned}
h^{-3}I_2 &\ll \int_{t/2}^{2t}\Big|Z\Big(\frac{1}{2}+iv\Big)\Big| \min\Big(\frac{1}{|t-v|}, \frac{|t-v|}{v}\Big)\,dv \\
&\ll \Big(\int_{t/2}^{2t}\Big|Z\Big(\frac{1}{2}+iv\Big)\Big|^2\,dv\Big)^{1/2}(j_1 + j_2 + j_3)^{1/2},
\end{aligned}
\tag{5.4}
$$

say. By (1.7), (3.4) and the definition of the μ-function,

$$\int_{t/2}^{2t}\Big|Z\Big(\frac{1}{2}+iv\Big)\Big|^2\,dv = \int_{t/2}^{2t}\Big|\zeta\Big(\frac{1}{2}+iv\Big)\Big|^2\Big|B\Big(\frac{1}{2}+iv\Big)\Big|^2\,dv \ll_\varepsilon t^{2\mu(1/2)+3/2+\varepsilon}. \tag{5.5}$$

Now

$$j_1 = \int_{t/2}^{t-\sqrt{t}} \frac{dv}{(t-v)^2} \ll \frac{1}{\sqrt{t}},$$

and the same bound holds for

$$j_3 = \int_{t+\sqrt{t}}^{2t} \frac{dv}{(t-v)^2}.$$

Further,

$$j_2 = \int_{t-\sqrt{t}}^{t+\sqrt{t}} (t-v)^2\frac{dv}{v^2} \ll \frac{1}{\sqrt{t}},$$

so that from (5.4) and (5.5) and the bounds for j_1, j_2 and j_3, we infer that

$$h^{-3}I_2 \ll_\varepsilon t^{1/2+\mu(1/2)+\varepsilon}. \tag{5.6}$$

The assertion of Theorem 2.2 follows from (5.3) and (5.6), since the first error term in (5.3) is absorbed by the right-hand side of (5.6), because $x^{1/2} \ll t^2$.

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